Calculation of Volatility in a Jump-Diffusion Model

Javier F. Navas[1]

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[1]Instituto de Empresa, Maria de Molina 13, Madrid 28006, Spain. Phone: +34 91 568-9600. Fax: +34 91 563-6859. E-mail: jfnavas@ie.edu. I would like to thank John J. McConnell, Stephen Figlewski (the editor), and an anonymous referee for their helpful comments and sugestions. The usual caveat applies.

Abstract

A common way to incorporate discontinuities in asset returns is to add a Poisson process to a Brownian motion. The jump-diffusion process provides probability distributions that typically fit market data better than those of the simple diffusion process. To compare the performance of these models in option pricing, the total volatility of the jump-diffusion process must be used in the Black-Scholes formula. A number of authors, including Merton (1976a & b), Ball and Torous (1985), Jorion (1988), and Amin (1993), miscalculate this volatility because they do not include the effect of uncertainty over the jump size. We calculate the volatility correctly and show how this affects option prices.

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The diffusion process is widely used to describe the evolution of asset returns over time. In option pricing it allows the use of Black and Scholes-type formulae to value European options on stocks, foreign currencies, interest rates, commodities and futures. Under this process, instantaneous asset returns are normally distributed. However, the distributions observed in the market exhibit non-zero skewness and higher kurtosis than the normal distribution[1], which produces pricing errors when the Black-Scholes formula is used.

A way of obtaining distributions consistent with market data is to assume that the stock price follows a jump-diffusion process. Merton (1976a) develops a model in which the arrival of normal information is modeled as a diffusion process, while the arrival of abnormal information is modeled as a Poisson process. The jump-diffusion process can potentially describe stock prices more accurately at the cost of making the market incomplete, since jumps in the stock price cannot be hedged using traded securities. If the market is incomplete, the payoffs of the option cannot be replicated, and the option cannot be priced. To overcome this problem, Merton assumes that the jump risk is diversifiable and, consequently, not priced in equilibrium. He then derives a closed-form expression for the price of a call option.

To study the performance of the jump-diffusion model, Merton (1976b) compares option prices computed with his model with those obtained with the Black-Scholes formula. For this comparison to be meaningful, the total volatility of the jump-diffusion process (ν) must be used in the Black-Scholes formula. Merton (1976a & b) miscalculates this volatility because he leaves out the effect of jump size uncertainty on return volatility, so his measure is smaller than ν. We refer to this measure as "Merton's volatility rate" (ν_M). He finds that there can be significant differences in option prices for deep out-of-the-money options and short maturity options when there are large and infrequent

[1]See, for example, Fama (1965), Jorion (1988), Hsieh (1988), Bates (1996), and Campa et al. (1997).

jumps.

Ball and Torous (1985) analyze a sample of NYSE listed common stocks and find the existence of lognormal jumps in most of the daily returns considered. They then study Merton and Black-Scholes call option prices and find small differences. However, they also miscalculate the total volatility of the jump-diffusion process, so that their results should be interpreted with care.

Other examples where the volatility of the jump-diffusion process is computed incorrectly are Jorion (1988), who adapts Merton's model to price foreign currency options, and Amin (1993), who develops a discrete-time option pricing model under a jump-diffusion process.

In this note we compute the total volatility of the jump-diffusion process correctly and show the effect of the miscalculation on option prices. We find that that the pricing error can be larger or smaller that what was previously reported, and that some options undervalued with ν_M are in fact overvalued with the correct volatility measure.

The rest of the paper is organized as follows. Next section briefly reviews the Merton (1976a) model and presents the total variance of the jump-diffusion process. Using this variance, we study in Section 2 the pricing error of an investor who uses the Black-Scholes formula to price options when the stock price follows a jump-diffusion process. Finally, we conclude with Section 3.

1 Merton's Jump-Diffusion Formula

We consider a continuous trading economy with trading interval $[0, \tau]$ for a fixed $\tau > 0$, in which there are three sources of uncertainty, represented by a standard Brownian motion $\{Z(t) : t \in [0, \tau]\}$ and a independent Poisson process $\{N(t) : t \in [0, \tau]\}$ with constant intensity λ and random jump size Y on the filtered probability space $(\Omega, \mathcal{F}, \mathrm{Q}, \{\mathcal{F}_{\mathrm{t}} : \mathrm{t} \in$

$[0, \tau]\})$.

