Australian Options

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Abstract

We study European options on the ratio of the stock price to its average and viceversa. Some of these options are traded in the Australian Stock Exchange since 1992, thus we call them Australian options. For geometric averages, we obtain closed-form expressions for option prices. For arithmetic means, we use different approximations that produce very similar results.

Keywords: Asian Options, Arithmetic Average, Geometric Average, Edgeworth Expansion, Lognormal Distribution, Gamma Distribution.

Journal of Economic Literature classification: G13, C15

1 Introduction

Asian options are options on the average of asset prices. It is usually argued that they provide the following advantages: (a) they are cheaper than standard European options, as the average is less volatile than the asset price itself, (b) they prevent manipulation of the underlying asset price at the maturity date and (c) they are the adequate hedging instrument for traders who act continuously over finite periods.

Options on the ratio of the stock price to its average (or viceversa) are particular cases of Asian options. They have recently appeared as special types of variable purchase options (VPOs). VPOs were first issued in 1992 and have been traded since then on the Australian Stock Exchange. A VPO is an option that gives its holder the right to buy at maturity a stochastic number of shares that depends on the terminal stock price. This option can have more complex features like caps and floors on the number of shares.

Handley (2000) provides a detailed description of VPOs as well as pricing formulas, which are tested in Handley (2003). In the first article, the author describes Asian VPOs, in which the number of shares that can be bought at maturity depends on the average stock price. These options are shown to be equivalent to options on the ratio of the stock price to its average. Alternatively, we could define Asian VPOs in such a way that they are equivalent to options on the ratio of the average of the stock price to the stock price itself.

In this paper we price options on these ratios using both geometric and arithmetic (discrete- and continuous-time) means of stock prices that are assumed to follow a lognormal process. When the average is computed on geometric basis, these ratios are lognormally distributed at maturity, thus we obtain formulas similar to those of Black and Scholes (1973).

However, when the average is computed on arithmetic basis, the risk-neutral distribution of these ratios is, in general, unknown and we can not obtain closed-form expressions for the prices of these options.[1] This happens because the arithmetic average is the convolution of correlated lognormal random variables and its distribution is not known.

This problem has been treated in the literature in different ways. Many studies use numerical techniques, such as the finite difference methods, as in Kemna and Vorst (1990) and Alziari *et al* (1997),[2] simulation, as in Kemna and Vorst (1990) or Vázquez-Abad and Dufresne (1998),[3] and the Fourier transform, as in Carverhill and Clewlow (1990) or Ju (1997).

A number of articles provide analytical solutions that approximate the price of these options. Examples of this literature include Yor (1992, 1993), Geman and Yor (1993), Curran (1994), De Schepper *et al* (1994), Eydeland and Geman (1995), Rogers and Shi (1995), Nielsen and Sandmann (1996, 1999, 2001), Fu *et al* (1999), Shirakawa (1999), and Dufresne (2000).

Jarrow and Rudd (1982) apply Edgeworth series expansion to option pricing when the risk-neutral distribution of the underlying asset at maturity is unknown. This method has been applied to Asian options by Turnbull and Wakeman (1991) and Ritchken *et al* (1993), among others. Some authors use only the first two moments in the Edgeworth series expansion, obtaining what is called the Wilkinson approximation. See, for example, Levy (1992) and Hansen and Jorgensen (2000).

Finally, Milevsky and Posner (1998) use the fact that the infinite sum of correlated lognormal random variables is reciprocal gamma distributed to obtain a closed-form solution for the value of arithmetic Asian options.[4] This formula is exact only when the average is computed continuously.

We price arithmetic Australian options using both the Wilkinson approximation and the gamma distribution. We also use Monte Carlo simulation with antithetic variables. The results show that option prices obtained with the three methods are quite similar. This is true even when the number of monitoring dates used to compute the average is small.

The rest of the paper is organized as follows. In Section 2 we introduce the pricing framework and derive closed-form expressions for the prices of geometric Australian options. In Section 3 we study different approximations to the value of arithmetic options. Section 4 summarizes and concludes. Technical details are relegated to the Appendix.

2 Geometric Australian Options

Following standard assumptions (see, for example, Duffie (1988), Chapter 22 for details), let the underlying asset price Z follow the following geometric Brownian motion in a risk-neutral world

$$dZ_t = \alpha_Z Z_t dt + \sigma_Z Z_t dW_t$$

where α_Z is the (constant) risk-neutral drift of the process, σ_Z is a positive constant and W_t is a standard Wiener process. Usually, σ_Z^2 is referred to as the logarithmic variance parameter of the asset. Then, Z_t follows the lognormal process

$$Z_t = Z_0 \exp\left\{\left(\alpha_Z - \frac{1}{2}\sigma_Z^2\right)t + \sigma_Z W_t\right\}. \tag{1}$$

It is well-known that the price at time 0 of an European call option on Z that matures at time T with strike price K is given by

$$C(Z, 0, T, K) = e^{-rT} E(Z_T) N(d_1) - K e^{-rT} N(d_2) \tag{2}$$

where $N(.)$ is the distribution function of a standard normal variable, $E(.)$ denotes the expectation under the risk-neutral measure, and

$$\begin{aligned} d_1 &= \frac{\ln\left(e^{-\alpha_Z T} E(Z_T)/K\right) + \left(\alpha_Z + \frac{1}{2}\sigma_Z^2\right) T}{\sigma_Z \sqrt{T}} \\ d_2 &= d_1 - \sigma_Z \sqrt{T} \end{aligned}$$

We now consider n monitoring dates so that the time interval $[0, T]$ is partitioned in the following way:

$$\{t_0 = 0 < t_1 < t_2 < \cdots < t_n = T\}, \ t_i - t_{i-1} = \frac{T}{n} = \Delta t, \ \forall \, i = 1, \cdots, n.$$

Let $S = \{ \ S_{t_i} \equiv S_i, \ i = 0, 1, \cdots, n \ \}$ be the price process for the stock. We suppose that S_t follows the risk-neutral process

$$dS_t = (r - q) S_t dt + \sigma S_t dW_t, \tag{3}$$

where q is the continuous dividend yield of the stock and σ is a positive constant. Then we can write

$$S_t = S_0 \exp\left\{ \left(r - q - \frac{1}{2}\sigma^2 \right) t + \sigma W_t \right\}. \tag{4}$$

We define the geometric mean of the n stock prices $S_1, \cdots, S_n$ as

$$G_n = \left(\prod_{i=1}^{n} S_i \right)^{\frac{1}{n}}, \quad G_0 \equiv S_0$$

Using (4), we have

$$G_n = S_0 \exp\left\{\left(r - q - \frac{1}{2}\sigma^2\right)\frac{n+1}{2}\Delta t + \frac{\sigma}{n}\sum_{i=1}^{n} W_{t_i}\right\} \tag{5}$$

Looking at (4) and (5), and using $t_n = n\Delta t$, we have

$$\frac{S_n}{G_n} = \exp\left\{\left(r - q - \frac{1}{2}\sigma^2\right)\frac{n-1}{2}\Delta t + \frac{\sigma}{n}\left[n\, W_{t_n} - \sum_{i=1}^{n} W_{t_i}\right]\right\} \tag{6}$$

$$\frac{G_n}{S_n} = \exp\left\{-\left(r - q - \frac{1}{2}\sigma^2\right)\frac{n-1}{2}\Delta t - \frac{\sigma}{n}\left[n\, W_{t_n} - \sum_{i=1}^{n} W_{t_i}\right]\right\} \tag{7}$$

It is clear from (5)-(7) that the geometric average and both ratios are lognormally distributed. Thus, we obtain the following result.

