On the Robustness of Least-Squares Monte Carlo (LSM) for Pricing American Derivatives

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Abstract

This paper analyses the robustness of Least-Squares Monte Carlo, a technique proposed by Longstaff and Schwartz (2001) for pricing American options. This method is based on least-squares regressions in which the explanatory variables are certain polynomial functions. We analyze the impact of different basis functions on option prices. Numerical results for American put options show that this approach is quite robust to the choice of basis functions. For more complex derivatives, this choice can slightly affect option prices.

Keywords: Least-Squares Monte Carlo, Option Pricing, American Options.

Journal of Economic Literature classification: C15, C60, G13.

Closed-form expressions for derivative prices exist in few cases. One example is an European stock option, whose price was derived by Black and Scholes (1973) and Merton (1973). For American options, some analytical expressions have also been found, but, in general, numerical methods are required.

One of the most popular numerical techniques in option pricing is Monte Carlo simulation. The Monte Carlo approach simulates paths for asset prices. An estimation of the option price is obtained by discounting the average of the option pay-offs computed for each path. This technique is appropriate to price European options with complex features (path-dependence, multiple stochastic processes, random volatility, jumps, ...).

This technique has not been widely applied to price American options. Carriere (1996) presents a backward induction algorithm and applies it to calculate the early exercise premium. He shows that the estimation of the decision rule to exercise early is equivalent to the estimation of a series of conditional expectations. The conditional expectations are estimated using splines and local regressions.

Tsitsiklis and Van Roy (1999) study perpetual American options and propose an stochastic algorithm that approximates the conditional expectations by a linear combination of basis functions. They also analyze the convergence properties of this algorithm. In general, there is no closed-form expression for the conditional expectation function, and they select a set of basis functions such that their weighted combination is “close” to the true function. The usefulness of this algorithm is illustrated by pricing a Bermuda option whose payoff is the gross return provided by a certain stock on the last hundred days.

Tsitsiklis and Van Roy (2001) analyze finite-horizon pricing problems. They present an algorithm for pricing complex American options that involves the evaluation of value functions at a finite set of “representative” elements of the state space. A linear combination of basis functions is fitted to the data via least-squares regressions, in order to approximate the conditional expectation over the entire state space. These authors provide convergence results and error bounds for their algorithm.

Longstaff and Schwartz (2001) also apply least-squares regressions and estimate the continuation values of a number of derivatives. They use only in-the-money path in the regressions to increase efficiency. They name this technique Least-Squares Monte Carlo (LSM).

The LSM method involves two approximations:

1. Replace the conditional expectation in the pricing algorithm by its orthogonal projection on the space generated by a finite set of functions.
2. Use Monte-Carlo simulations and least-squares regressions to estimate numerically the conditional expectation function.

An interesting question is to analyze the convergence properties of the LSM technique. Longstaff and Schwartz (2001) state some partial convergence results. Clément, Lamberton, and Protter (2001) analyze it in more detail. They prove that the estimated conditional expectation approaches (with probability one) the true conditional expectation as the number of basis functions goes to infinity. They also determine convergence rates and prove that the normalized estimation error is asymptotically Gaussian.

In this paper, we take a different approach and analyze the robustness of the LSM approach for pricing American options numerically. For a standard put option, we find that LSM technique is very robust to the choice of the type and the number of basis functions. Although we find that for more than 20 polynomial terms, put prices can change significantly due to optimization problems of the least-squares routines. For more complex derivatives, however, we find that the number and the type of basis functions can slightly affect option prices.

This article is organized as follows. Section 1 reviews numerical methods for pricing American-style options. In Section 2, we briefly present the LSM technique. Section 3 describes the set of basis functions used in this paper and studies the pricing of some American derivatives. In Section 4, we analyze the out-of-sample performance of LSM. Finally, Section 5 concludes the paper.

1 A Review of American Option Pricing

As mentioned before, analytical expressions for American option prices have been found in some cases. For example, pricing formulas for perpetual American options are provided by McKean (1965). Roll (1977), Geske (1979), and Whaley (1981) have derived closed-form solutions for the price of American call options with discrete dividends.

Several authors[1] obtain closed-form solutions for approximations to the original pricing problem. Ait-Sahlia and Carr (1997) and Ju (1998) compare some of these approximations.

To price complex options, in general, numerical techniques are required. Lattice methods are based on the discretization of the risk-neutral processes followed by the relevant variables. Then, backward induction in time is used to find the option price. The most basic method is the binomial model, introduced by Cox, Ross, and Rubinstein (1979) and Rendleman and Bartter (1979), that is based on the random walk approximation to the Brownian motion. In some cases, trinomial trees, originally proposed by Parkinson (1977), are used to increase the accuracy.

An alternative technique is the "finite difference" method. After building a grid of mesh points, an approximate solution of the partial differential equation for the option price is obtained by replacing the partial derivatives with finite differences. Depending on how these differences are computed, the fully explicit, fully implicit or Crank-Nicolson method, is obtained.[2]

American options can also be priced without approximating the stochastic process for the underlying asset or the partial differential equation for the option price. This is the case of quadrature techniques, which are based on approximating the integral that gives the option price. Examples of this technique are the trapezoidal and Simpson's rules.

Some authors use the "integral representation" method, which consists of decomposing the price of an American put option into the price of an European put option plus the early exercise premium, that is expressed as an integral (see Kim (1990), Jacka (1991), and Carr, Jarrow, and Mynemi (1992)). Different approximations of this integral have been proposed. Ju (1998) shows numerically that his approximation together with the method in Broadie and Detemple (1996) and the randomization technique of Carr (1998) are the most accurate methods for pricing American options.

More recently, Bunch and Johnson (2000) have derived exact expressions for the critical stock price function and the American put price in the perpetual and finite cases. Ait-Sahlia (1996) and Ait-Sahlia and Lai (1996, 2000) also have obtained closed-form expressions for the optimal exercise boundary.

Monte Carlo simulation, introduced in option pricing by Boyle (1977), provides an intuitive way of valuing options. For a survey on aplications of this technique to option pricing see Boyle, Broadie, and Glasserman (1997). This method is suitable for path-dependent European options with multiple state variables, stochastic volatility, jumps,.... Its major disadvantage is that it is computationally intensive and inefficient. To mitigate this problem, variance reduction techniques, such as antithetic variables and control variates, have been developed. Applications to the valuation of American options are based on the parameterization of the transitional density function or the early exercise boundary, and can be found in Bossaerts (1989), Barraquand and Martineau (1995), Broadie and Glasserman (1997a), Broadie, Glasserman, and Jain (1997), and Raymar and Zwecher (1997), among others.