We define $\mathcal{F}_{\mathrm{t}} \equiv \mathcal{F}_{\mathrm{t}}^{\mathrm{Z}} \vee \mathcal{F}_{\mathrm{t}}^{\mathrm{N}}$ and $\mathcal{F} \equiv \mathcal{F}_{\tau}$, where $\mathcal{F}_{\mathrm{t}}^{\mathrm{Z}}$ and $\mathcal{F}_{\mathrm{t}}^{\mathrm{N}}$ are the smallest right-continuous complete σ-algebras generated by $\{Z(s) : s \leq t\}$ and $\{N(s) : s \leq t\}$ respectively.

Merton (1976a) assumes that the stock price dynamics are described by the following stochastic differential equation

$$\frac{dS(t)}{S(t-)} = (\alpha - \lambda\kappa)\, dt + \sigma dZ(t) + (Y(t) - 1)\, dN(t), \tag{1}$$

where $\kappa \equiv E\{Y(t) - 1\}$ is the expected relative jump of $S(t)$.

Assuming that jump sizes are lognormally distributed with parameters μ and δ, and that the jump risk is diversifiable, Merton shows that the option price is given by

$$F(S(t), \tau, K, \sigma^2, r; \mu, \delta^2, \lambda) = \sum_{n=0}^{\infty} \frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!} W(S(t), \tau, K, \sigma_n^2, r_n) \tag{2}$$

where $W(S(t), \tau, K, \sigma_n^2, r_n)$ is the Black-Scholes option price for a European option with exercise price K and maturity τ on a non-dividend stock. The volatility of stock returns is σ, the risk-free interest rate is r, and

$$\begin{aligned}
r_n &\equiv r + \frac{n}{\tau}\left(\mu + \frac{\delta^2}{2}\right) - \lambda\kappa, \\
\sigma_n^2 &\equiv \sigma^2 + \frac{n}{\tau}\delta^2, \\
\lambda' &\equiv \lambda(1 + \kappa) = \lambda \exp\left(\mu + \frac{\delta^2}{2}\right).
\end{aligned}$$

Unlike in the diffusion case, when changes in stock prices are given by expression (1) the instantaneous stock returns are not normally distributed. The distribution will have non-zero skewness and will exhibit leptokurtosis when compared to a Gaussian distribution, which is consistent with the empirical evidence. There will not be, in general, a closed-form expression for the density function of the distribution, but we can easily compute the moments of stock returns. Applying Itô's formula to (1) we have that, under Q, the expected natural logarithm of the stock price is given by

$$\begin{aligned} E\left\{\ln\frac{\mathrm{S(t)}}{\mathrm{S(0)}}\right\} &= \left(\alpha - \lambda\kappa - \frac{\sigma^2}{2}\right)t + E\left\{E\left\{\ln\mathrm{Y(n(t))}|\mathcal{F}_{\mathrm{T}}^{\mathrm{N}}\right\}\right\} \\ &= \left(\alpha - \lambda\kappa - \frac{\sigma^2}{2}\right)t + E\left\{n(t)\mu\right\}, \\ &= \left(\alpha - \lambda\kappa - \frac{\sigma^2}{2}\right)t + \lambda t\mu, \end{aligned}$$

where $Y(n) \equiv \prod_{i=1}^{n} Y_i$, $\{Y_i, i = 1, 2, \ldots, n\}$ are the random jump amplitudes, and $n(t)$ is a Poisson random variable with parameter λt.

Since the Brownian motion and the Poisson process are independent by construction (see Protter (1995)), to calculate the variance, we can write

$$Var\left\{\ln\frac{\mathrm{S(t)}}{\mathrm{S(0)}}\right\} = Var\left\{\sigma Z(t)\right\} + Var\left\{\ln\mathrm{Y(n(t))}\right\} \tag{3}$$