Proposition 1 *We consider European call options on S_n/G_n and G_n/S_n that mature at time T and with strike price K. The prices at time 0 of these options are given by expression (2), where the expected value and the logarithmic variance of the asset at maturity are given by the following table:*[5]

Z_n	$E(Z_n)$	$\sigma_Z^2 T$
G_n	$S_0 \exp\left\{\left(r - q - \frac{n-1}{6n}\sigma^2\right)\frac{n+1}{2n}T\right\}$	$\frac{(n+1)\left(n+\frac{1}{2}\right)}{3n^2}\,\sigma^2 T$
S_n/G_n	$\exp\left\{\left(r - q - \frac{n+1}{6n}\sigma^2\right)\frac{n-1}{2n}T\right\}$	$\frac{(n-1)\left(n-\frac{1}{2}\right)}{3n^2}\,\sigma^2 T$
G_n/S_n	$\exp\left\{-\left(r - q - \frac{5n-1}{6n}\sigma^2\right)\frac{n-1}{2n}T\right\}$	$\frac{(n-1)\left(n-\frac{1}{2}\right)}{3n^2}\,\sigma^2 T$

Proof:

These moments are obtained using the properties of the lognormal distribution, and Lemma 2 in the Appendix. □

Note that the prices of Australian options do not depend on the current stock price, S_0. When $Z_n = G_n$, we get the price derived by Turnbull and Wakeman (1991) and Ritchken *et al* (1993).

Table 1 shows call[6] prices (multiplied by 100) for different cases and monitoring dates. Option prices are computed using Proposition 1. The interest rate is 10% and the stock dividend yield is 3%. We include the stock price (S_n) and its geometric average (G_n) as underlying assets as a reference. We assume that the initial stock price (S_0) is 1.

[Insert Table 1 about here]

We see that option prices do not necessarily increase with time to maturity or with volatility. This is also true for standard geometric Asian options. For example, when $T = 0.5, K = 0.8$, and $n = 1,000$, the call option on G_n has a value of 20.548 and 20.538 for $\sigma = 0.2$ and 0.4, respectively.[7]

As additional reference, the Black-Scholes call option prices (dividend yield = 0) in the four cases studied in Table 1 are 24.027, 27.993, 26.081, and 3.743, respectively.

We now define the continuous geometric average of the stock price over the interval $[0, T]$ as

$$G_T = \exp\left\{\frac{1}{T}\int_0^T \ln(S_t)\, dt\right\}, \quad G_0 \equiv S_0$$

Using (4), we have

$$G_T = S_0 \exp\left\{\frac{1}{2}\left(r - q - \frac{1}{2}\sigma^2\right)T + \frac{\sigma}{T}\int_0^T W_t dt\right\} \tag{8}$$

Looking at (4) and (8), we have

$$\frac{S_T}{G_T} = \exp\left\{\frac{1}{2}\left(r - q - \frac{1}{2}\sigma^2\right)T + \frac{\sigma}{T}\left[T\, W_T - \int_0^T W_t dt\right]\right\} \tag{9}$$

$$\frac{G_T}{S_T} = \exp\left\{-\frac{1}{2}\left(r - q - \frac{1}{2}\sigma^2\right)T - \frac{\sigma}{T}\left[T\, W_T - \int_0^T W_t dt\right]\right\} \tag{10}$$

From (8)-(10) we see that the geometric average and both ratios are lognormally distributed. Thus, to price options on these assets, we just need the moments of their risk-neutral distributions.

Proposition 2 *We consider European call options on S_T/G_T and G_T/S_T that mature at time T and with strike price K. The prices at time 0 of these options are given by expression (2), where the expected value and the logarithmic variance of the asset at maturity are given by the following table:*[8]

Z_T	$E(Z_T)$	$\sigma_Z^2 T$
S_T	$S_0 \exp\{(r-q)T\}$	$\sigma^2 T$
G_T	$S_0 \exp\left\{\frac{1}{2}\left(r-q-\frac{1}{6}\sigma^2\right)T\right\}$	$\frac{\sigma^2}{3}T$
S_T/G_T	$\exp\left\{\frac{1}{2}\left(r-q-\frac{1}{6}\sigma^2\right)T\right\}$	$\frac{\sigma^2}{3}T$
G_T/S_T	$\exp\left\{-\frac{1}{2}\left(r-q-\frac{5}{6}\sigma^2\right)T\right\}$	$\frac{\sigma^2}{3}T$

Proof: These moments are obtained using (8)-(10), the properties of the lognormal distribution, and Lemma 3 in the Appendix. □

Notice that:

- The logarithmic variances of both ratios are equal to the one derived by Kemna and Vorst (1990) for the continuous geometric average. The intuition for this result is that, with infinite monitoring dates, the volatility of the ratio depends only on the volatility of the average. This value increases with σ and is one third of the variance in the Black-Scholes formula.

- The expected value of G_T is S_0 times the expected value of S_T/G_T.

- The expected values of S_T/G_T and G_T/S_T do not depend on the current stock price, S_0.

- The expected values of G_T is smaller than that of S_T.

The last column in Table 1 presents geometric option prices for continuous monitoring. As we are taking $S_0 = 1$, the option on S_T/G_T is equivalent to that on G_T. We see that option prices are very similar to those obtained with 1,000 monitoring dates.

It is easy to see that the expected values of G_T and S_T/G_T decrease with σ, while that of G_T/S_T increases with σ.[9] Since the logarithmic variance of the assets studied increase with σ and T and their expected values also depend on these parameters, we have that option prices can decrease with volatility or time to maturity. This result is described in more detail in the next lemma that shows the theta and vega for these options.

Lemma 1

1. *The theta of a call option on Z_T is given by*

$$\theta_C(Z_T) = e^{-rT}\left[(\alpha_{Z_T} - r)E(Z_T)N(d_1) + KrN(d_2) + \frac{\sigma_Z}{2\sqrt{T}}E(Z_T)N'(d_1)\right] \tag{11}$$

2. *The theta of a put option on Z_T is given by*

$$\theta_P(Z_T) = e^{-rT}\left[(r - \alpha_{Z_T})E(Z_T)N(-d_1) - KrN(-d_2) + \frac{\sigma_Z}{2\sqrt{T}}E(Z_T)N'(d_1)\right] \tag{12}$$

3. *The vega of a call option on Z_T is given by*

$$\nu_C(Z_T) = e^{-rT}E(Z_T)\sqrt{T}\left[\frac{\partial \alpha_Z}{\partial \sigma}\sqrt{T}N(d_1) + \frac{\partial \sigma_Z}{\partial \sigma}N'(d_1)\right] \tag{13}$$

4. The vega of a put option on Z_T is given by

$$\nu_P(Z_T) = e^{-rT}E(Z_T)\sqrt{T}\left[\frac{\partial\sigma_Z}{\partial\sigma}N'(d_1) - \frac{\partial\alpha_Z}{\partial\sigma}\sqrt{T}N(-d_1)\right] \qquad (14)$$

Proof: The results for call options are obtained differentiating with respect to T or σ in expression 2. For put options, the put-call parity is used. □

Figure 1 plots geometric Australian option prices as a function of time to maturity. The averages are computed with infinite monitoring dates. The exercise price is $K = 0.8$ for calls and $K = 1.2$ for puts. The other parameters are $r = 0.1, q = 0.03, \sigma = 0.2$.

