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The aggregate-monotonic core [☆]

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Abstract

We introduce the aggregate-monotonic core as the set of allocations of a transferable utility cooperative game attainable by single-valued solutions that satisfy core-selection and aggregate-monotonicity. We provide a necessary and sufficient condition for the coincidence of the core and the aggregate-monotonic core. Finally, we introduce upper and lower aggregate-monotonicity for set-valued solutions, and characterize the aggregate-monotonic core using core-selection and upper and lower aggregate-monotonicity.

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1. Introduction

The *core* (Gillies, 1959) of a *transferable utility cooperative game* (a game) is the set of feasible outcomes that cannot be improved upon by any coalition of players. Since the core of a game may be empty, generalizations and modifications have been considered from the beginning (for details see Kannai, 1992). A single-valued solution satisfies *core-selection* if it selects a core element for any game with a non-empty core.

In the study of games, several monotonicity properties of single-valued solutions have been introduced. Megiddo (1974) studies *aggregate-monotonicity*, which states that when the worth of the grand coalition increases whereas the worths of all other coalitions remain the same, then everyone's payoff should weakly increase. Young (1985) considers a stronger property, called *coalitional-monotonicity*: if the worth of a given coalition increases whereas the worths of all other coalitions remain the same, then the payoff of every member of that coalition should weakly increase. He provides a five-agent example showing that this property and core-selection are incompatible. Later, Housman and Clark (1998) show the same incompatibility for a four-agent game.

On the other hand, core-selection and aggregate-monotonicity are compatible on the domain of all games. The per-capita prenucleolus, a variant of the classical prenucleolus, defined by means of the per-capita excesses instead of

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the classical excesses, satisfies both properties (see for example Young et al., 1982 or Moulin, 1988). Nevertheless, for an arbitrary game there may be core elements that are never selected by single-valued solutions satisfying both properties. Consider for instance the three-player game¹: $v(i) = 0$ for all $i \in \{1, 2, 3\}$, $v(12) = v(13) = 1$, $v(23) = 0$ and $v(123) = 3$, and the core element $(0, 1, 2)$. This core allocation is never attainable by a single-valued solution that satisfies core-selection and aggregate-monotonicity, since in any such solution on game v , agent 1 gets at least payoff 1. To see this, notice that for game v' , defined as v except for $v'(123) = 1$, the allocation $(1, 0, 0)$ is the unique core element.

In this paper we introduce the *aggregate-monotonic core* of a game, defined as the set of allocations attainable by single-valued solutions that satisfy core-selection and aggregate-monotonicity. Firstly, we provide an explicit description of the aggregate-monotonic core (Theorem 5), which emphasizes the fact that the aggregate-monotonic core is non-empty for any game and it consists of a subset of the core for any game with a non-empty core. Secondly, in our main result (Theorem 7), we identify those games for which the core and the aggregate-monotonic core coincide, that is, those cases where there is no trade-off between core stability and aggregate-monotonicity. Finally, in Theorem 16, we characterize the aggregate-monotonic core seen as a set-valued solution. For this purpose, we use the natural extensions to set-valued solutions of core-selection and aggregate-monotonicity properties. We distinguish between *upper* and *lower aggregate-monotonicity* for set-valued solutions.

2. The aggregate-monotonic core

We denote by $N = \{1, \dots, n\}$ a finite set of players. A coalition is a subset of N and we denote by 2^N the set of all coalitions in N . By $|S|$ we denote the cardinality of the coalition $S \subseteq N$. We use the symbol \subset for strict set inclusions and \subseteq for weak set inclusions. By e_S we denote the characteristic vector of \mathbb{R}^N associated to coalition $S \subseteq N$, $S \neq \emptyset$, i.e. $e_{S,i} = 1$ if $i \in S$ and $e_{S,i} = 0$ if $i \notin S$; we use e_i instead of $e_{\{i\}}$ if no confusion arises.