2 Least-Squares Monte Carlo

Longstaff and Schwartz (2001) also use simulation to value American options, estimating directly the conditional expectation function. This expectation represents the continuation value of the option at each exercise date. These authors estimate the continuation value by a least-squares regression jointly with the cross-sectional information provided by Monte Carlo simulation. In this regression, they use a set of basis functions in the underlying asset prices. The fitted values are taken as the expected continuation values. Comparing these estimations with the immediate exercise values, they identify the optimal exercise decision. This procedure is repeated recursively going back in time. Discounting the obtained cash-flows to time zero, the price of the American option is found.

More formally, they assume a finite time horizon, $[0,T]$, in which they define a probability space,[3] $(\Omega, \mathbb{F}, P)$, and an equivalent martingale measure, Q. Let $C(\omega, s; t, T)$, $\omega \in \Omega$, $s \in (t, T]$ denote the path of option cash-flows, conditional on (a) the option being exercised after t and (b) the optionholder following the optimal stopping strategy at every time after t.

The American option is approximated by its Bermuda counterpart, assuming a finite number of exercise dates $0 < t_1 < t_2 < \ldots < t_K = T$. The continuation value is, under no-arbitrage conditions, the risk-neutral expectation of the future discounted cash flows $C(\omega, s; t_i, T)$:

$$F(\omega; t_i) = E_Q \left[\sum_{j=i+1}^{K} \exp\left(-\int_{t_i}^{t_j} r(\omega, s)\, ds \right) \; C(\omega, t_j; t_i, T) \mid \mathbb{F}_{t_i} \right],$$

where $r(.)$ is the risk-free interest rate and $\mathbb{F}_{t_i}$ is the information set at time t_i.

The idea underlying the LSM algorithm is that this conditional expectation can be approximated by a least-squares regression for each exercise date. At time t_{K-1}, it is assumed that $F(\omega; t_{K-1})$ can be expressed as a linear combination of orthonormal basis functions $(p_j(X))$ such as Laguerre, Hermite, Legendre or

Jacobi polynomials. That is

$$F(\omega; t_{K-1}) = \sum_{j=0}^{\infty} a_j p_j(X),\ a_j \in I\!R$$

which is approximated by

$$F_M(\omega; t_{K-1}) = \sum_{j=0}^{M} a_j p_j(X),\ a_j \in I\!R.$$

This procedure is repeated backwards until the first exercise date.

3 Numerical Results on the Robustness of LSM

We now analyze the effect on option prices of a change in the type or the number of basis functions. A common choice of basis functions is a set of polynomials. We consider the following ones (with up to twenty terms):[4]

Name	$f_n(x)$	Name	$f_n(x)$
Power	$W_n(x)$	Chebyshev 1st kind A	$T_n(x)$
Legendre	$P_n(x)$	Chebyshev 1st kind B	$C_n(x)$
Laguerre	$L_n(x)$	Chebyshev 1st kind C	$T_n^*(x)$
Hermite A	$H_n(x)$	Chebyshev 2nd kind A	$U_n(x)$
Hermite B	$H_{e_n}(x)$	Chebyshev 2nd kind B	$S_n(x)$

where $n \geq 0$ denotes the degree of the polynomial.

These polynomials can be expressed in three alternative ways:

1. Explicit expression

$$f_n(x) = d_n \sum_{m=0}^{N} c_m g_m(x),$$

where N takes different values for different polynomials (see Table 1).

2. Rodrigues' formula

$$f_n(x) = \frac{1}{a_n\, g(x)} \frac{\partial^n}{\partial x^n} \left[\rho(x)\ (g(x))^n\right]$$

3. Recurrence law

$$a_{n+1}\, f_{n+1}(x) = (a_n + b_n\, x)\, f_n(x) - a_{n-1}\, f_{n-1}(x)$$

The coefficients and functions included in these expressions are shown in Tables 1, 2, and 3, respectively.

[Insert Tables 1, 2, and 3 about here]

For illustration purporses, in Figure 1 we plot some polynomials for different degrees. The remaining ones can be found in Abramowitz and Stegun (1972).

[Insert Figure 1 about here]

From a theoretical point of view, it would be desirable to use an orthonormal basis of functions on which to project continuation values. This means that

$$\int_a^b f_n(x)\, f_m(x)\, dx = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

The values for the limits of this integral vary with the polynomials. See Abramowitz and Stegun (1972) for details. In most cases, the range of underlying prices (X) is different from the interval $[a, b]$ so that the basis functions will not be orthonormal. Consequently, we should increase the number of terms used in the regressions.

A linear regression can be interpreted as the projection of the dependent variable on the span generated by the independent ones. Consequently, to study the robustness of LSM with respect to the above polynomials, we must ensure that they do not generate the same span.

Abramowitz and Stegun (1972) provide the following relationships between some of these polynomials:

$$\begin{aligned} H_{e_n}(x) &= 2^{-n/2} H_n\left(\frac{x}{\sqrt{2}}\right) \\ T_n(x) &= \frac{1}{2}[U_n(x) - U_{n-2}(x)] \\ C_n(x) &= 2\,T_n\left(\frac{x}{2}\right) \\ S_n(x) &= U_n\left(\frac{x}{2}\right) \end{aligned}$$

Additionally, in Table 1 it is easily seen that $T_n^*(x) = 2^{1-n}\,T_n(x)$.

Thus, we only need to study the effect of using $W_n(x), P_n(x), L_n(x), H_{e_n}(x)$, or $T_n(x)$ on option prices.

Furthermore, as can be seen in Tables 22.3-22.10 of Abramowitz and Stegun (1972), the coefficients of each of these polynomials with respect to power functions form a non-singular matrix. This implies that the span generated by each of them is the same as the one generated by the power functions. As a consecuence, the LSM technique should provide identical option prices for different polynomials.

3.1 Valuation of the Standard Put Option

We now price an American put option with strike 40 on a non-dividend stock that trades currently at 40. The risk-free interest rate is 0.06, and the volatility of the stock return is 0.2. We approximate this option assuming that there are 70 exercise dates during the life of the option.