[Insert Figure 1 about here]

We see that, in this case, the price of a call option on S_T/G_T increases with T. However, the price of a call option on G_T/S_T decreases with T. The latter result is due to the fact that $r > \frac{1}{3}\left(q + \frac{5}{6}\sigma^2\right)$, so that $\alpha_{Z_T} - r < 0$ and $\partial C(.)/\partial T$ can be negative.

Since the exercise price for the put options is relatively high ($K = 1.2$), we see that the price of the put option on S_T/G_T decreases with T. The same occurs for a put option on G_T/S_T when time to maturity is small (between 0 and 0.65 years). For higher T, the put price increases. When $T > 1.5$, the put price decreases again. Interestingly, if we reduce the exercise price to $K = 1.1$, the put price increases for all T.

It is easy to see that the vega of call options on G_T or S_T/G_T is negative iff

$$\frac{N'(d_1)}{\sigma N(d_1)} < \frac{1}{2}\sqrt{\frac{T}{3}}$$

It is also clear that

$$\nu_P\left(\frac{G_T}{S_T}\right) < 0 \Leftrightarrow \frac{N'(d_1)}{\sigma N(-d_1)} < \frac{5}{2}\sqrt{\frac{T}{3}}$$

As shown in Figure 2, both inequalities can hold for reasonable parameter values. This figure exhibits geometric Australian option prices as a function of volatility. The averages are computed with infinite monitoring dates. The exercise prices are $K = 0.8$ and $K = 1.1$ for calls and puts, respectively. The remaining parameters are: $r = 0.1, q = 0.03, T = 0.1$.

[Insert Figure 2 about here]

We see that the price of the call option on S_T/G_T first decreases and then increases with volatility. Its vega is zero when $\sigma = 0.67$. The price of the call option on G_T/S_T always increases with σ. The same is true for a put option on S_T/G_T. However, the price of the put option on G_T/S_T first decreases and then increases with volatility. Its vega reaches zero for $\sigma = 0.36$.

3 Arithmetic Australian Options

We define the discrete arithmetic mean of the n stock prices $S_1, \cdots, S_n$ as

$$A_n = \frac{1}{n}\sum_{i=1}^{n} S_i, \quad A_0 \equiv S_0 \tag{15}$$

The continuous counterpart is given by

$$A_T = \frac{1}{T}\int_0^T S_t \, dt \tag{16}$$

As mentioned before, the distribution of A_n is unknown. Therefore, we can not apply expression 2 to price this type of options. Two ways to overcome this problem are:

- To approximate the true distribution with an alternative one.
- To approximate the distribution of A_n with that of A_T.

3.1 Pricing the Options with the Edgeworth / Wilkinson Approximation

To price options, the risk-neutral distribution of he underlying asset at maturity is approximated with a tractable distribution. The true distribution is expanded around the approximating one. This approach is called generalized Edgeworth series expansion. The coefficients of this expansion are functions of the moments of the true and approximating distribution. Considering up to four terms in this expansion and specifying the approximating distribution to be lognormal, the option price is equal to the Black-Scholes price plus three adjustment terms. These terms depend, respectively, on the difference between the variance, skewness, and kurtosis of the true and the lognormal distribution. The intuition is that the first four moments of the distribution are enough to reflect the effects of the distribution on option prices.

More concretely, we approximate the true probability distribution, $F(s)$, with an approximating distribution, $A(s)$. It is assumed that both distributions have continuous density functions, $f(s)$ and $a(s)$. We employ the following notation:

$$\begin{aligned}\alpha_j(F) &= \int_{-\infty}^{\infty} s^j f(s)ds \\ \mu_j(F) &= \int_{-\infty}^{\infty} (s-\alpha_1(F))^j f(s)ds \\ \Psi(F,t) &= \int_{-\infty}^{\infty} e^{its} f(s)ds, \quad i=\sqrt{-1}\end{aligned}$$

where $\alpha_j(F)$ and $\mu_j(F)$ are, respectively, the j-th non-central and central moments of F and $\Psi(F,t)$ is the characteristic function of F.[10]

Following Stuart and Ord (1987), the cumulants $k_j(F)$ of the distribution F are defined by the identity in t

$$\ln \Psi(F,t) = \sum_{j=1}^{\infty} k_j(F) \frac{(it)^j}{j!}$$

For practical purposes, we only need the first four cumulants in the Edgeworth series expansion. These cumulants are, respectively, the mean, the variance, the coefficient of skewness and the excess of kurtosis:

$$\begin{aligned} k_1(F) &= \alpha_1(F), \quad k_2(F) = \mu_2(F) \\ k_3(F) &= \mu_3(F), \quad k_4(F) = \mu_4(F) - 3\mu_2^2(F) \end{aligned}$$

Jarrow and Rudd (1982) prove the following series expansion for $f(s)$ around $a(s)$:

$$\begin{aligned} f(s) &= a(s) + \frac{k_2(F) - k_2(A)}{2!} \frac{d^2a(s)}{ds^2} - \frac{k_3(F) - k_3(A)}{3!} \frac{d^3a(s)}{ds^3} \\ &+ \frac{k_4(F) - k_4(A) + 3(k_2(F) - k_2(A))^2}{4!} \frac{d^4a(s)}{ds^4} + \varepsilon(s) \end{aligned} \quad (17)$$

where, by construction, $k_1(F)$ is set equal to $k_1(A)$.

The difference between $f(s)$ and $a(s)$ depends on the cumulants of both distributions with weighting factors given by the derivatives of $a(s)$. The terms on the right-hand side of (17) reflect any difference in variance, skewness and kurtosis and variance between $f(s)$ and $a(s)$. The residual error, $\varepsilon(s)$, includes any remaining difference. For a numerical analysis of this error term, see Section 5 in Jarrow and Rudd (1982).

Now, we employ (17) to obtain an approximate option price. Using $f(s)$ as the true distribution of the asset price at maturity, we obtain the expected value at maturity of an option on this asset. Then, this expansion provides an approximated expected value for the option at maturity in terms of the approximating distribution, $a(s)$.