A *transferable utility cooperative game* (a game) is a pair (N, v) (v , for short) where N is the set of players and $v: 2^N \rightarrow \mathbb{R}$ is the characteristic function with $v(\emptyset) = 0$. The number $v(S)$ is the worth of coalition S , that is, what S can achieve on its own. The subgame (S, v^S) of a game (N, v) is the game with player set $S \subseteq N$, $S \neq \emptyset$, and characteristic function $v^S(T) = v(T)$ for all $T \subseteq S$. The set of all games with player set N is denoted by G^N .

A vector $x \in \mathbb{R}^N$ is usually named a payoff vector or allocation, and each component x_i is interpreted as the allotment to player $i \in N$. Given a payoff vector $x \in \mathbb{R}^N$ and a coalition $S \subseteq N$, $S \neq \emptyset$, we write $x(S) = \sum_{i \in S} x_i$ for the payoff to coalition S , with $x(\emptyset) = 0$. We denote by \leq in \mathbb{R}^N the standard partial order, i.e. $x \leq y$ if $x_i \leq y_i$ for all $i \in N$. Given two non-empty subsets $A, B \subseteq \mathbb{R}^N$ and $k \in \mathbb{R}$, we define $A + B = \{x + y: x \in A \text{ and } y \in B\}$ and $kA = \{kx: x \in A\}$. By Δ_N we denote the unit-simplex in \mathbb{R}^N , i.e. $\Delta_N = \{x \in \mathbb{R}_+^N: x_1 + \dots + x_n = 1\}$. Finally, we say that x is an extreme point of the convex set $A \subseteq \mathbb{R}^N$ if $y, z \in A$ and $x = \frac{1}{2}y + \frac{1}{2}z$ imply $y = z$.

The *pre-imputation set* of the game (N, v) consists of those payoff vectors that allocate the worth of the grand coalition: $I^*(v) = \{x \in \mathbb{R}^N: x(N) = v(N)\}$. The *core* of the game (N, v) (Gillies, 1959) consists of those pre-imputations satisfying that every coalition $S \subset N$ receives at least its worth: $C(v) = \{x \in I^*(v): x(S) \geq v(S) \text{ for all } S \subset N\}$. By B^N we denote the set of all *balanced* games with player set N , that is, those with a non-empty core. A game (N, v) is *totally balanced* if every subgame is balanced.

A *single-valued solution* on G^N is a function $\alpha: G^N \rightarrow \mathbb{R}^N$ such that $\alpha(v) \in I^*(v)$ for all $v \in G^N$.

In this paper, we deal with two properties of single-valued solutions on G^N : *core-selection* and *aggregate-monotonicity* (Megiddo, 1974).

Definition 1. A single-valued solution, $\alpha: G^N \rightarrow \mathbb{R}^N$, satisfies *core-selection* if $\alpha(v) \in C(v)$ for all $v \in B^N$.

Definition 2. A single-valued solution, $\alpha: G^N \rightarrow \mathbb{R}^N$, satisfies *aggregate-monotonicity* if for all $v, v' \in G^N$, with $v(S) = v'(S)$ for all $S \subset N$ and $v(N) < v'(N)$, it holds that $\alpha(v) \leq \alpha(v')$.

By \mathbb{S}_{AC}^N we denote the set of all single-valued solutions on G^N that satisfy core-selection and aggregate-monotonicity. Notice that $\mathbb{S}_{AC}^N \neq \emptyset$, since the per-capita prenucleolus satisfies both properties (see Moulin, 1988

¹ As usual, we write $v(i), v(12), \dots$, instead of $v(\{i\}), v(\{1, 2\}), \dots$.

or Young et al., 1982). We are interested in the set of allocations of a game (N, v) that a single-valued solution in \mathbb{S}_{AC}^N may select.

Definition 3. The *aggregate-monotonic core* of the game (N, v) is the set of pre-imputations of the game (N, v) attainable by single-valued solutions on G^N that satisfy core-selection and aggregate-monotonicity. Formally, $\mathcal{AC}(v) = \{\alpha(v): \alpha \in \mathbb{S}_{AC}^N\}$.