Notice that $\sum_{i=1}^{n}\ln\mathrm{Y_i}$ is conditionally distributed as a Gaussian with mean $n\mu$ and variance $n\delta^2$. However, this does not imply that $Var\left\{\ln\mathrm{Y(n(t))}\right\} = \lambda t\delta^2$ since now $n(t)$ is random. Using the fact that for any random variable $u, Var\{u\} = -E\{u\}^2 + E\left\{u^2\right\}$ we have that

$$\begin{aligned} Var\left\{\ln\mathrm{Y(n(t))}\right\} &= -E\left\{\ln\mathrm{Y(n(t))}\right\}^2 + E\left\{\left[\ln\mathrm{Y(n(t))}\right]^2\right\} \\ &= -(\lambda t\mu)^2 + E\left\{E\left\{\left[\ln\mathrm{Y(n(t))}\right]^2|\mathcal{F}_{\mathrm{T}}^{\mathrm{N}}\right\}\right\} \\ &= -(\lambda t\mu)^2 + E\left\{n(t)\delta^2 + n(t)^2\mu^2\right\} \\ &= -(\lambda t\mu)^2 + \delta^2\lambda t + \mu^2\left(\lambda t + \lambda^2 t^2\right) \end{aligned}$$

$$= \lambda \left(\mu^2 + \delta^2\right) t. \quad (4)$$

Hence, the total variance of the natural logarithm of the stock price under a jump-diffusion process is given by

$$Var\left\{\ln \frac{\mathrm{S(t)}}{\mathrm{S(0)}}\right\} = \left(\sigma^2 + \lambda\left(\mu^2 + \delta^2\right)\right) t. \quad (5)$$

A similar result can be found in Press (1967), Navas (1994), and Das and Sundaram (1999). Merton (1976a & b), Ball and Torous (1985), Jorion (1988), and Amin (1993) miscalculate this variance as $(\sigma^2 + \lambda\delta^2)t$, which will only be correct when $\mu = 0$. However, they assume that the expected jump size is zero, i.e. $E\{Y - 1\} = 0$, which is equivalent to take $\mu = -\frac{\delta^2}{2}$. Obviously, when δ^2 increases the error in the variance will increase.

2 Implications for Option Pricing

If an investor prices options according to the Black-Scholes model but the market prices options according to Merton's jump-diffusion process, she will underprice out-of-the-money and in-the-money options and she will overprice at-the-money (ATM) options, since the true stock return distribution will have fatter tails than the Gaussian distribution. Hence, if the investor estimates the volatility parameter of the diffusion process implicitly from market prices, she will observe the well known volatility smile. In Figure 1 we price a call option with different exercise prices with Merton's (1976a) model using the parameters $\tau = 0.5, \sigma^2 = 0.05, r = 0.10, \mu = -0.025, \delta^2 = 0.05$, and $\lambda = 1$. Then, we take those values as market prices and compute implied volatilities from the Black-Scholes formula. We observe that in order to fit "market" prices correctly, the volatility of the diffusion process in the Black-Scholes model must be set higher for out-of-the-money (OTM) and ITM options than for ATM options. The curve is not symmetric because of the presence of skewness in the distribution of stock returns.

Insert Figure 1 about here.

We now study in more detail the pricing "error" that our investor will face when she uses the Black-Scholes formula to price options when the true model is Merton's jump-diffusion one. We assume that the investor estimates the volatility of stock returns from the time series generated by the jump-diffusion process. In this case, her estimate of the volatility of stock returns will be an unbiased estimate of the total volatility of the jump-diffusion process (see Merton (1976b) for further details).

In Table 1 we compare Merton and Black-Scholes prices for different parameters of the jump-diffusion process. The table drawns upon Table IV of Ball and Torous (1985). They value a 6-month stock call option with exercise price 35 when the stock is trading at 38. The variance of the diffusion part is $\sigma^2 = 0.05$, and the interest rate is $r = 0.10$.

Insert Table 1 about here.

In the first and second rows, we follow Merton (1976a & b) and Ball and Torous (1985) and study the case where the expected relative jump size of the stock (κ) is 0, i.e. we take $\mu = -\frac{\delta^2}{2}$. To be consistent with their notation, we represent Merton prices by F, that is $F \equiv F(S(t), \tau, K, \sigma^2, r; \mu, \delta^2, \lambda)$. We denote the investor appraised option value using Black-Scholes formula by $F_e(\nu^2)$ and $F_e(\nu_M^2)$, depending on the variance rate used,[2] that is $F_e(\nu^2) \equiv W(S(t), \tau, K, \nu^2, r)$ and $F_e(\nu_M^2) \equiv W(S(t), \tau, K, \nu_M^2, r)$.