In a risk-neutral world, the true price of the call option, $C(F)$, is obtained by discounting its expected value at the risk-free rate:

$$C(F) = e^{-rT} \int_{-\infty}^{\infty} \max\{S_T - K, 0\} \, dF(S_T)$$

Using (17) and a little algebra, the call price becomes

$$\begin{aligned} C(F) &= C(A) + e^{-rT} \frac{k_2(F) - k_2(A)}{2!} a(K) - e^{-rT} \frac{k_3(F) - k_3(A)}{3!} \left. \frac{da}{dS_T} \right|_K \\ &\quad + e^{-rT} \frac{k_4(F) - k_4(A) + 3(k_2(F) - k_2(A))^2}{4!} \left. \frac{d^2 a}{dS_T^2} \right|_K + \varepsilon(K) \qquad (18) \end{aligned}$$

where

$$C(A) = e^{-rT} \int_{-\infty}^{\infty} \max\{S_T - K, 0\} \, dA(S_T)$$

Although (17) is valid for any approximating distribution $a(s)$, a natural candidate is the lognormal one. In this case, (18) shows that the true option price is equal to the Black-Scholes price plus three adjustment terms.

As mentioned in the introduction, the Wilkinson approximation is a particular case of the Edgeworth expansion, where just the first two cumulants are used.

3.2 Pricing the Options with the Gamma Distribution

It is known that the infinite sum of lognormal distributions is a reciprocal gamma distribution.[11] Using this distribution as state-price density function, Milevsky and Posner (1998) obtain an expression for the price of arithmetic

Asian options. The solution is the same as the Black-Scholes formula where the normal distribution is replaced by the gamma one.

In more detail, let X be gamma distributed with parameters α and β, that is, $X \sim \Gamma(\alpha, \beta)$. It is straightforward to show that

$$\alpha = \frac{2M_2 - M_1^2}{M_2 - M_1^2}, \quad \beta = \frac{M_2 - M_1^2}{M_1 M_2} \tag{19}$$

where M_1 and M_2 denote the first two non-central moments of $Y = \frac{1}{X}$.

Hence, to price options, we must obtain the first two risk-neutral moments (M_1, M_2) of the underlying asset at maturity. The details of this derivation are relegated to the Appendix (Lemmas 7 and 8). The next step is to compute α and β using (19). Finally, we use the cumulative density function of the gamma distribution as $N(.)$ in the Black-Scholes formula.

Table 2 shows arithmetic call[12] option prices (multiplied by 100) for different monitoring dates. The interest rate is 10% and the stock dividend yield is 3%. We price options on $A_n, S_n/A_n$ and A_n/S_n with three methods: Monte Carlo simulation,[13] Wilkinson approximation, and gamma distribution.

[Insert Table 2 about here]

We see that derivative prices with the three methods are very close. For example, when $\sigma = 0.20, T = 0.5, K = 0.8$, and $n = 1,000$, the values of call options on A_n/S_n are 18.319, 18.324, and 18.321, respectively. Thus, Edgeworth expansions do not seem to be needed.

To price options on S_n/A_n with both the Wilkinson approximation and the gamma distribution, we have computed its moments using the approximation of Mood *et al* (1974) (see Lemma 6 in the Appendix). In this table we see that those prices are very similar to those obtained with Monte Carlo, so that the approximations seem to work pretty well. For example, when $\sigma = 0.2, T = 0.5, K = 0.8$, and $n = 1,000$, the values of call options using the Wilkinson approximation and the gamma distribution are 20.375 and 20.374, respectively, while the value obtained with Monte Carlo simulation is 20.377.

To understand better why the three method produce very similar results, we plot the risk-neutral probability density function of the arithmetic stock price average in Figure 3.

[Insert Figure 3 about here]

The parameter values are: $r = 0.1,\ q = 0, \sigma = 0.2, T = 1, S_0 = 100$ and $n = \infty$. The expected value of the average price is 105.17, and the variance 152.74. For $n = \infty$ the true density function is reciprocal gamma, with parameters $\alpha = 74.42$ and $\beta = 1.29\text{E-}4$. This function is approximated with a lognormal distribution with the same moments. The density function is also estimated with Monte Carlo simulation, using a set of 50 runs of 10,000 paths with 1,000 time steps. We see that, for the parameter values used, the density functions are quite similar, hence the price of options on arithmetic stock prices must be close.

4 Conclusions

Australian options are options on the ratio of the stock price to its average or viceversa. They show up in variable purchase options, recently studied by Handley (2000, 2003).

If the stock price follows a geometric Brownian motion and the average is defined on geometric basis, these ratios also follow a geometric Brownian motion. Thus, we obtain closed-form expressions for the price of the options. However, when the average is defined on arithmetic basis, the risk-neutral distributions of these ratios at maturity are unknown. Hence, to price the options we use a particular case of Edgeworth expansion (known as Wilkinson approximation) as well as a gamma approximation (following Milevsky and Posner (1998)). We compare the results with those obtained with Monte Carlo simulations, and we find that option prices are very similar in the three cases. Hence, in practice, it does not seem to be necessary to use high order moments in the Edgeworth expansion nor to require a large number of monitoring dates in the gamma approximation for pricing these claims.

Appendix

We now state two lemmas that will be useful to obtain the moments of the geometric price average and its associated ratios, for both discrete and continuous monitoring.

Lemma 2 *Given the Brownian motions* W_{t_i}, $i = 1, \cdots, n$, *it is verified that*

$$\begin{aligned} V\left(\sum_{i=1}^{n} W_{t_i}\right) &= \frac{(n+1)\left(n+\frac{1}{2}\right)n}{3}\,\Delta t \\ V\left(n\,W_{t_n} - \sum_{i=1}^{n} W_{t_i}\right) &= \frac{(n-1)\left(n-\frac{1}{2}\right)n}{3}\,\Delta t \end{aligned}$$

Proof:

For $n \geq 2$, we have

$$\begin{aligned} V\left(\sum_{i=1}^{n} W_{t_i}\right) &= V\left(\sum_{i=1}^{n-1} W_{t_i} + W_{t_n}\right) = V\left(\sum_{i=1}^{n-1} W_{t_i}\right) + t_n + 2\sum_{i=1}^{n-1} \mathrm{Cov}(W_{t_i}, W_{t_n}) \\ &= V\left(\sum_{i=1}^{n-1} W_{t_i}\right) + n\Delta t + 2\sum_{i=1}^{n-1} i\Delta t = V\left(\sum_{i=1}^{n-1} W_{t_i}\right) + n^2\Delta t \end{aligned}$$

By induction, we get

$$V\left(\sum_{i=1}^{n} W_{t_i}\right) = \sum_{i=1}^{n} i^2\,\Delta t = \frac{(n+1)\left(n+\frac{1}{2}\right)n}{3}\,\Delta t$$

$$\begin{aligned} V\left(n\,W_{t_n} - \sum_{i=1}^{n} W_{t_i}\right) &= V\left(\sum_{i=1}^{n-1} [W_{t_n} - W_{t_i}]\right) \\ &= V\left(\sum_{i=1}^{n-1} W_{t_i}\right) + \sum_{i=1}^{n-1}\sum_{i=1}^{n-1} \mathrm{Cov}(W_{t_n}, W_{t_n} - 2W_{t_i}) \\ &= V\left(\sum_{i=1}^{n-1} W_{t_i}\right) = \sum_{i=1}^{n-1} i^2\,\Delta t = \frac{n\left(n-\frac{1}{2}\right)(n-1)}{3}\,\Delta t \end{aligned}$$

□

Lemma 3 *Given the Brownian motions W_t, $t \in [0, T]$, it is verified that*

$$V\left(\int_0^T W_t dt\right) = V\left(T\, W_T - \int_0^T W_t dt\right) = \frac{T^3}{3}$$

Proof:

Integrating by parts, we have

$$\int_0^T (T-t) dW_t = \int_0^T W_t dt, \quad \int_0^T t dW_t = TW_T - \int_0^T W_t dt$$

Then, we get

$$V\left(\int_0^T W_t dt\right) = V\left(\int_0^T (T-t) dW_t\right) = \int_0^T (T-t)^2 dt = -\frac{(T-t)^3}{3}\bigg|_0^T = \frac{T^3}{3}$$

$$V\left(TW_T - \int_0^T W_t dt\right) = V\left(\int_0^T t dW_t\right) = \int_0^T t^2 dt = \frac{t^3}{3}\bigg|_0^T = \frac{T^3}{3}$$

□

The following two lemmas are used to obtain the moments of the arithmetic price average (discrete and continuous) and its associated ratios.