Notice that $\mathcal{AC}(v) \neq \emptyset$ for all $v \in G^N$ and $\mathcal{AC}(v) \subseteq C(v)$ for all $v \in B^N$. To give an intuitive and simple description of this set we introduce some notation. Let (N, v) be an arbitrary game. We denote by B_v^N the set of balanced games that can be obtained from v increasing or decreasing only the worth of the grand coalition: $B_v^N = \{v' \in B^N: v'(S) = v(S) \text{ for all } S \subset N\}$. Notice that for all $v \in G^N$, B_v^N is a non-empty subset of B^N .

Definition 4. The *root game* (N, v_r) associated to the game (N, v) is the smallest game in B_v^N , i.e. $v_r \in B_v^N$ and $v_r(N) \leq w(N)$ for all $w \in B_v^N$. Additionally, a game (N, v) is said to be *rooted* if $v = v_r$.

Notice that by definition, $C(v_r) \neq \emptyset$ for all $v \in G^N$. As a consequence, any single-valued solution on G^N satisfying core-selection must pick an allocation in the core of any rooted game. This will be used in the next theorem.

Theorem 5. Let (N, v) be a game. Then, $\mathcal{AC}(v) = C(v_r) + (v(N) - v_r(N))\Delta_N$.

Proof. Take $x \in C(v_r) + (v(N) - v_r(N))\Delta_N$, that is, $x = y + (v(N) - v_r(N))z$, where $y \in C(v_r)$ and $z \in \Delta_N$. We must prove that there exists $\alpha \in \mathbb{S}_{AC}^N$ such that $\alpha(v) = x$.

To define α on G^N , consider an arbitrary game $w \in G^N$. If $w_r \neq v_r$, let $x^{w_r} \in C(w_r)$ and then, define $\alpha(w) = x^{w_r} + \frac{w(N) - w_r(N)}{n} e_N$. If $w_r = v_r$, define $\alpha(w) = y + \frac{w(N) - y(N)}{x(N) - y(N)}(x - y)$ if $v \neq v_r$ and $\alpha(w) = x + \frac{w(N) - v_r(N)}{n} e_N$ if $v = v_r$.

It is easy to see that α is a single-valued solution on G^N that satisfies core-selection and aggregate-monotonicity and $\alpha(v) = x$.

The reverse inclusion is straightforward and it is left to the reader. \square

From Theorem 5, it is obvious that $\mathcal{AC}(v)$ is a non-empty, compact, and convex set for all $v \in G^N$. Notice also that, from the proof of Theorem 5, it is actually not difficult to find single-valued solutions on G^N other than the per-capita prenucleolus that are both aggregate-monotonic and core-selection. In other words, \mathbb{S}_{AC}^N seems to be quite large.

Remark. Another notion, based on completely different considerations, is the concept of grand subcore, introduced in Zakharov and Kwon (1998). Theorem 5 makes it possible to compare the grand subcore with the aggregate-monotonic core. They coincide for balanced games and differ in the class of non-balanced games, where the grand subcore is always empty. Zakharov and Kwon study consistency properties of the grand subcore with respect to different reduction processes. Later, Zakharov and Dementieva (2004) use the grand subcore in a dynamic cooperative setting.

It is clear that the aggregate-monotonic core coincides with the core for every rooted game. However, the aggregate-monotonic core of a game may be a proper subset of the core, even for a *convex* game.² In fact, for the three-player convex game given in Introduction, it holds that $\mathcal{AC}(v) \subset C(v)$.

In the main theorem of the paper, Theorem 7, we show that the coincidence of the core and the aggregate-monotonic core of an arbitrary non-rooted game depends on the largeness of the core of its root game. Sharkey (1982) introduces the concept of a *large core*, based on the notion of aspiration. An aspiration of the game (N, v) is a vector $x \in \mathbb{R}^N$ such that $x(S) \geq v(S)$ for all $S \subseteq N$. We denote by $A(v)$ the set of aspirations of the game v .

Definition 6. The core of the game (N, v) is *large* if for all $y \in A(v)$ there exists $x \in C(v)$ such that $x \leq y$.

² A game v is convex if and only if $v(T) + v(S) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \subseteq N$ (Shapley, 1971).