The value of this option using the binomial method of Cox, Ross, and Rubinstein (1979) (with 1,000 steps) is 2.31928. The value of the corresponding European option, using simulation, binomial trees, and the Black and Scholes (1973) formula, are 2.06193, 2.06560, and 2.06640, respectively.

To avoid numerical problems, we standardize the option dividing by the strike price and we use double-precision variables. We also employ SVDFIT, a Numerical Recipes routine that performs linear least-squares fits using the singular value decomposition technique. This routine avoids solving the normal equations of linear least-squares problems, that in many cases involve nearly singular matrices.

We show the results of the LSM algorithm with up to 20 polynomial terms in Table 4. For completness, we also report the option prices obtained for different polynomials.

We use 100,000 simulations, half of them with antithetic variables. Notice that this implies that we have to store (100,000 × 2) matrices.

[Insert Table 4 about here]

As expected, for a fixed number of terms, we obtain similar option prices for different basis functions, and the differences are due to numerical errors.

For a given polynomial, we see that option values do not increase monotonically with the number of terms. For up to five terms, option prices typically increase. With more terms, option values can decrease and increase later. This result illustrates the difficulties of implementing the convergence criterion suggested by Longstaff and Schwartz (2001).[5] For example, if we use $T_n(x)$ as basis functions, their rule suggests that five terms are enough to value the option. In this case, the option price would be 2.30689. However, option values increase again taking 7, 8, or 10 terms (for 8 terms, the option price is 2.30854).

The standard errors for the simulated values are remarkably low, ranging between 0.631 and 0.911 cents, and are very similar to those obtained by Longstaff and Schwartz (2001).

Notice that the computed option prices are lower than those obtained with the binomial tree. This is not surprising since we are considering only 70 exercise dates. A more reasonable benchmark is the binomial price of the Bermuda option, 2.31153. We use a binomial tree with 1050 steps in which the option can be exercised every 15 steps. Interestingly, LSM seems to slightly underprice the option.

For the sake of brevity, we have presented the results for just one set of parameter values. Similar tables for different cases are available upon request. The results do not change qualitatively.

When the number of terms is large (20 or more), numerical problems can appear for some polynomials, especially for out-of-the money put options ($E = 30$) with high volatility ($\sigma = 0.5$), as shown in Figure 2. This is not due to the LSM algorithm but to the least-squares routine. We see that, for the Hermite A polynomial, the sum of square errors increase with the number of terms, worsening the fit of the regressions.

[Insert Figure 2 about here]

3.2 Option on the maximum of five assets

We now analyze a Bermuda call option on the maximum of five uncorrelated assets. The volatility of asset returns is 0.2, the risk-free interest rate is 0.05, the dividend yield is 0.1, the maturity of the option is three years, and there are three exercise times per year. The strike price is 100 and the initial assets prices are 100 for the five assets.

This option has been priced by Broadie and Glasserman (1997b), using the stochastic mesh method. They find that the 90% confidence interval for the price of this option is $[26.101, 26.211]$.

Longstaff and Schwartz (2001) value this option using the LSM approach with 19 basis functions: a constant, five Hermite B ($H_{e_n}(x)$) polynomials in the maximum of the five assets, the second to the fifth maximums and their square values, the four products of consecutive pairs of maximums, and the product of the five assets. Using 50,000 paths, their option value is 26.182, which is within the interval given by Broadie and Glasserman (1997b).

We use different basis functions to price this option. In Table 5, we report option prices obtained when the Hermite B polynomial is replaced by the Chebyshev polynomial of the first kind, class A ($T_n(x)$), with up to ten terms (i.e. between 14 and 24 basis functions). As mentioned before, the results for the two polynomials should be very similar, since they generate the same span. We simulate 50,000 + 50,000 antithetic paths.

[Insert Table 5 about here]

Using very few terms (0, 1, or 2), we obtain values which are outside the interval given by Broadie and Glasserman (1997b). We see that option prices increase with up to four or five terms. With more terms, these values can decrease and increase again.

Interestingly, with more than 7 terms for $T_n(x)$, the option price decreases significantly, reaching a value of 23.89298 for 10 terms. Figure 3 shows that this is due to numerical problems of the least-squares optimization. We see that, for the polynomial $T_n(x)$, the least-squares fit deteriorates when using more than five terms.

[Insert Figure 3 about here]

In all the cases, the standard errors of the price estimates are very small (around 6 cents). Notice that while standard errors are about ten times those obtained for put options, option prices are also about ten times higher.

A final remark is that using five Hermite B polynomials, the option price is 26.187, which is very close to the price given by Longstaff and Schwartz (2001). The value of the corresponding European option is 23.098, so that the early exercise premium is higher than 3.

Now, we set the polynomials equal to $H_{e_n}(x)$, and we change the remaining basis functions. The results are shown in Table 6.

[Insert Table 6 about here]

The second column presents the prices obtained without including the squares of the second to the fifth maximums. Dropping out those values has little impact on option prices and standard errors. As before, the option value increases monotonically only with up to five terms. In the third column, we also leave out the products of consecutive maximums. In this case, option prices are outside the range given by Broadie and Glasserman (1997b) except when we use five terms. In the following column, we work with the polynomials $H_{e_n}(x)$, the second to the fifth maximums, and their squares. Now, all the option prices are outside the interval. Finally, the fifth column shows the prices obtained with the same basis functions as in Table 5 plus the third power of the second to the fifth maximums. Compared to the second column of Table 5, we find that option prices and standard errors are similar in both cases.

In all instances, fixed the number of terms, the differences in option prices do not exceed 0.3%.

3.3 American-Bermuda-Asian option

Following Longstaff and Schwartz (2001), we now price a call option on the average of the stock price during a given time horizon. This option can be exercised at any time after some initial period. It matures in two years and it cannot be exercised during the first quarter. The average stock price is the continuous arithmetic mean from three months before the valuation date to time t, where $0.25 \leq t \leq 2$. The risk-free interest rate is 0.06 and the volatility of the stock return is 0.2.

Table 7 presents the in-sample results for 50,000 simulations (25,000 plus 25,000 antithetic) and 100 time steps per year. The third column replicates part of Table 3 in Longstaff and Schwartz (2001). To price the option, we use eight basis functions: a constant, the first two Laguerre polynomials in the stock price, the first two Laguerre polynomials in the average stock price, and the cross products of these polynomials up to third degree.