Lemma 4 *For $i, j = 1, 2, \cdots, n$ and $k \in \mathbb{R}$, we have*

$$E(S_i^k) = S_0^k \exp\left\{k\left(r - q + \frac{k-1}{2}\sigma^2\right) i\Delta t\right\} \quad (20)$$

$$E(S_i S_j^k) = E(S_i)E(S_j^k)\exp\{k\sigma^2 \min\{i, j\}\Delta t\} \quad (21)$$

$$E(S_i S_j S_n^k) = E(S_i S_j)E(S_n^k)\exp\{k\sigma^2(i+j)\Delta t\} \quad (22)$$

Moreover, for $k \in \mathbb{R}$, we have

$$E(A_n S_n^k) = \frac{S_0}{n}\, E(S_n^k)\, h_1(r^*) \quad (23)$$

$$E(A_n^2 S_n^k) = \left(\frac{S_0}{n}\right)^2 E(S_n^k)[2f_1(r^* + \sigma^2)(h_1(2r^* + \sigma^2) - h_1(r^*)) - h_1(2r^* + \sigma^2)] \quad (24)$$

where $r^* = r - q + k\sigma^2$ and $h_1(.)$ and $f_1(.)$ as given by (33)-(34).

Proof:

$E(S_i^k)$ is obtained applying properties of lognormal distributions to (4).

Using (4), (20), and a little algebra, we have, for $a, b, k \in \mathbb{R}$,

$$E(S_i^a S_j^b S_n^k) = E(S_i^a)E(S_j^b)E(S_n^k)\exp\{[ab\min\{i,j\} + k(ai+bj)]\sigma^2\Delta t\}$$

For $k = 0$, we have

$$E(S_i^a S_j^b) = E(S_i^a)E(S_j^b)\exp\{ab\sigma^2\min\{i,j\}\Delta t\} \tag{25}$$

and, then,

$$E(S_i^a S_j^b S_n^k) = E(S_i^a S_j^b)E(S_n^k)\exp\{k\sigma^2(ai+bj)\Delta t\} \tag{26}$$

Using (25) with $a = 1, b = k$ and (26) with $a = b = 1$ provides $E(S_i S_j^k)$ and $E(S_i S_j S_n^k)$, respectively.

- **Mean of** $A_n S_n^k$: Apply (15), (20) for $k = 1$, and (21) for $j = n$.

- **Mean of** $A_n^2 S_n^k$: Applying (15) and (22), we get

$$E(A_n^2 S_n^k) = \left(\frac{1}{n}\right)^2 E(S_n^k) z_n^*, \quad z_n^* = \sum_{i,j=1}^{n} E(S_i S_j) e^{k\sigma^2(i+j)\Delta t} \tag{27}$$

After some algebra, we get the recurrence law

$$z_n^* = z_{n-1}^* + S_0^2\left[2f_1(r^*+\sigma^2)\left(e^{(2r^*+\sigma^2)n\Delta t} - e^{r^* n\Delta t}\right) - e^{(2r^*+\sigma^2)n\Delta t}\right], \ z_0^* = 0$$

with $f_1(.)$ as given by (34).

Applying this recurrence law for different values of n, we obtain

$$z_n^* = S_0^2\left[2f_1(r^*+\sigma^2)(h_1(2r^*+\sigma^2) - h_1(r^*)) - h_1(2r^*+\sigma^2)\right]$$

Plugging this expression into (27), we get the final value for $E(A_n^2 S_n^k)$.

□

Lemma 5 *For $k \in \mathbb{R}$, we have*

$$
\begin{aligned}
E(A_T S_T^k) &= \frac{S_0}{T} E(S_T^k)\Phi(r^*) \\
E(A_T^2 S_T^k) &= 2\left(\frac{S_0}{T}\right)^2 E(S_T^k)\frac{\Phi(2r^* + \sigma^2) - \Phi(r^*)}{r^* + \sigma^2}
\end{aligned}
$$

with $\Phi(.)$ as given by (40) and r^* as given by Lemma 4.

Proof:

$E(A_T S_T^k)$ is obtained using (16) and applying (20) with $k = 1$ and (21) with $j = n$.

To compute $E(A_T^2 S_T^k)$, use (16) and apply (21) with $k = 1$ and (22). □

The following Lemma will be useful to compute the moments of ratios involving arithmetic average asset prices:

Lemma 6 *Let X and Y be two random variables. Then, it is verified that*

$$
\begin{aligned}
E\left(\frac{X}{Y}\right) &\simeq \frac{E(X)}{E(Y)} - \frac{1}{(E(Y))^2}Cov(X,Y) + \frac{E(X)}{(E(Y))^3}V(Y) \\
V\left(\frac{X}{Y}\right) &\simeq \left(\frac{E(X)}{E(Y)}\right)^2\left(\frac{V(X)}{(E(X))^2} + \frac{V(Y)}{(E(Y))^2} - 2\frac{Cov(X,Y)}{E(X)E(Y)}\right)
\end{aligned}
$$

Proof: See Mood *et al* (1974), p. 181. □

We now present the moments of the arithmetic mean price and its associated ratios (for both discrete and continuous monitoring) in the next two lemmas.

Lemma 7

1. *The moments of the variable A_n are given by*

$$
\begin{aligned}
E(A_n) &= \frac{S_0}{n}\, h_1(r-q) && (28)\\
Cov(A_n, S_n) &= \frac{S_0}{n} E(S_n)[h_1(r-q+\sigma^2) - h_1(r-q)] && (29)\\
V(A_n) &= \left(\frac{S_0}{n}\right)^2 [2f_1(r-q+\sigma^2)[h_1(2(r-q)+\sigma^2) - h_1(r-q)] \\
&\quad -h_1(2(r-q)+\sigma^2) - [h_1(r-q)]^2] && (30)
\end{aligned}
$$

2. *The moments of the variable S_n/A_n, $n \geq 2$ can be approximated by*

$$
\begin{aligned}
E\left(\frac{S_n}{A_n}\right) &\simeq \frac{E(S_n)}{E(A_n)} - \frac{1}{(E(A_n))^2} Cov(A_n, S_n) + \frac{E(S_n)}{(E(A_n))^3} V(A_n)\\
V\left(\frac{S_n}{A_n}\right) &\simeq \left(\frac{E(S_n)}{E(A_n)}\right)^2 \left(\frac{V(S_n)}{(E(S_n))^2} + \frac{V(A_n)}{(E(A_n))^2} - 2\frac{Cov(A_n, S_n)}{E(S_n)E(A_n)}\right)
\end{aligned}
$$

with $E(A_n)$, $Cov(A_n, S_n)$, and $V(A_n)$ as given by (28)-(30).