Sharkey (1982) gives sufficient conditions for the largeness of the core, one of which is the convexity of the game. If a given game $v \in G^N$ has a large core, any other game obtained from v by increasing only the value of the grand coalition has a large core, too. van Gellekom et al. (1999) study this and other prosperity properties and connect the largeness of the core of a game v with the extreme points of $U(v)$, where $U(v) = \{y \in \mathbb{R}^N: y(S) \geq v(S) \text{ for all } S \subset N\}$. Notice that in fact $U(v) = A(v_r)$ for all $v \in G^N$. In particular, they state that given a balanced game (N, v) , its core, $C(v)$, is large if and only if $z(N) \leq v(N)$ for all extreme points z of $A(v_r)$.

Theorem 7. *Let (N, v) be a non-rooted balanced game. Then, $\mathcal{AC}(v) = C(v)$ if and only if (N, v_r) has a large core.*

Proof. The “if” part is trivial and left to the reader.

Let us prove the “only if” part. Suppose (N, v_r) does not have a large core. Then, by van Gellekom et al. (1999), there exists an extreme point y^* of $A(v_r)$ with $y^*(N) > v_r(N)$. Let $\mathcal{S}(y^*) = \{S \subset N: y^*(S) = v_r(S)\}$. As y^* is an extreme point of $A(v_r)$ there exists a subset of coalitions $\{S_1, \dots, S_n\}$ of $\mathcal{S}(y^*)$ such that the vectors e_{S_1}, \dots, e_{S_n} form a basis of \mathbb{R}^N . Hence, for all $i \in N$ there is a coalition $S \in \mathcal{S}(y^*)$ with $i \in S$. We claim that there is no $x \in C(v_r)$ with $x \leq y^*$ or, equivalently $y^* \notin C(v_r) + \mathbb{R}_+^N$. Suppose on the contrary that $x \leq y^*$ for some $x \in C(v_r)$. Take $i \in N$ and $S \in \mathcal{S}(y^*)$ such that $i \in S$. Then, $v_r(S) \leq x(S) \leq y^*(S) = v_r(S)$, and since $x \leq y^*$ it follows that $x_k = y_k^*$ for all $k \in S$. In particular, $x_i = y_i^*$. Therefore, $x = y^*$ and $y^*(N) = x(N) = v_r(N)$ which involves a contradiction with $y^*(N) > v_r(N)$.

In the sequel, we distinguish between the following two cases:

Case I. Assume $y^*(N) \geq v(N)$. By a separating hyperplane theorem (see for example Webster, 1994) applied to the fact that $y^* \notin C(v_r) + \mathbb{R}_+^N$, there exists $q \in \mathbb{R}^N$ satisfying $q \cdot y^* < q \cdot x + q \cdot h'$ for all $x \in C(v_r)$ and all $h' \in \mathbb{R}_+^N$. Let $x^* \in C(v_r)$ be such that $q \cdot x^* \leq q \cdot x$ for all $x \in C(v_r)$. For any $\lambda \in (0, 1]$, we define $z_\lambda = \lambda y^* + (1 - \lambda)x^*$. Notice that $z_\lambda \in A(v_r)$. Moreover, $q \cdot z_\lambda = \lambda q \cdot y^* + (1 - \lambda)q \cdot x^* < \lambda(q \cdot x + q \cdot h') + (1 - \lambda)q \cdot x^* \leq \lambda(q \cdot x + q \cdot h') + (1 - \lambda)q \cdot x \leq q \cdot x + q \cdot (\lambda h')$ for all $x \in C(v_r)$ and all $h' \in \mathbb{R}_+^N$. Hence, $q \cdot z_\lambda < q \cdot x + q \cdot h$ for all $x \in C(v_r)$ and all $h \in \mathbb{R}_+^N$. Therefore, $z_\lambda \notin C(v_r) + \mathbb{R}_+^N$. Trivially, there exists $\lambda_* \in (0, 1]$ such that $z_{\lambda_*}(N) = v(N)$, thus $z_{\lambda_*} \in C(v)$, and $z_{\lambda_*} \notin \mathcal{AC}(v)$. This finishes case I.