[Insert Table 7 about here]

We obtain values slightly below those reported by Longstaff and Schwartz (2001). For example, when the initial average value of the stock, A, is 100 and the underlying stock price, S, is 120, our option price is 23.60899 versus 23.775. However, standard errors are almost identical to those of Longstaff and Schwartz (2001).

To study the numerical stability of these prices, in the fourth column we replace Laguerre polynomials ($L_n(x)$) with Hermite B polynomials ($H_{e_n}(x)$) of degree five and their cross products up to third degree.

We find that changing the type and the degree of the polynomials affects slightly option prices. Now we undervalue the option relative to the previous column,[6] but the price difference never exceeds 1%.

In Table 8, we analyze the sensitivity of the option price with respect to the degree of the Hermite B polynomial.[7] We choose an option ($A = 110, S = 120$) and we use up to ten terms in both the underlying stock price and its average. Now, the cross products are not considered.

[Insert Table 8 about here]

We see that the value of the option increases with the number of terms up to degree six. With a higher number of terms, the option value decreases and increases again. In any case, the price differences are always smaller than 0.2%.

Finally, we analyze the impact of cross products on the option price in Table 9. We value the previous option with Hermite B polynomials of degrees two and three.

[Insert Table 9 about here]

In Case I, we do not use cross products. Therefore, these values are taken from Table 8. In Case II, we use the same cross products as in Longstaff and Schwartz (2001). Apparently, adding cross products does not influence option prices. However, as we can see in Case III, using more cross products slightly affects the option value.[8]

4 Out-of-sample Performance of LSM

Following Longstaff and Schwartz (2001), we now perform a diagnostic test to assess the forecasting ability of the LSM algorithm. We simulate two sets of Monte Carlo paths and apply the LSM method to the first one (in-sample), determining the optimal stopping rule. This rule is then applied to the second set of paths (out-of-sample).

Out-of-sample prices for the standard American put option are included in Table 10. We see they are very close to those obtained in-sample (the differences are smaller than one cent). Standard errors are also very similar, ranging from 0.630 to 0.906 cents.

[Insert Table 10 about here]

Table 11 shows the results of applying the LSM algorithm to out-of-sample paths for a Bermuda call option on the maximum of five uncorrelated assets.

[Insert Table 11 about here]

We see that, again, in-sample and out-of-sample prices and standard errors are very similar. Notice, however, that when using 10 terms of the polynomial $T_n(x)$, the in-sample option value is significantly different from the out-sample one (23.86298 versus 24.77399, respectively). This is due to the numerical problems mentioned before.

In Table 12, we present out-of-sample prices for the American-Bermuda-Asian call option. We see that out-of-sample prices are very close to in-sample ones (the largest difference, in absolute value, is 0.4 %).

[Insert Table 12 about here]

5 Conclusions

Monte Carlo simulation is widely used for pricing European options. However, its application for valuing American derivatives is not straightforward.

Recently, Longstaff and Schwartz (2001) have developed the Least-Squares Monte Carlo (LSM) technique, that uses simple regressions to price American options. At each exercise date, they estimate the continuation value of the option regressing the expected cash-flows on basis functions of the underlying asset price. The forecasting performance of this technique can be assessed applying the stopping rule (obtained with a certain set of simulated paths) to a different set of paths.

This paper analyzes the robustness of the LSM approach relative to the type and number of basis functions. We apply this algorithm to price an American put option, a Bermuda call option on the maximum of five assets, and an American-Bermuda-Asian option. We have computed in- and out-of sample option prices as well standard errors for different types and numbers of basis functions.

The overall conclusion is that this technique is very robust when pricing the American put option. In this case, the choice of basis functions is clear, since it reduces to select the degree of a polynomial on the stock price. We find that, for a given degree, using different polynomials produces very similar results, which is not surprising since we can express any polynomial as a linear combination of others. Moreover, in- and out-of sample option prices are very similar and standard errors are low in both cases. For a reasonable polynomial degree (between

3 and 20), we obtain very similar option prices. Using more terms, however, can conduce to numerical problems in the least-squares regressions.

For complex options, the choice of basis functions is not clear, since we have to combine polynomials with other functions to represent the information set at each exercise date. For these options, we find that the robustness does not seem to be guaranteed and the type and number of basis functions can slightly affect option prices.

Footnotes

1. See Johnson (1983), Geske and Johnson (1984), Barone-Adesi and Whaley (1987), Bunch and Johnson (1992), Broadie and Detemple (1996), Ho, Stapleton, and Subrahmanyam (1997), and Ju and Zhong (1999), among others.

2. The first two methods were introduced by Schwartz (1977) and Brennan and Schwartz (1977, 1978) while the Crank-Nicolson method was first used in option pricing by Courtadon (1982).

3. This is a triple consisting of Ω, the set of all possible sample paths (ω), $\mathbb{F}$, the sigma-algebra of events at time T, and P, a probability measure defined on the elements of $\mathbb{F}$.

4. See Demidowitsch, Maron, and Schuwalowa (1980) for details on $T_n^*(x)$ and Abramowitz and Stegun (1972) for the remaining ones.

5. This criterion indicates that, to price accurately the option, we should increase the number of basis functions until the option price no longer increases.

6. For example, when $A = 100$ and $S = 120$, the option price is 23.48875 versus 23.60899.

7. We do not report the result for degree one because of numerical difficulties when pricing the option.

8. To compute these cross products, we use all the possible pairs of the basis functions employed in the second row. Thus, there are 11 and 22 functions for degrees two and three, respectively.