3. *The moments of the variable A_n/S_n, $n \geq 2$ are given by*

$$
\begin{aligned}
E\left(\frac{A_n}{S_n}\right) &= \frac{1}{n}\exp\{-n(r-q-\sigma^2)\Delta t\}\; h_1(r-q-\sigma^2) && (31)\\
V\left(\frac{A_n}{S_n}\right) &= \left(\frac{1}{n}\right)^2 \exp\{-n(2(r-q)-3\sigma^2)\Delta t\}\\
&\quad \times[2f_1(r-q-\sigma^2)[h_1(2(r-q)-3\sigma^2) - h_1(r-q-2\sigma^2)]\\
&\quad -h_1(2(r-q)-3\sigma^2) - \exp\{-n\sigma^2\Delta t\} h_1^2(r-q-\sigma^2)] && (32)
\end{aligned}
$$

with

$$
\begin{aligned}
h_1(x) &= \sum_{i=1}^{n} e^{xi\Delta t} = f_1(x)\left(e^{xn\Delta t} - 1\right),\; x \neq 0, \quad h_1(0) = n && (33)\\
f_1(x) &= \frac{e^{x\Delta t}}{e^{x\Delta t} - 1},\; x \neq 0 && (34)
\end{aligned}
$$

Proof:

1. Moments of the arithmetic average A_n

   (a) Mean of A_n:

   Apply (23) with $k = 0$.

   (b) Covariance of A_n with S_n:

   Apply (23) for $k = 1$ and (28).

   (c) Variance of A_n:

   Apply (24) for $k = 0$ and (28).

2. Moments of the variable S_n/A_n:

   Apply part 2 in Lemma 6 with $X = S_n,\ Y = A_n$.

3. Moments of the variable A_n/S_n

   (a) Mean of A_n/S_n:

   Apply (20) for $i = n, k = -1$ and (23) for $k = -1$.

   (b) Variance of A_n/S_n:

   Apply (20) for $i = n, k = -2$, (24) for $k = -2$, and (31). □

From (34), we have that $f_1(0) = \infty$. As the function $f_1(.)$ appears in $V(A_n)$ and $V\left(\frac{A_n}{S_n}\right)$, the next remark provides the values of these expressions when the argument of $f_1(.)$ becomes null.

Remark 1 *Particular cases:*

1. If $r = q - \sigma^2$, the moments of the variable A_n are given by

$$V(A_n) = 2\left(\frac{S_0}{n}\right)^2 f_1(\sigma^2) e^{-(n+1)\sigma^2 \Delta t} \sum_{i=1}^{n} \Big(\cosh(\sigma^2 i \Delta t) - 1\Big)$$

2. *If $r = q + \sigma^2$, the moments of the variable A_n/S_n, $n \geq 2$ are given by*

$$V\left(\frac{A_n}{S_n}\right) = \left(\frac{1}{n}\right)^2 \left[(2f_1(\sigma^2) - 1)\left(e^{-\sigma^2 \Delta t} h_1(\sigma^2) - n\right) - n(n-1)\right]$$

Proof: It involves tedious algebra and is omitted for brevity. □

Lemma 8

1. *The moments of the variable A_T are given by*

$$E(A_T) = \frac{S_0}{T}\Phi(r-q) \tag{35}$$

$$Cov(A_T, S_T) = \frac{S_0}{T}E(S_T)[\Phi(r-q+\sigma^2) - \Phi(r-q)] \tag{36}$$

$$V(A_T) = \left(\frac{S_0}{T}\right)^2 \left[2\frac{\Phi(2(r-q)+\sigma^2) - \Phi(r-q)}{r-q+\sigma^2} - (\Phi(r-q))^2\right] \tag{37}$$

2. *The moments of the variable S_T/A_T can be approximated by*

$$E\left(\frac{S_T}{A_T}\right) \simeq \frac{E(S_T)}{E(A_T)} - \frac{1}{(E(A_T))^2}Cov(A_T, S_T) + \frac{E(S_T)}{(E(A_T))^3}V(A_T)$$

$$V\left(\frac{S_T}{A_T}\right) \simeq \left(\frac{E(S_T)}{E(A_T)}\right)^2 \left(\frac{V(S_T)}{(E(S_T))^2} + \frac{V(A_T)}{(E(A_T))^2} - 2\frac{Cov(A_T, S_T)}{E(S_T)E(A_T)}\right)$$

with $E(A_T)$, $Cov(A_T, S_T)$, and $V(A_T)$ as given by (35)-(37).

3. *The moments of the variable A_T/S_T are given by*

$$E\left(\frac{A_T}{S_T}\right) = \frac{1}{T}\Phi(\sigma^2 - (r-q)) \tag{38}$$

$$\begin{aligned} V\left(\frac{A_T}{S_T}\right) &= \left(\frac{1}{T}\right)^2 \exp\{-(2(r-q) - 3\sigma^2)T\} \\ &\quad \times \left[2\frac{\Phi(2(r-q) - 3\sigma^2) - \Phi(r-q-2\sigma^2)}{r-q-\sigma^2}\right. \\ &\quad \left. - \exp\{-\sigma^2 T\}\Phi^2(r-q-\sigma^2)\right] \end{aligned} \tag{39}$$

with

$$\Phi(x) = \frac{\exp\{xT\} - 1}{x}, \; x \neq 0, \quad \Phi(0) = T \tag{40}$$

Proof: It is similar to that of Lemma 7 and is omitted. □

Looking at (37), we see that $V(A_T)$ is undetermined when $r = q - \sigma^2$. The same happens with $V\left(\frac{A_T}{S_T}\right)$ when $r = q + \sigma^2$. As before, we study these special cases in the next remark.