Case II. Assume $y^*(N) < v(N)$. Let $i_* \in N$ be such that

$$|\{S \in \mathcal{S}(y^*): i_* \in S\}| \leq |\{S \in \mathcal{S}(y^*): j \in S\}| \quad \text{for all } j \in N, \tag{1}$$

that is, player i_* is one of the players belonging to the least number of coalitions in $\mathcal{S}(y^*)$.

Now, define $\bar{y}^* = y^* + (v(N) - y^*(N)) \cdot e_{i_*}$. Clearly $\bar{y}^* \in C(v)$. We will show that $\bar{y}^* \notin \mathcal{AC}(v)$. Suppose there exists $x \in C(v_r)$ with $x \leq \bar{y}^*$.

Let $R = \bigcup_{S \in \mathcal{S}(y^*): i_* \notin S} S$. It is easy to check that $R \neq \emptyset$ by (1), and $N \setminus R \neq \emptyset$. We claim that $x_i = y_i^* = \bar{y}_i^*$ for all $i \in R$. Let i be an arbitrary player in R . The equality $y_i^* = \bar{y}_i^*$ follows by definition of \bar{y}^* and the fact that $i_* \notin R$. To show that $x_i = \bar{y}_i^*$, first notice that there exists a coalition $S \in \mathcal{S}(y^*)$ with $i \in S$ and $i_* \notin S$, since $i \in R$. Each player in S belongs to R too, and consequently $\bar{y}^*(S) = y^*(S) = v_r(S)$. By assumption, $x \in C(v_r)$ with $x \leq \bar{y}^*$, and so $x(S) \geq v_r(S)$ and $x(S) \leq \bar{y}^*(S)$. Hence, $x(S) = \bar{y}^*(S)$, but since $x \leq \bar{y}^*$ it follows that $x_i = \bar{y}_i^*$.

Furthermore, we also claim that $N \setminus R \subseteq S$ for all $S \in \mathcal{S}(y^*)$ such that $i_* \in S$. To see this, let $j \in N \setminus R$ and $T \in \mathcal{S}(y^*)$ with $i_* \in T$. Since $j \in N \setminus R$, for all $S \in \mathcal{S}(y^*)$ such that $i_* \notin S$ we have $j \notin S$, which implies $|\{S \in \mathcal{S}(y^*): i_* \in S\}| \geq |\{S \in \mathcal{S}(y^*): j \in S\}|$. Now, by (1),

$$|\{S \in \mathcal{S}(y^*): i_* \in S\}| = |\{S \in \mathcal{S}(y^*): j \in S\}|. \tag{2}$$

If $j \in T$ we are done. If not, we have $T \in \mathcal{S}(y^*)$ with $i_* \in T$ and $j \notin T$, and it follows from (2) that there must exist $T' \in \mathcal{S}(y^*)$ with $i_* \notin T'$ and $j \in T'$, which implies a contradiction with the fact that $j \in N \setminus R$.

Now, let $S \in \mathcal{S}(y^*)$ be such that $i_* \in S$. Then, $x(S) = x(S \cap R) + x(S \cap (N \setminus R)) = x(S \cap R) + x(N \setminus R) \geq v_r(S) = y^*(S) = y^*(S \cap R) + y^*(S \cap (N \setminus R)) = y^*(S \cap R) + y^*(N \setminus R)$, which implies $x(N \setminus R) \geq y^*(N \setminus R)$ since $x_i = \bar{y}_i^*$ for all $i \in R$.

Finally, $x(N) = x(R) + x(N \setminus R) = y^*(R) + x(N \setminus R) \geq y^*(R) + y^*(N \setminus R) = y^*(N) > v_r(N)$, which involves a contradiction with the fact that $x \in C(v_r)$. This finishes case II and ends the proof. \square

From Theorem 7 it follows that the coincidence of the aggregate-monotonic core and the core of a non-rooted balanced game is inherited by all those balanced games obtained from the initial game by changing only the worth of the grand coalition. The next corollary states this relation.