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Table 1: **Explicit expressions of the basis functions.**

$f_n(x)$	N	d_n	c_m	$g_m(x)$
$W_n(x)$	0	1	1	x^n
$P_n(x)$	$[n/2]$	2^{-n}	$(-1)^m \begin{pmatrix} n \\ m \end{pmatrix} \begin{pmatrix} 2n-2m \\ n \end{pmatrix}$	x^{n-2m}
$L_n(x)$	n	1	$\frac{(-1)^m}{m!} \begin{pmatrix} n \\ n-m \end{pmatrix}$	x^m
$H_n(x)$	$[n/2]$	$n!$	$(-1)^m \frac{1}{m!\,(n-2m)!}$	$(2x)^{n-2m}$
$H_{e_n}(x)$	$[n/2]$	$n!$	$(-1)^m \frac{1}{m!\,(n-2m)!}$	x^{n-2m}
$T_n(x)$	$[n/2]$	$n/2$	$(-1)^m \frac{(n-m-1)!}{m!\,(n-2m)!}$	$(2x)^{n-2m}$
$C_n(x)$	$[n/2]$	n	$(-1)^m \frac{(n-m-1)!}{m!\,(n-2m)!}$	x^{n-2m}
$T_n^*(x)$	$[n/2]$	$2^{-n}\, n$	$(-1)^m \frac{(n-m-1)!}{m!\,(n-2m)!}$	$(2x)^{n-2m}$
$U_n(x)$	$[n/2]$	1	$(-1)^m \frac{(n-m)!}{m!\,(n-2m)!}$	$(2x)^{n-2m}$
$S_n(x)$	$[n/2]$	1	$(-1)^m \frac{(n-m)!}{m!\,(n-2m)!}$	x^{n-2m}

The basis functions are particular cases of the following expression

$$f_n(x) = d_n \sum_{m=0}^{N} c_m g_m(x),$$

where $n \geq 0$ denotes the degree of the polynomial.

Table 2: **Expressions of the basis functions using Rodrigues' formula.**

$f_n(x)$	a_n	$\rho(x)$	$g(x)$
$W_n(x)$	$\frac{(2n)!}{n!}$	x^{2n}	1
$P_n(x)$	$(-1)^n\ 2^n\ n!$	1	$1-x^2$
$L_n(x)$	$n!$	e^{-x}	x
$H_n(x)$	$(-1)^n$	e^{-x^2}	1
$H_{e_n}(x)$	$(-1)^n$	$e^{-x^2/2}$	1
$T_n(x)$	$(-1)^n\ 2^n\ \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}}$	$(1-x^2)^{-1/2}$	$1-x^2$
$C_n(x)$	$(-1)^n\ 2^n\ \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}}$	$\left(1-\frac{x^2}{4}\right)^{-1/2}$	$1-\frac{x^2}{4}$
$T_n^*(x)$	$(-1)^n\ 2^{2n-1}\ \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}}$	$(1-x^2)^{-1/2}$	$1-x^2$
$U_n(x)$	$\frac{(-1)^n\ 2^{n+1}\ \Gamma\left(n+\frac{3}{2}\right)}{(n+1)\ \sqrt{\pi}}$	$(1-x^2)^{1/2}$	$1-x^2$
$S_n(x)$	$\frac{(-1)^n\ 2^{n+1}\ \Gamma\left(n+\frac{3}{2}\right)}{(n+1)\ \sqrt{\pi}}$	$\left(1-\frac{x^2}{4}\right)^{1/2}$	$1-\frac{x^2}{4}$

The basis functions are especial cases of Rodrigues' formula which is given by

$$f_n(x) = \frac{1}{a_n\ g(x)}\ \frac{\partial^n}{\partial x^n}\ [\rho(x)\ (g(x))^n],$$

where $n \geq 0$ denotes the degree of the polynomial.

Table 3: **Recurrence law for the basis functions.**

$f_n(x)$	a_{n+1}	a_n	b_n	a_{n-1}	$f_0(x)$	$f_1(x)$
$W_n(x)$	1	0	1	0	1	x
$P_n(x)$	$n+1$	0	$2n+1$	n	1	x
$L_n(x)$	$n+1$	$2n+1$	-1	n	1	$1-x$
$H_n(x)$	1	0	2	$2n$	1	$2x$
$H_{e_n}(x)$	1	0	1	n	1	x
$T_n(x)$	1	0	2	1	1	x
$C_n(x)$	1	0	1	1	2	x
$T_n^*(x)$	1	0	1	$1/4$	1	x
$U_n(x)$	1	0	2	1	1	$2x$
$S_n(x)$	1	0	1	1	1	$2x$

The general expression for the recurrence law is given by

$$a_{n+1}\; f_{n+1}(x) = (a_n + b_n\; x)\; f_n(x) - a_{n-1}\; f_{n-1}(x),$$

where $n \geq 0$ denotes the degree of the polynomial.

Table 4: **American put option prices.**

Number of terms	$W_n(x)(s.e.)$	$P_n(x)$ (s.e.)	$L_n(x)$ (s.e.)	$H_n(x)$ (s.e.)	$H_{e_n}(x)$ (s.e.)
1	2.09671 (.00631)	2.09671 (.00631)	2.11487 (.00647)	2.09671 (.00631)	2.09671 (.00631)
2	2.26853 (.00911)	2.26853 (.00911)	2.29251 (.00888)	2.26864 (.00911)	2.26853 (.00911)
3	2.29770 (.00866)	2.29880 (.00866)	2.29801 (.00866)	2.29892 (.00866)	2.29848 (.00864)
4	2.30739 (.00859)	2.30620 (.00852)	2.30761 (.00859)	2.30644 (.00850)	2.30680 (.00849)
5	2.30777 (.00853)	2.30677 (.00851)	2.30898 (.00855)	2.30780 (.00854)	2.30816 (.00852)
6	2.30803 (.00854)	2.30690 (.00848)	2.30926 (.00857)	2.30812 (.00853)	2.30650 (.00851)
7	2.30817 (.00854)	2.30803 (.00850)	2.30818 (.00855)	2.30755 (.00858)	2.30694 (.00852)
8	2.30544 (.00852)	2.30791 (.00847)	2.30989 (.00854)	2.30814 (.00859)	2.30805 (.00854)
9	2.30585 (.00850)	2.30899 (.00852)	2.30802 (.00858)	2.30806 (.00858)	2.30770 (.00858)
10	2.30779 (.00852)	2.30760 (.00849)	2.30877 (.00859)	2.30750 (.00856)	2.30708 (.00859)
15	2.30823 (.00850)	2.30915 (.00843)	2.30899 (.00858)	2.30561 (.00853)	2.30719 (.00857)
20	2.30965 (.00853)	2.30898 (.00842)	2.30856 (.00855)	2.30381 (.00849)	2.30429 (.00859)

The parameters of the option are: $\sigma = 0.2$, $r = 0.06$, $T = 1$ year, $S_0 = E = 40$. The value of the American option using the binomial method with 1,000 steps is 2.31928. The values of the corresponding European option, using 100,000 (50,000 plus 50,000 antithetic) simulations, binomial trees, and the Black and Scholes (1973) formula, are 2.06193, 2.06560, and 2.06640, respectively. We approximate the American option considering 70 exercise dates. For this Bermuda option, we also use a binomial tree, obtaining a value of 2.31153. The first row shows the ten polynomials used. $n \geq 0$ denotes the degree of the polynomial. We report the standard errors (s.e.) of the simulated values in parentheses.