Remark 2 *Particular cases:*

1. *If $r = q - \sigma^2$, the moments of the variable A_T are given by*

$$V(A_T) \;=\; 2\left(\frac{S_0}{T}\right)^2 \frac{e^{-\sigma^2 T}}{\sigma^4}(\sinh(\sigma^2 T) - \sigma^2 T)$$

2. *If $r = q + \sigma^2$, the moments of the variable A_T/S_T are given by*

$$V\left(\frac{A_T}{S_T}\right) \;=\; \left(\frac{1}{T}\right)^2 \left[2\,\frac{\Phi(\sigma^2) - T}{\sigma^2} - T^2\right]$$

Proof: It involves tedious algebra and is omitted for brevity. □

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Footnotes

1. As will be indicated later, the distribution of the continuous-time average is known, allowing us to obtain exact analytical expressions for option prices.
2. Other examples are Dewynne and Wilmott (1995a, 1995b), He and Takahashi (1996), Zvan *et al* (1998) or Shreve and Več̃er (2000).
3. See also Haykov (1993), Corwin *et al* (1996) and Nielsen and Sandmann (1996).
4. Dufresne (1990) and Yor (1993) are examples of papers that deal with the gamma distribution.
5. For completeness, this table includes the geometric average.
6. For the sake of brevity, we omit the corresponding table for put options. It is available from the authors upon request.
7. Recall that these prices have been multiplied by 100.
8. For completeness, this table includes the stock price and its geometric average. The formula for the geometric Asian option was first derived by Kemna and Vorst (1990).
9. It can be shown that for σ large enough ($\sigma > \sqrt{6/5}\sqrt{2T\ln(S_0) + 3(r-q)}$), this expected value is higher than $E(S_T)$.
10. Analogous notation is employed for the approximating distribution A.
11. A variable Y follows a reciprocal gamma distribution if $X = 1/Y$ follows a gamma distribution. The main characteristics of the gamma distribution can be found, for instance, in Johnson *et al* (1994), Chapter 17.
12. As before, the corresponding table for put options has been omitted.
13. We use 10,000 simulations and antithetic variables to reduce standard errors.

Table 1. Geometric Call option prices.

Parameters			Asset	Number of monitoring dates (n)				
$\sigma(\%)$	T	K	Z_n	1	10	100	1,000	∞
20	0.5	0.8	S_n	22.576	22.576	22.576	22.576	22.576
			G_n	22.576	20.696	20.561	20.548	20.546
			S_n/G_n	19.025	20.400	20.531	20.545	20.546
			G_n/S_n	19.025	18.133	18.161	18.166	18.166
20	1	0.8	S_n	25.187	25.187	25.187	25.187	25.187
			G_n	25.187	21.361	21.084	21.056	21.053
			S_n/G_n	18.097	20.756	21.023	21.050	21.053
			G_n/S_n	18.097	16.491	16.580	16.593	16.594
40	0.5	0.8	S_n	24.801	24.801	24.801	24.801	24.801
			G_n	24.801	20.766	20.558	20.538	20.536
			S_n/G_n	19.025	20.332	20.514	20.534	20.536
			G_n/S_n	19.025	20.266	20.908	20.979	20.987
20	0.5	1.1	S_n	3.175	3.175	3.175	3.175	3.175
			G_n	3.175	0.905	0.737	0.721	0.719
			S_n/G_n	0	0.549	0.701	0.717	0.719
			G_n/S_n	0	0.282	0.374	0.384	0.385

Prices are multiplied by 100. The interest rate is 10% and the dividend yield 3%. For options on S_n and G_n we take $S_0 = 1$. For options on G_n, derivative prices can be computed with the Merton's (1973) formula when $n = 1$, and with the Kemna and Vorst's (1990) formula when $n = \infty$.

Table 2. Arithmetic Call option prices.

Parameters			Asset	Number of monitoring dates (n)				
$\sigma(\%)$	T	K	Z_n	1	10	100	1,000	∞
20	0.5	0.8	A_n MC	22.551	20.737	20.731	20.723	-
			A_n W	22.576	20.885	20.729	20.714	20.712
			A_n GD	22.535	20.883	20.728	20.711	20.711
			S_n/A_n MC	19.025	20.329	20.381	20.377	-
			S_n/A_n W	19.025	20.209	20.360	20.375	20.377
			S_n/A_n GD	19.025	20.208	20.358	20.374	20.375
			A_n/S_n MC	19.025	18.332	18.317	18.319	-
			A_n/S_n W	19.025	18.389	18.330	18.324	18.324
			A_n/S_n GD	19.025	18.388	18.327	18.321	18.321
20	1	0.8	A_n MC	25.089	21.431	21.366	21.387	-
			A_n W	25.187	21.736	21.418	21.387	21.384
			A_n GD	25.059	21.716	21.404	21.373	21.370
			S_n/A_n MC	18.097	20.642	20.730	20.766	-
			S_n/A_n W	18.097	20.376	20.682	20.713	20.717
			S_n/A_n GD	18.097	20.366	20.667	20.698	20.701
			A_n/S_n MC	18.097	16.841	16.884	16.899	-
			A_n/S_n W	18.097	16.964	16.887	16.880	16.883
			A_n/S_n GD	18.097	16.947	16.862	16.855	16.857

Prices are multiplied by 100. The interest rate is 10% and the dividend yield 3%. For options on A_n we take $S_0 = 1$. MC, W, and GD refer to Monte Carlo simulation, Wilkinson approximation and gamma distribution, respectively.

Table 2. Arithmetic Call option prices (cont.).

Parameters			Asset	Number of monitoring dates (n)				
$\sigma(\%)$	T	K	Z_n	1	10	100	1,000	∞
40	0.5	0.8	A_n MC	24.687	21.141	21.111	21.206	-
			A_n W	24.801	21.468	21.188	21.161	21.158
			A_n GD	24.400	21.358	21.099	21.074	21.071
			S_n/A_n MC	19.025	19.853	20.011	19.987	-
			S_n/A_n W	19.025	19.641	19.918	19.947	19.951
			S_n/A_n GD	19.025	19.562	19.819	19.846	19.849
			A_n/S_n MC	19.025	21.432	21.620	21.535	-
			A_n/S_n W	19.025	21.276	21.569	21.599	21.620
			A_n/S_n GD	19.025	21.213	21.489	21.517	21.535
20	0.5	1.1	A_n MC	3.141	0.802	0.756	0.757	-
			A_n W	3.176	0.933	0.778	0.761	0.759
			A_n GD	3.198	0.980	0.802	0.785	0.782
			S_n/A_n MC	0	0.593	0.612	0.682	-
			S_n/A_n W	0	0.528	0.681	0.698	0.699
			S_n/A_n GD	0	0.549	0.705	0.721	0.722
			A_n/S_n MC	0	0.389	0.404	0.413	-
			A_n/S_n W	0	0.302	0.387	0.396	0.401
			A_n/S_n GD	0	0.320	0.409	0.419	0.424

Prices are multiplied by 100. The interest rate is 10% and the dividend yield 3%. For options on A_n we take $S_0 = 1$. MC, W and GD refer to Monte Carlo simulation, Wilkinson approximation and gamma distribution, respectively.