Corollary 8. Let (N, v) be a non-rooted game. Then, the following three assertions are equivalent:

1. (N, v_r) has a large core.
2. $\mathcal{AC}(w) = C(w)$ for all $w \in B_v^N$.
3. There exists $w \in B_v^N$, $w \neq v_r$ such that $\mathcal{AC}(w) = C(w)$.

Proof. It follows easily by applying Theorem 7. \square

Notice that from Corollary 8 follows that it is sufficient that the aggregate-monotonic core does not coincide with the core of one non-rooted balanced game v to ensure that this coincidence will not hold for any other game $w \in B_v^N$, $w \neq v_r$.

In the proof of Theorem 7 we use the fact that if the root game (N, v_r) of an arbitrary balanced game (N, v) does not have a large core then it is always possible to find an aspiration $y \in A(v_r)$ with $y(N) = v(N)$ such that there is no $x \in C(v_r)$ with $x \leq y$. This observation can be extended to an arbitrary balanced game that does not have a large core.

Corollary 9. Let (N, v) be a balanced game and $k \in \mathbb{R}$, $k > v(N)$. If $C(v)$ is not large, there exists an aspiration $y \in A(v)$ with $y(N) = k$ such that there is no $x \in C(v)$ with $x \leq y$.

Proof. Suppose (N, v) does not have a large core. By van Gellekom et al. (1999), there exists an extreme point y^* of $A(v_r)$ with $y^*(N) > v(N)$. Repeating the arguments used in the proof of Theorem 7, there is no $x \in C(v)$ with $x \leq y^*$ or, equivalently $y^* \notin C(v) + \mathbb{R}_+^N$.

We distinguish between the following two cases:

Case I. Assume $v(N) < k \leq y^*(N)$. By a separating hyperplane theorem there exists $q \in \mathbb{R}^N$ satisfying $q \cdot y^* < q \cdot x + q \cdot h$ for all $x \in C(v)$ and all $h \in \mathbb{R}_+^N$. Let $x^* \in C(v)$ be such that $q \cdot x^* \leq q \cdot x$ for all $x \in C(v)$. For any $\lambda \in (0, 1]$, we define $z_\lambda = \lambda y^* + (1 - \lambda)x^*$. Repeating the arguments used in case I of the proof of Theorem 7, there exists $\lambda_* \in (0, 1]$ such that $z_{\lambda_*}(N) = k$. Now, taking $y = z_{\lambda_*}$, we obtain $y \in A(v)$, $y(N) = k$, and there is no $x \in C(v)$ with $x \leq y$.

Case II. Assume $y^*(N) < k$. Define $\bar{y}^* = y^* + (k - y^*(N)) \cdot e_{i_*}$ where $i_* \in N$ is such that $|\{S \in \mathcal{S}(y^*): i_* \in S\}| \leq |\{S \in \mathcal{S}(y^*): j \in S\}|$ for all $j \in N$ and $\mathcal{S}(y^*) = \{S \subset N: y^*(S) = v(S)\}$. Repeating the arguments used in case II of the proof of Theorem 7 and taking $y = \bar{y}^*$, we obtain $y \in A(v)$, $y(N) = k$, and there is no $x \in C(v)$ with $x \leq y$. \square

As a consequence of Corollary 9 we obtain a relation between the cores of two games that differ exclusively in the worth of the grand coalition.

Corollary 10. Let (N, v) and (N, w) be two balanced games such that $v(S) = w(S)$ for all $S \subset N$ and $v(N) < w(N)$. Then, $C(w) = C(v) + (w(N) - v(N))\Delta_N$ if and only if (N, v) has a large core.

Proof. The “if” part is trivial. The “only if” part follows easily from Corollary 9 considering $k = w(N)$. \square

Moulin (1990) introduces the concept of a *totally large core*. A game (N, v) has a totally large core if each one of its subgames has a large core, i.e. (S, v^S) has a large core for all $S \subseteq N$, $S \neq \emptyset$. Having a totally large core turns out to be equivalent to the convexity of the game.