Table 4. (cont.) **American put option prices.**

Number of terms	$T_n(x)$ (s.e.)	$C_n(x)$ (s.e.)	$T^*_n(x)$ (s.e.)	$U_n(x)$ (s.e.)	$S_n(x)$ (s.e.)
1	2.09671 (.00631)	2.09671 (.00631)	2.09671 (.00631)	2.09671 (.00631)	2.09671 (.00631)
2	2.26853 (.00911)	2.26828 (.00910)	2.26853 (.00911)	2.26864 (.00911)	2.26853 (.00911)
3	2.29903 (.00865)	2.29628 (.00860)	2.29912 (.00866)	2.29822 (.00865)	2.29848 (.00864)
4	2.30624 (.00850)	2.30546 (.00850)	2.30652 (.00850)	2.30703 (.00853)	2.30651 (.00849)
5	2.30689 (.00851)	2.30780 (.00853)	2.30744 (.00851)	2.30744 (.00852)	2.30809 (.00852)
6	2.30634 (.00848)	2.30646 (.00850)	2.30647 (.00848)	2.30760 (.00849)	2.30670 (.00850)
7	2.30720 (.00847)	2.30758 (.00851)	2.30734 (.00850)	2.30761 (.00851)	2.30705 (.00849)
8	2.30854 (.00849)	2.30765 (.00848)	2.30735 (.00848)	2.30776 (.00848)	2.30622 (.00848)
9	2.30795 (.00850)	2.30728 (.00849)	2.30784 (.00847)	2.30838 (.00852)	2.30696 (.00849)
10	2.30800 (.00846)	2.30709 (.00847)	2.30785 (.00850)	2.30828 (.00850)	2.30805 (.00847)
15	2.30905 (.00844)	2.30810 (.00848)	2.30761 (.00849)	2.30842 (.00845)	2.30893 (.00848)
20	2.30938 (.00841)	2.30755 (.00843)	2.30845 (.00845)	2.30900 (.00843)	2.30638 (.00840)

The parameters of the option are: $\sigma = 0.2$, $r = 0.06$, $T = 1$ year, $S_0 = E = 40$. The value of the American option using the binomial method with 1,000 steps is 2.31928. The values of the corresponding European option, using 100,000 (50,000 plus 50,000 antithetic) simulations, binomial trees, and the Black and Scholes (1973) formula, are 2.06193, 2.06560, and 2.06640, respectively. We approximate the American option considering 70 exercise dates. For this Bermuda option, we also use a binomial tree, obtaining a value of 2.31153. The first row shows the ten polynomials used. $n \geq 0$ denotes the degree of the polynomial. We report the standard errors (s.e.) of the simulated values in parentheses.

Table 5: **Bermuda call option on the maximum of five assets.**

Number of terms	$H_{e_n}(x)$ (s.e.)	$T_n(x)$ (s.e.)
0	25.73070 (.06383)	25.73070 (.06383)
1	25.87009 (.06582)	25.87009 (.06582)
2	26.10928 (.06249)	26.11103 (.06252)
3	26.17398 (.06187)	26.17694 (.06196)
4	26.19432 (.06171)	26.19424 (.06187)
5	26.19673 (.06180)	26.18403 (.06214)
6	26.19645 (.06214)	26.18781 (.06225)
7	26.19626 (.06213)	26.11228 (.06261)
8	26.19068 (.06227)	25.99370 (.06246)
9	26.17984 (.06225)	25.87901 (.06106)
10	26.17448 (.06256)	23.89298 (.05349)

The characteristics of the option are: $\sigma = 0.2$ (for the five assets), $r = 0.05$, the dividend yield is 0.1, $T = 3$ years, $E = 100$, there are three exercise times per year, and the initial assets prices are 100 for the five assets. We use 100,000 (50,000 plus 50,000 antithetic) simulations and the following basis functions: a constant, the second to the fifth maximums and their squares, the four products of consecutive pairs of maximums, the product of the five assets, and zero to ten terms of the polynomials indicated in the first row. $n \geq 0$ denotes the degree of the polynomial. We report the standard errors (s.e.) of the simulated values in parentheses.

Table 6: **Effect of the choice of basis functions on the Bermuda call option on the maximum of five assets.**

Number of terms	Case I (s.e.)	Case II (s.e.)	Case III (s.e.)	Case IV (s.e.)
0	25.75969 (.06354)	25.66929 (.05474)	25.36842 (.05165)	25.76960 (.06398)
1	25.86988 (.06582)	25.76242 (.06570)	25.84218 (.06587)	25.88012 (.06595)
2	26.07051 (.06239)	26.03418 (.06327)	26.05393 (.06342)	26.10515 (.06250)
3	26.15420 (.06193)	26.06087 (.06323)	26.06920 (.06314)	26.18469 (.06198)
4	26.16086 (.06185)	26.08385 (.06341)	26.09032 (.06313)	26.19798 (.06179)
5	26.16341 (.06195)	26.10152 (.06329)	26.08691 (.06298)	26.20064 (.06187)
6	26.14974 (.06231)	26.08556 (.06392)	26.07915 (.06349)	26.18932 (.06210)
7	26.15517 (.06231)	26.09288 (.06399)	26.07881 (.06350)	26.19892 (.06212)
8	26.14555 (.06250)	26.09031 (.06420)	26.07253 (.06371)	26.18422 (.06223)
9	26.14921 (.06255)	26.08191 (.06417)	26.07572 (.06372)	26.17811 (.06229)
10	26.11754 (.06303)	26.08579 (.06416)	26.07383 (.06369)	26.18058 (.06239)

The characteristics of the option are: $\sigma = 0.2$ (for the five assets), $r = 0.05$, the dividend yield is 0.1, $T = 3$ years, $E = 100$, there are three exercise times per year, and the initial assets prices are 100 for the five assets. We use 100,000 (50,000 plus 50,000 antithetic) simulations. In Case I we use the following basis functions: a constant, the second to the fifth maximums, the four products of consecutive pairs of maximums, and the product of the five assets. Case II is Case I without the products of consecutive maximums. Case III considers the second to the fifth maximums and their squares. Finally, Case IV uses a constant, the second to the fifth maximums, their squares, their third powers, the products of consecutive pairs of maximums, and the product of the five assets. In all the cases, we also use the polynomials $H_{e_n}(x)$ with up to ten terms. We report the standard errors (s.e.) of the simulated values in parentheses.