In the first row, we assume that the variance of the log of the jump size is equal to the variance of the diffusion ($\delta^2 = 0.05$), and that there is only one jump per year ($\lambda = 1$). This implies that the variance of the diffusion process is practically half of the total

[2] Recall that the total variance of the process per unit of time, ν^2, is $\sigma^2 + \lambda\left(\mu^2 + \delta^2\right)$, while ν_M^2 is defined as $\sigma^2 + \lambda\delta^2$

variance of the process. With these values, the Merton option price is $F = 5.9713$. As a reference, the Black-Scholes price *if* there were no jumps would be $F_e(\sigma^2) = 5.3396$. However, this will not be the appraised option value of an investor who incorrectly believes that the stock price follows a diffusion process and estimates the volatility from historical data. As stated before, her estimation of the variance will be $\nu^2 = 0.10062$ and her appraised option value $F_e(\nu^2) = 6.0711$. In this case the investor will overprice the option by only 1.7%. However, the pricing error increases as there are larger and fewer jumps per year. For instance, when $\delta^2 = 0.5$ and $\lambda = 0.1$ (second row), Merton option price is $F = 5.6979$, while the investor appraised value is $F_e(\nu^2) = 6.1447$ (a 7.8% higher). The table also shows the appraised option values incorrectly computed by Ball and Torous (1985) using ν_M, $F_e(\nu_M^2)$.

An interesting question is whether the lack of significant differences between the Black-Scholes and Merton model prices found by Ball and Torous (1985) could be due to their miscalculation of the total volatility of the process. Using their parameter estimates, we find that the difference between ν and ν_M is insignificant in most of the stocks. Thus, using the correct volatility measure apparently will not change their results. However, rows 3-10 of Table 1 show that if we relax Ball and Torous' assumption that the mean relative jump size is zero, the difference in option prices can be large. For example, when investors expect one jump per year that will produce a mean relative jump in stock price of -20% (row 9), $F = 6.6872$, $F_e(\nu_M^2) = 6.0628$, and $F_e(\nu^2) = 6.8066$. That is, an investor using incorrectly the Black-Scholes formula and miscalculating the volatility of the process will underprice the option by 9.3%, while that using the correct volatility she will overprice it by 1.8%. Hence, the restriction imposed on the jump size together with the miscalculation of the volatility could partially explain the results of Ball and Torous (1985).

We can further analyze the pricing errors standardizing the variables as in Merton (1976b). In this way, we reduce from eight to four the number of parameters needed to compute option prices in the jump-diffusion model. As other authors, Merton considers the special case in which $\mu = -\frac{\delta^2}{2}$. Substituting $\kappa = 0$ into equation (2), the option pricing formula simplifies to

$$F = \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} W(S(t), \tau, K, \sigma_n^2, r), \tag{6}$$

where $\sigma_n^2 = \sigma^2 + \frac{n}{\tau}\delta^2$.

If we define $W'(S, \tau) \equiv W(S, \tau, 1, 1, 0)$, it is easy to show that

$$W'(X, \tau_n) = \frac{W(S, \tau, K, \sigma_n^2, r)}{Ke^{-r\tau}},$$

where $X \equiv \frac{S}{Ke^{-r\tau}}$ and $\tau_n \equiv \sigma_n^2\tau$.

With this notation, we rewrite equation (6) as

$$F = Ke^{-r\tau} \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} W'(X, \tau_n). \tag{7}$$

Similarly, we can express the investor appraisal value of the option using ν_M as

$$F_e(\nu_M^2) \equiv W(S, \tau, K, \nu_M^2, r) = Ke^{-r\tau} W'(X, \bar{\tau}), \tag{8}$$

where $\bar{\tau} \equiv \nu_M^2\tau$, and can be considered as a maturity measure of the option.

Merton (1976b) defines two other standardized variables: Γ, a variance measure[3], and Λ, the expected number of jumps during the life of the option divided by the maturity

[3]Notice that Γ does not represent the variance of the jump process divided by the total variance of the jump-diffusion process, as stated by Merton (1976b).

measure $\bar{\tau}$. Their expressions are given by

$$\Gamma \equiv \frac{\lambda\delta^2}{\nu_M^2},$$
$$\Lambda \equiv \lambda\frac{\tau}{\bar{\tau}}.$$

From these definitions it is easy to show that $\tau_n = (1-\Gamma)\bar{\tau} + n\frac{\Gamma}{\Lambda}$ and $\lambda\tau = \Lambda\bar{\tau}$.