Given a game, we are interested in the coincidence of the core and the aggregate-monotonic core of all its subgames. The convexity of the root game of an arbitrary game (N, v) is not enough to ensure that $\mathcal{AC}(v^S) = C(v^S)$ for all $S \subseteq N$, $S \neq \emptyset$, as the next example shows.

Example 11. Let us consider the following four-player game: $v(i) = 0$ for all $i \in \{1, 2, 3, 4\}$, $v(12) = v(13) = v(124) = v(134) = 1$, $v(123) = v(1234) = 2$ and $v(S) = 0$ otherwise. The game v is convex, and $v = v_r$. However, for the subgame associated to coalition $S = \{1, 2, 3\}$, it is easy to check that $C(v^S) = co\{(2, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ where co means the convex hull, while $C(v_r^S) = \{(1, 0, 0)\}$, since $v_r^S(123) = 1$. Hence, $\mathcal{AC}(v^S) \subset C(v^S)$.

The next corollary analyzes the coincidence of the core and the aggregate-monotonic core for all the subgames of a game.

Corollary 12. Let (N, v) be a totally balanced game. The following two assertions are equivalent:

1. The root game of every subgame of (N, v) has a large core.
2. The root game of every subgame of (N, v) is convex.

Moreover, each of the above assertions implies $\mathcal{AC}(v^S) = C(v^S)$ for all $S \subseteq N$, $S \neq \emptyset$.

Proof. 1 \rightarrow 2. Let $S \subseteq N$, $S \neq \emptyset$, be an arbitrary coalition, and let $T \subset S$, $T \neq \emptyset$, be an arbitrary subcoalition of S . By assumption, $C(v_r^S)$ and $C(v_r^T)$ are large, and $C(v^T)$ is also large since, because v^T is balanced, we have $v^T(T) \geq v_r^T(T)$. By Moulin (1990), (S, v_r^S) is convex.

2 \rightarrow 1. Let $S \subseteq N$, $S \neq \emptyset$, be an arbitrary coalition. By assumption, (S, v_r^S) is convex, then $C(v_r^S)$ is large.

Applying Theorem 7 it follows that each of the above assertions implies $\mathcal{AC}(v^S) = C(v^S)$ for all $S \subseteq N$, $S \neq \emptyset$. \square

Notice that $\mathcal{AC}(v^S) = C(v^S)$ for all $S \subseteq N$, $S \neq \emptyset$, does not imply either assertion 1 or assertion 2, since the core of a rooted subgame might not be large. Take for example the three-player game: $v(i) = 0$ for all $i \in \{1, 2, 3\}$, $v(12) = v(13) = 1$, $v(23) = 0$ and $v(123) = 1$. Then, $\mathcal{AC}(v^S) = C(v^S)$ for all $S \subseteq N$, $S \neq \emptyset$. However, $C(v)$ is not large.

A set-valued solution on G^N is a correspondence $\Gamma : G^N \rightarrow \mathbb{R}^N$ such that $\Gamma(v) \subseteq I^*(v)$ for all $v \in G^N$. The main objective of the last part of the paper is to characterize the aggregate-monotonic core as a set-valued solution on G^N .

For this purpose, we introduce the core-selection and aggregate-monotonicity properties for set-valued solutions.

Definition 13. A set-valued solution, $\Gamma : G^N \rightarrow \mathbb{R}^N$, satisfies core-selection if $\Gamma(v) \subseteq C(v)$ for all $v \in B^N$.

To extend the aggregate-monotonicity property, we distinguish between *upper* and *lower aggregate-monotonicity*.

Definition 14. A set-valued solution, $\Gamma : G^N \rightarrow \mathbb{R}^N$, satisfies *upper aggregate-monotonicity* if for all $v, v' \in G^N$, with $v(S) = v'(S)$ for all $S \subset N$ and $v(N) < v'(N)$, it holds that for all $x \in \Gamma(v)$ there exists $y \in \Gamma(v')$ such that $y \geq x$.