Table 7: **American-Bermuda-Asian option prices with Laguerre and Hermite B polynomials.**

A	S	$L_n(x)$ (s.e.)	$H_{e_n}(x)$ (s.e.)
90	100	7.52344 (.04514)	7.44979 (.04491)
90	110	14.18221 (.05883)	14.13072 (.05835)
90	120	22.15082 (.06781)	22.04639 (.06723)
100	100	8.29836 (.04625)	8.23392 (.04598)
100	110	15.37693 (.05835)	15.31818 (.05771)
100	120	23.60899 (.06591)	23.48875 (.06542)
110	100	9.45722 (.04525)	9.41530 (.04484)
110	110	17.12380 (.05501)	17.03744 (.05428)
110	120	25.29639 (.06281)	25.20921 (.06223)

The characteristics of the option are: $\sigma = 0.2$, $r = 0.06$, $T = 2$, $E = 100$. The initial value of the average and the underlying stock price are denoted by A and S, respectively. The average stock price is the continuous arithmetic mean from three months before the valuation date to time t, where $0.25 \leq t \leq 2$. The option cannot be exercised during the first quarter. We use 50,000 (25,000 plus 25,000 antithetic) simulations and approximate the American option considering 100 exercise dates per year. We use a constant, the first two Laguerre ($L_n(x)$) or Hermite B ($H_{e_n}(x)$) polynomials in the stock price, the first two polynomials in the average stock price, and the cross products of these polynomials up to third degree. We report the standard errors (s.e.) of the simulated values in parentheses.

Table 8: **Sensitivity of the American-Bermuda-Asian option prices with respect to the degree of Hermite B polynomials.**

Degree	Option price (s.e.)
2	25.16948 (.06163)
3	25.17115 (.06246)
4	25.19076 (.06255)
5	25.21053 (.06225)
6	25.22129 (.06222)
7	25.15755 (.06210)
8	25.19111 (.06229)
9	25.21444 (.06217)
10	25.21348 (.06213)

The characteristics of the option are: $\sigma = 0.2$, $r = 0.06$, $T = 2$, $E = 100$. The initial value of the average is $A = 110$ and the underlying stock price is $S = 120$. The average stock price is the continuous arithmetic mean from three months before the valuation date to time t, where $0.25 \leq t \leq 2$. The option can only be exercised after the first quarter. We use 50,000 (25,000 plus 25,000 antithetic) simulations and approximate the American option considering 100 exercise dates per year. We use Hermite B polynomials in the stock price and its average as basis functions. We report the standard errors (s.e.) of the simulated values in parentheses.

Table 9: **Sensitivity of the American-Bermuda-Asian option prices with respect to cross products of Hermite B polynomials.**

	Degree 2 (s.e.)	Degree 3 (s.e.)
Case I	25.16948 (.06163)	25.17115 (.06246)
Case II	25.16996 (.06163)	25.17099 (.06246)
Case III	25.14129 (.06233)	25.26195 (.06253)

The characteristics of the option are: $\sigma = 0.2$, $r = 0.06$, $T = 2$, $E = 100$. The initial value of the average is $A = 110$ and the underlying stock price is $S = 120$. The average stock price is the continuous arithmetic mean from three months before the valuation date to time t, where $0.25 \leq t \leq 2$. The option can only be exercised after the first quarter. We use 50,000 (25,000 plus 25,000 antithetic) simulations and approximate the American option considering 100 exercise dates per year. We use Hermite B polynomials and their cross products in the stock price and its average as basis functions. In Case I, we do not use cross products. Case II employs the same cross products as in Longstaff and Schwartz (2001). In Case III, we use all the possible pairs of the basis functions employed in Case I. We report the standard errors (s.e.) of the simulated values in parentheses.

Table 10: **Out-of-sample American put option prices.**

Number of terms	$W_n(x)(s.e.)$	$P_n(x)$ (s.e.)	$L_n(x)$ (s.e.)	$H_n(x)$ (s.e.)	$H_{e_n}(x)$ (s.e.)
1	2.10361 (.00632)	2.10104 (.00630)	2.11904 (.00647)	2.10104 (.00630)	2.10104 (.00630)
2	2.26870 (.00906)	2.26870 (.00906)	2.29046 (.00885)	2.26888 (.00906)	2.26870 (.00906)
3	2.29645 (.00861)	2.29676 (.00860)	2.29722 (.00866)	2.29693 (.00859)	2.29624 (.00856)
4	2.31097 (.00872)	2.30881 (.00866)	2.30808 (.00870)	2.30850 (.00863)	2.30809 (.00862)
5	2.31217 (.00869)	2.31345 (.00872)	2.31224 (.00864)	2.31062 (.00870)	2.31201 (.00873)
6	2.31290 (.00871)	2.31437 (.00872)	2.31251 (.00869)	2.31264 (.00870)	2.31240 (.00872)
7	2.31323 (.00874)	2.31362 (.00871)	2.31269 (.00868)	2.30973 (.00869)	2.31162 (.00870)
8	2.31469 (.00873)	2.31279 (.00866)	2.31287 (.00866)	2.30920 (.00871)	2.31261 (.00869)
9	2.31357 (.00871)	2.31301 (.00869)	2.31265 (.00868)	2.30938 (.00869)	2.30996 (.00869)
10	2.31479 (.00871)	2.31478 (.00870)	2.31224 (.00868)	2.30854 (.00866)	2.30956 (.00871)
15	2.31286 (.00868)	2.31568 (.00864)	2.31241 (.00867)	2.30721 (.00864)	2.30974 (.00866)
20	2.31426 (.00871)	2.31792 (.00864)	2.31250 (.00867)	2.30238 (.00856)	2.31083 (.00872)

Option prices are computed applying the stopping rule for one set of paths to a different one. The parameters of the option are: $\sigma = 0.2$, $r = 0.06$, $T = 1$ year, $S_0 = E = 40$. The value of the American option using the binomial method with 1,000 steps is 2.31928. The values of the corresponding European option, using 100,000 (50,000 plus 50,000 antithetic) simulations, binomial trees, and the Black and Scholes (1973) formula, are 2.06193, 2.06560, and 2.06640, respectively. We approximate the American option considering 70 exercise dates. For this Bermuda option, we also use a binomial tree, obtaining a value of 2.31153. The first row shows the ten polynomials used. $n \geq 0$ denotes the degree of the polynomial. We report the standard errors (s.e.) of the simulated values in parentheses.