Finally, from equations (7) and (8) we can express the option prices normalized by the present value of the exercise price as

$$f = \sum_{n=0}^{\infty} \frac{e^{-\Lambda\bar{\tau}}(\Lambda\bar{\tau})^n}{n!} W'\left(X, (1-\Gamma)\bar{\tau} + n\frac{\Gamma}{\Lambda}\right), \text{and} \tag{9}$$

$$f_{eM} \equiv \frac{F_e(\nu_M^2)}{Ke^{-r\tau}} = W'(X, \bar{\tau}). \tag{10}$$

Figure 2 shows the absolute and relative normalized pricing errors, defined as $f - f_e$ and $\frac{f-f_e}{f_e}$ respectively, for different stock prices. We take $\bar{\tau} = 0.0492$, $\Gamma = 0.3670$, and $\Lambda = 1.8084$. This represents, for example, a one-year call option with an annual Merton's volatility rate (ν_M) of 0.2218, when the number of jumps per year (λ) is 0.0890 and the volatility of the diffusion part (σ) is 0.1765. These values are obtained from Andersen and Andreasen (1999), who estimate the jump-diffusion parameters for a sample of S&P 500 index options, without assuming that $\mu = -\frac{\delta^2}{2}$. Notice that, in this case, the total volatility of the process (ν) is 0.3459, considerably higher than Merton's volatility rate. An investor using ν_M in the Black-Scholes formula will observe a pricing error that is larger (in absolute value) for deep in-the-money options and smaller for most of the other options than what she would actually observe in the market (using ν). Moreover, if the normalized stock price (X) is greater than 0.94, the Black-Scholes model with ν_M underprices the option, while with ν the model overprices the option. For example, when $S = 100$ and $K = 100$ ($X = 1.10$) an investor using ν_M in the Black-Scholes formula will underprice the option by 7.85%, when in fact the model overprices it by 18.33%. Thus, the practical relevance of the miscalculation of the total variance of the jump-diffusion process can be significant.

Insert Figure 2 about here.

3 Conclusions

If the true process describing the dynamics of stock returns is a jump-diffusion process, but an investor incorrectly believes that stock returns follow a diffusion process, she will use the Black-Scholes formula to price options. If she estimates the volatility of stock returns from time series data, she will actually estimate the total volatility of the jump-diffusion process. When she uses this volatility in the Black-Scholes formula, she will observe a pricing error.

Many authors miscalculate this error, since they compute the total volatility of the jump-diffusion process incorrectly.

In this paper we compute the total volatility of the process correctly and recalculate the pricing error. We show that, for reasonable parameter values, the pricing error can behave very differently than what was previously reported.

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Table 1: **Call option prices for different parameters of the jump-diffusion process**.

	δ^2	λ	μ	ν_M^2	ν^2	F	$F_e(\sigma^2)$	$F_e(\nu_M^2)$	$F_e(\nu^2)$
$\kappa = 0$	0.05	1.00	-0.025	0.10	0.10062	5.9713	5.3396	6.0628	6.0711
	0.50	0.10	-0.250	0.10	0.10625	5.6979	5.3396	6.0628	6.1447
$\kappa = 0.1$	0.05	1.00	0.070	0.10	0.10494	5.9647	5.3396	6.0628	6.1277
	0.50	0.10	-0.155	0.10	0.10239	5.6826	5.3396	6.0628	6.0944
$\kappa = 0.2$	0.05	1.00	0.157	0.10	0.12475	6.1554	5.3396	6.0628	6.3778
	0.50	0.10	-0.068	0.10	0.10046	5.6758	5.3396	6.0628	6.0689
$\kappa = -0.1$	0.05	1.00	-0.130	0.10	0.11699	6.2055	5.3396	6.0628	6.2817
	0.50	0.10	-0.355	0.10	0.11263	5.7234	5.3396	6.0628	6.2266
$\kappa = -0.2$	0.05	1.00	-0.248	0.10	0.16158	6.6872	5.3396	6.0628	6.8066
	0.50	0.10	-0.473	0.10	0.12239	5.7603	5.3396	6.0628	6.3488

This table relys on Table IV of Ball and Torous (1985). They assume that the mean relative jump size of the stock (κ) is zero (rows one and two). The other parameters of the model are $S(0) = 38, \tau = 0.5, K = 35, \sigma^2 = 0.05$, and $r = 0.10$, where σ is the volatility of the diffusion process. F represents the Merton model price, and F_e is the investor appraised option value using the Black-Scholes formula. The total volatility per unit of time of the jump-diffusion process (needed to compute F_e) is denoted by ν, whereas ν_M is the volatility used by Ball and Torous when computing Black-Scholes prices.