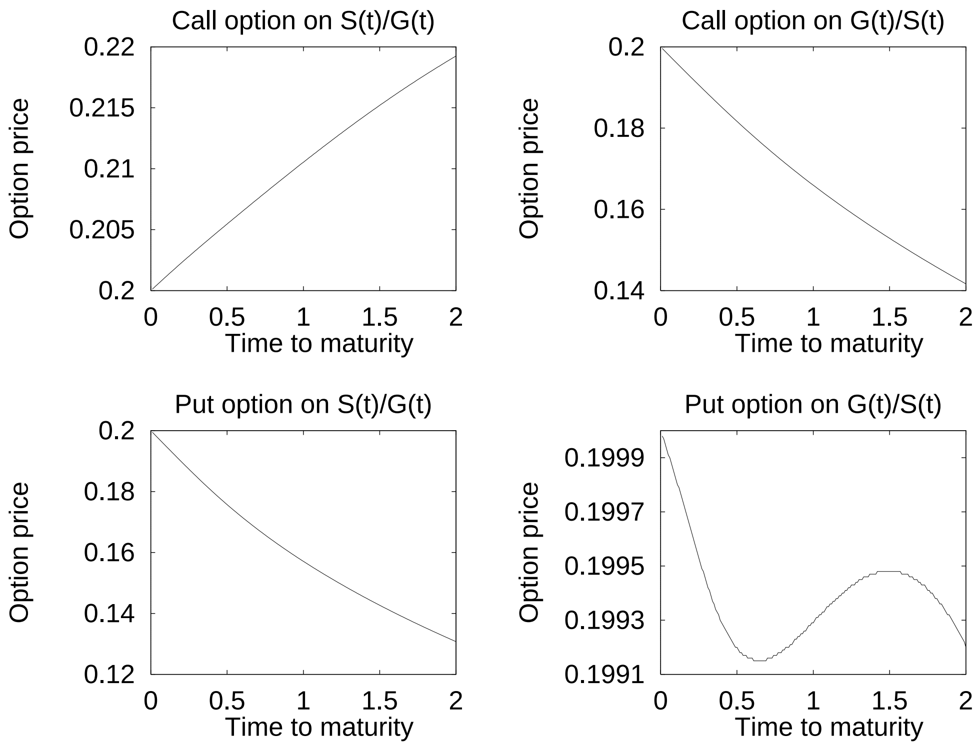

Figure 1: **Geometric Australian option prices as a function of time to maturity**. The exercise price is $K = 0.8$ for calls and $K = 1.2$ for puts. The other parameter values are: $r = 0.1, q = 0.03, \sigma = 0.2$, and $n = \infty$.

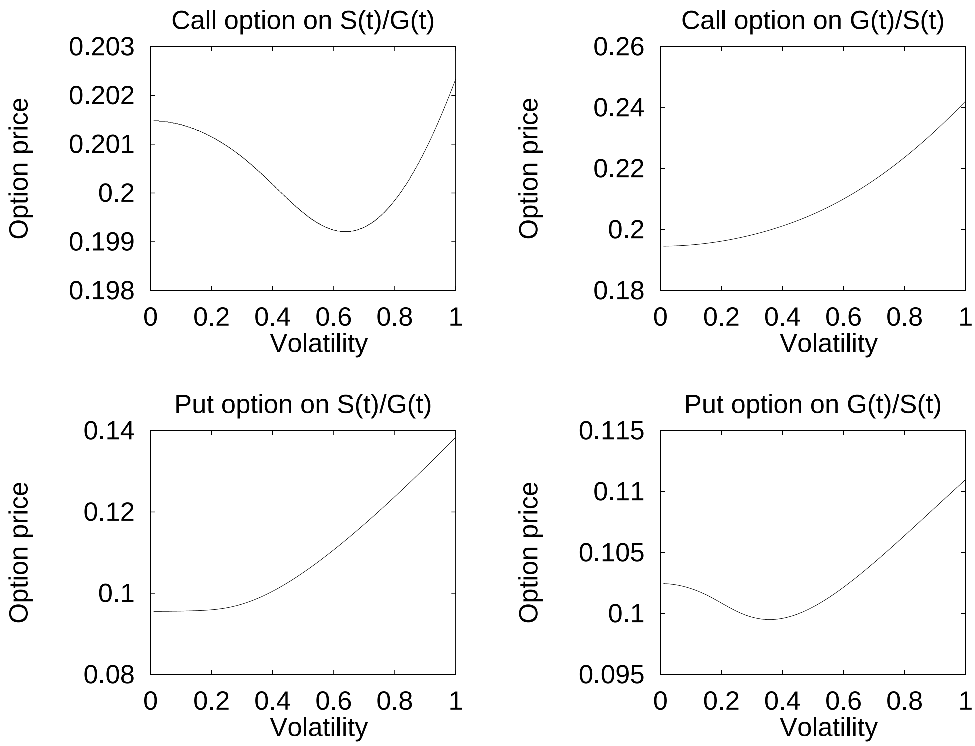

Figure 2: **Geometric Australian option prices as a function of volatility (σ).** The exercise price is $K = 0.8$ for calls and $K = 1.1$ for puts. The other parameter values are: $r = 0.1, q = 0.03, T = 0.1$, and $n = \infty$.

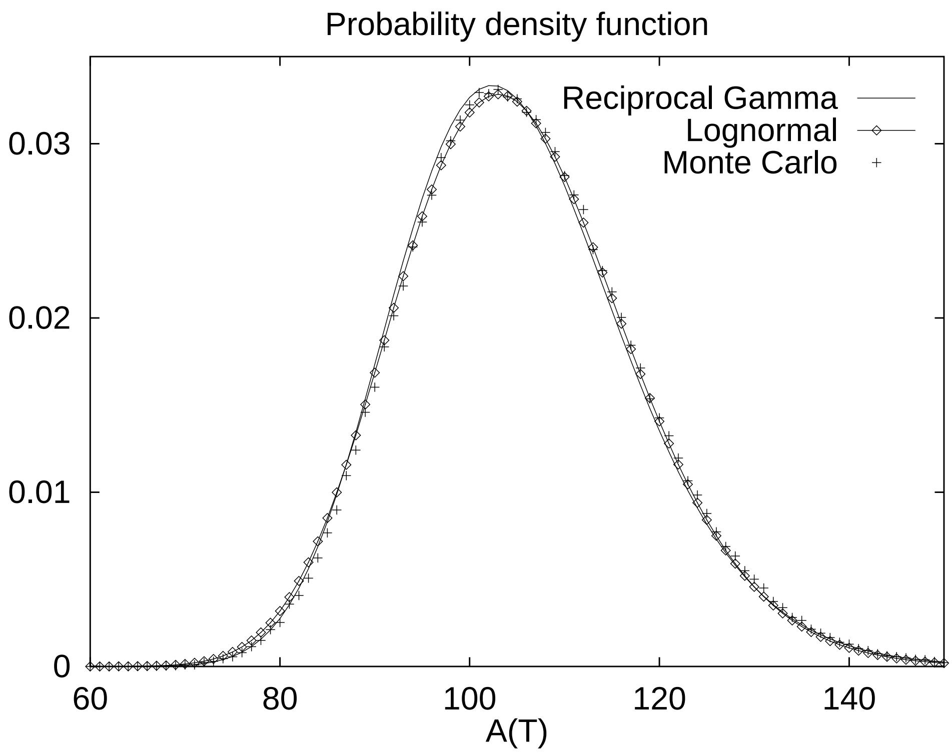

Figure 3: **Approximation to the risk-neutral probability density function of the arithmetic stock price average at maturity**. The parameter values are: $r = 0.1,\ q = 0, \sigma = 0.2, T = 1, S_0 = 100$ and $n = \infty$. The expected value of the average price is 105.17, and the variance 152.74. For $n = \infty$ the true density function is reciprocal gamma, with parameters $\alpha = 74.42$ and $\beta = 1.29\text{E-4}$. This function is approximated with a lognormal distribution with the same moments. The density function is also estimated with Monte Carlo simulation, using a set of 50 runs of 10,000 paths with 1,000 time steps.