Definition 15. A set-valued solution, $\Gamma : G^N \rightarrow \mathbb{R}^N$, satisfies *lower aggregate-monotonicity* if for all $v, v' \in G^N$, with $v(S) = v'(S)$ for all $S \subset N$ and $v(N) < v'(N)$, it holds that for all $y \in \Gamma(v')$ there exists $x \in \Gamma(v)$ such that $x \leq y$.

Notice that either form of aggregate-monotonicity is equivalent to aggregate-monotonicity when we refer to a single-valued solution.

The upper aggregate-monotonicity for set-valued solutions is the monotonicity property that Megiddo introduces in his seminal paper in 1974. Later, Ichiishi (1990) develops a comparative study of the core of two arbitrary games with respect to two different welfare criteria, defined similarly to the upper and lower aggregate monotonicity properties.

There are set-valued solutions on G^N satisfying only one of these properties. In particular, the core satisfies upper aggregate-monotonicity, but not lower aggregate-monotonicity. The set-valued solution Γ that assigns to every game $v \in G^N$ the core of its dual game, $\Gamma(v) = C(v^*)$, where $v^*(S) = v(N) - v(N \setminus S)$ for all $S \subseteq N$, satisfies lower aggregate-monotonicity, but not upper aggregate-monotonicity.

The aggregate-monotonic core can be characterized on G^N by core-selection, and upper and lower aggregate-monotonicity.

Theorem 16. The aggregate-monotonic core is the 'largest' (w.r.t. inclusion) set-valued solution on G^N that satisfies core-selection, upper aggregate-monotonicity and lower aggregate-monotonicity.

Proof. Clearly, the aggregate-monotonic core satisfies core-selection. We show that it satisfies upper and lower aggregate-monotonicity. Let $v, v' \in G^N$ be such that $v(S) = v'(S)$ for all $S \subset N$ and $v(N) < v'(N)$. Obviously, v and v' have the same associated root game, v_r .

If $v_r(N) \leq v(N) < v'(N)$, then it is easy to check that $\mathcal{AC}(v') = C(v_r) + (v(N) - v_r(N))\Delta_N + (v'(N) - v(N))\Delta_N = \mathcal{AC}(v) + (v'(N) - v(N))\Delta_N$. Similarly, if $v(N) < v'(N) \leq v_r(N)$ then it follows easily that $\mathcal{AC}(v) =$

$C(v_r) + (v'(N) - v_r(N))\Delta_N + (v(N) - v'(N))\Delta_N = \mathcal{AC}(v') + (v(N) - v'(N))\Delta_N$. Consequently, for these cases upper and lower aggregate-monotonicity are satisfied.

If $v(N) < v_r(N) < v'(N)$, let $x \in \mathcal{AC}(v)$. Then, there exists $z \in C(v_r)$ such that $z \geq x$, and there exists $y \in \mathcal{AC}(v')$ such that $y \geq z$. Consequently, $y \geq x$ and upper aggregate-monotonicity is satisfied. Analogously, lower aggregate-monotonicity is also satisfied.

To finish the proof, it is enough to show that if Γ is an arbitrary set-valued solution on G^N that satisfies core-selection, and upper and lower aggregate-monotonicity, then $\Gamma(v) \subseteq \mathcal{AC}(v)$ for all $v \in G^N$. Let v be an arbitrary game and let v_r be its root game. If $v = v_r$, by core-selection it follows that $\Gamma(v) \subseteq C(v) = \mathcal{AC}(v)$.

If $v(N) > v_r(N)$ then by core-selection we have $\Gamma(v_r) \subseteq C(v_r)$. Now, let $x \in \Gamma(v)$. By lower aggregate-monotonicity there exists a $y \in \Gamma(v_r)$ such that $y \leq x$. Therefore, $x \in \mathcal{AC}(v)$.

Finally, if $v(N) < v_r(N)$, again by core-selection $\Gamma(v_r) \subseteq C(v_r)$. Now, let $x \in \Gamma(v)$. By upper aggregate-monotonicity there exists a $y \in \Gamma(v_r)$ such that $y \geq x$. Therefore, $x \in \mathcal{AC}(v)$. \square

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