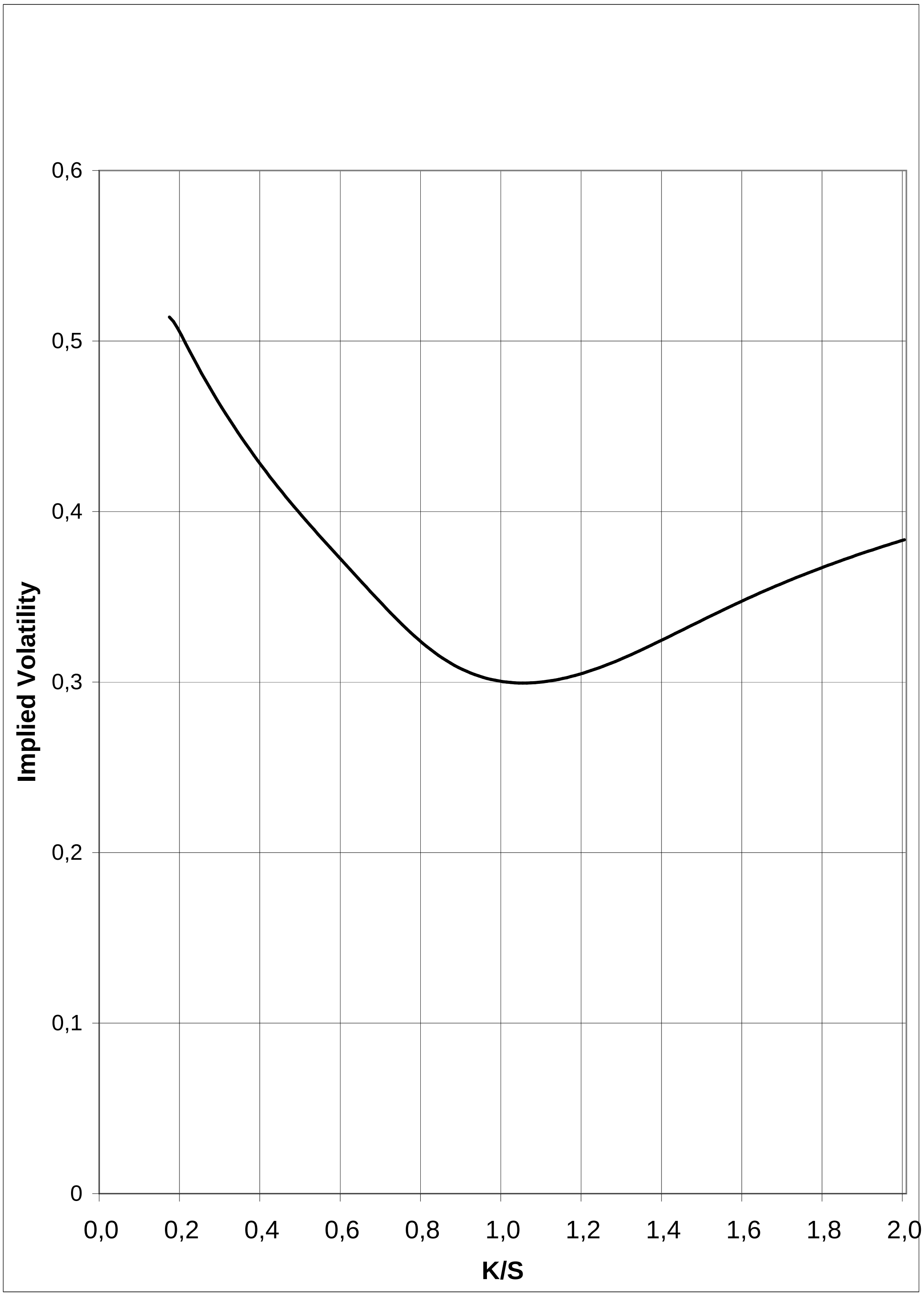

Figure 1: **Implied Black-Scholes Volatility from Merton (1976a) call option prices**. The parameters of the model are $\tau = 0.5, \sigma^2 = 0.05, r = 0.10, \mu = -0.025, \delta^2 = 0.05$, and $\lambda = 1$.

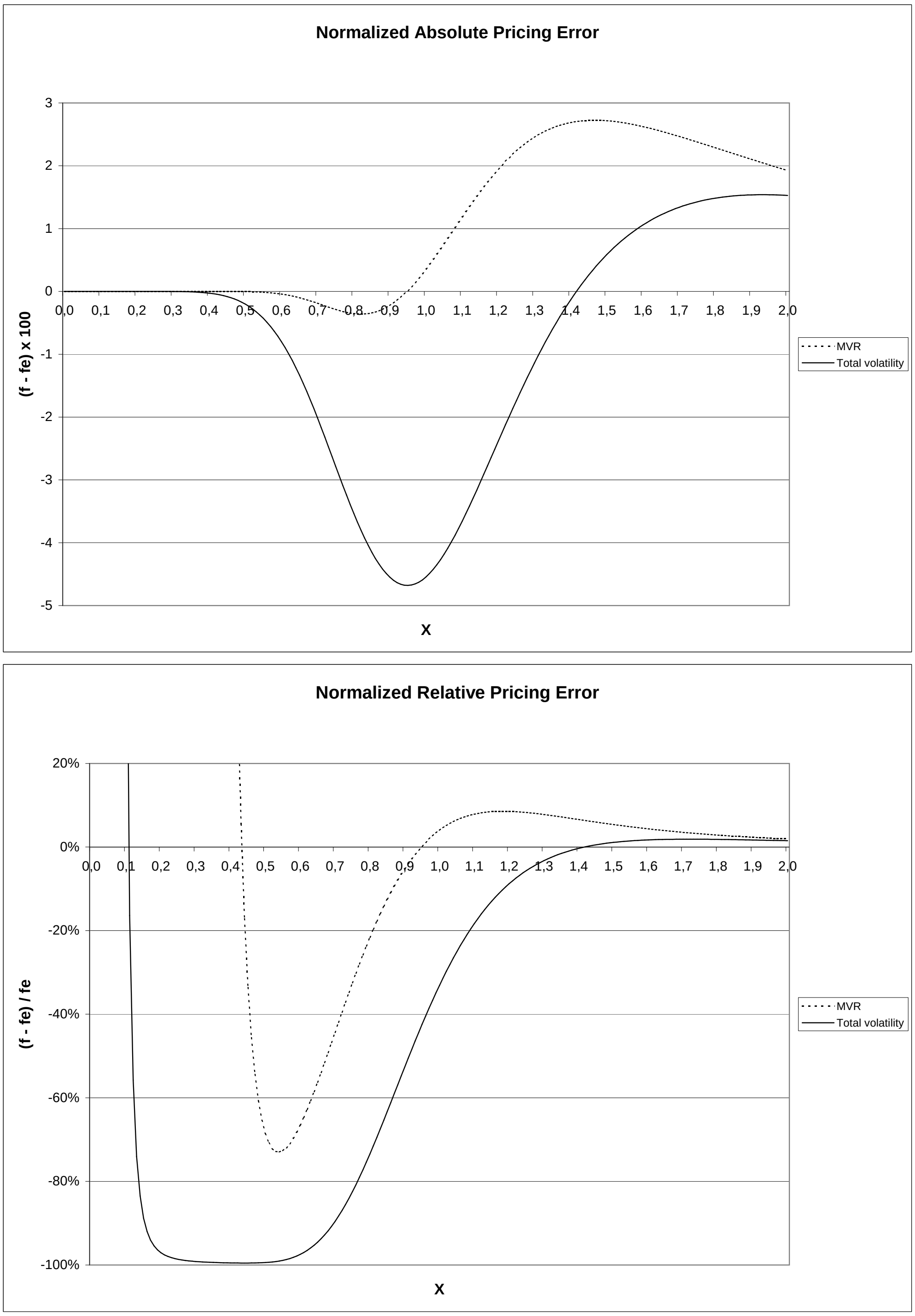

Figure 2: **Option price error for actual parameter values**. MVR stands for Merton's volatility rate. Stock and option prices are expressed in units of the present value of the exercise price. X is the normalized stock price, f is the normalized Merton (1976b) call option price, and f_e is the normalized investor appraisal of the option value when using the Black-Scholes formula. The parameters of the model are $\bar{\tau} = 0.0492, \Gamma = 0.367$ and $\Lambda = 1.808$, and are obtained from Andersen and Andreasen (1999).