Table 10 (cont.) **Out-of-sample American put option prices.**

Number of terms	$T_n(x)$ (s.e.)	$C_n(x)$ (s.e.)	$T_n^*(x)$ (s.e.)	$U_n(x)$ (s.e.)	$S_n(x)$ (s.e.)
1	2.10104 (.00630)	2.10104 (.00630)	2.10104 (.00630)	2.10104 (.00630)	2.10104 (.00630)
2	2.26870 (.00906)	2.26880 (.00906)	2.26870 (.00906)	2.26888 (.00906)	2.26870 (.00906)
3	2.29704 (.00859)	2.29745 (.00853)	2.29683 (.00859)	2.29697 (.00860)	2.29624 (.00856)
4	2.30903 (.00864)	2.30582 (.00861)	2.30879 (.00863)	2.30794 (.00866)	2.30781 (.00862)
5	2.31305 (.00872)	2.31217 (.00873)	2.31380 (.00872)	2.31286 (.00871)	2.31200 (.00872)
6	2.31325 (.00871)	2.31394 (.00872)	2.31379 (.00871)	2.31397 (.00871)	2.31489 (.00873)
7	2.31399 (.00871)	2.31463 (.00873)	2.31387 (.00872)	2.31328 (.00870)	2.31380 (.00871)
8	2.31281 (.00868)	2.31404 (.00870)	2.31487 (.00872)	2.31340 (.00867)	2.31475 (.00871)
9	2.31332 (.00869)	2.31358 (.00871)	2.31359 (.00866)	2.31327 (.00868)	2.31431 (.00871)
10	2.31507 (.00871)	2.31373 (.00870)	2.31345 (.00868)	2.31460 (.00868)	2.31313 (.00868)
15	2.31619 (.00864)	2.31427 (.00868)	2.31484 (.00868)	2.31614 (.00862)	2.31442 (.00869)
20	2.31831 (.00864)	2.31443 (.00868)	2.31441 (.00868)	2.31772 (.00861)	2.31368 (.00865)

Option prices are computed applying the stopping rule for one set of paths to a different one. The parameters of the option are: $\sigma = 0.2$, $r = 0.06$, $T = 1$ year, $S_0 = E = 40$. The value of the American option using the binomial method with 1,000 steps is 2.31928. The values of the corresponding European option, using 100,000 (50,000 plus 50,000 antithetic) simulations, binomial trees, and the Black and Scholes (1973) formula, are 2.06193, 2.06560, and 2.06640, respectively. We approximate the American option considering 70 exercise dates. For this Bermuda option, we also use a binomial tree, obtaining a value of 2.31153. The first row shows the ten polynomials used. $n \geq 0$ denotes the degree of the polynomial. We report the standard errors (s.e.) of the simulated values in parentheses.

Table 11: **Bermuda call option on the maximum of five assets. Out-of-sample values.**

Number of terms	$H_{e_n}(x)$ (s.e.)	$T_n(x)$ (s.e.)
0	25.71728 (.06327)	25.71728 (.06327)
1	25.83671 (.06531)	25.83671 (.06531)
2	26.07751 (.06190)	26.07692 (.06192)
3	26.14010 (.06139)	26.13720 (.06149)
4	26.14024 (.06130)	26.14446 (.06149)
5	26.15865 (.06150)	26.14179 (.06171)
6	26.14962 (.06174)	26.13779 (.06179)
7	26.14758 (.06174)	26.09857 (.06207)
8	26.14826 (.06185)	26.02797 (.06324)
9	26.14113 (.06182)	25.85449 (.06102)
10	26.13227 (.06197)	24.77399 (.05692)

Option prices are computed applying the stopping rule for one set of paths to a different one. The characteristics of the option are: $\sigma = 0.2$ (for the five assets), $r = 0.05$, the dividend yield is 0.1, $T = 3$ years, $E = 100$, there are three exercise times per year, and the initial assets prices are 100 for the five assets. We use 100,000 (50,000 plus 50,000 antithetic) simulations and the following basis functions: a constant, the second to the fifth maximums and their squares, the four products of consecutive pairs of maximums, the product of the five assets, and zero to ten terms of the polynomials indicated in the first row. $n \geq 0$ denotes the degree of the polynomial. We report the standard errors (s.e.) of the simulated values in parentheses.

Table 12: **American-Bermuda-Asian option prices with Laguerre and Hermite B polynomials. Out-of-sample values.**

A	S	$L_n(x)$ (s.e.)	$H_{e_n}(x)$ (s.e.)
90	100	7.55461 (.04532)	7.47624 (.04503)
90	110	14.19743 (.05901)	14.15329 (.05857)
90	120	22.16454 (.06805)	22.04900 (.06751)
100	100	8.32834 (.04645)	8.25856 (.04612)
100	110	15.38253 (.05856)	15.32942 (.05793)
100	120	23.59644 (.06620)	23.47711 (.06570)
110	100	9.47838 (.04547)	9.43118 (.04500)
110	110	17.11952 (.05518)	17.03020 (.05442)
110	120	25.28705 (.06304)	25.19874 (.06248)

Option prices are computed applying the stopping rule for one set of paths to a different one. The characteristics of the option are: $\sigma = 0.2$, $r = 0.06$, $T = 2$, $E = 100$. The initial value of the average and the underlying stock price are denoted by A and S, respectively. The average stock price is the continuous arithmetic mean from three months before the valuation date to time t, where $0.25 \leq t \leq 2$. The option cannot be exercised during the first quarter. We use 50,000 (25,000 plus 25,000 antithetic) simulations and approximate the American option considering 100 exercise dates per year. We use a constant, the first two Laguerre ($L_n(x)$) or Hermite B ($H_{e_n}(x)$) polynomials in the stock price, the first two polynomials in the average stock price, and the cross products of these polynomials up to third degree. We report the standard errors (s.e.) of the simulated values in parentheses.

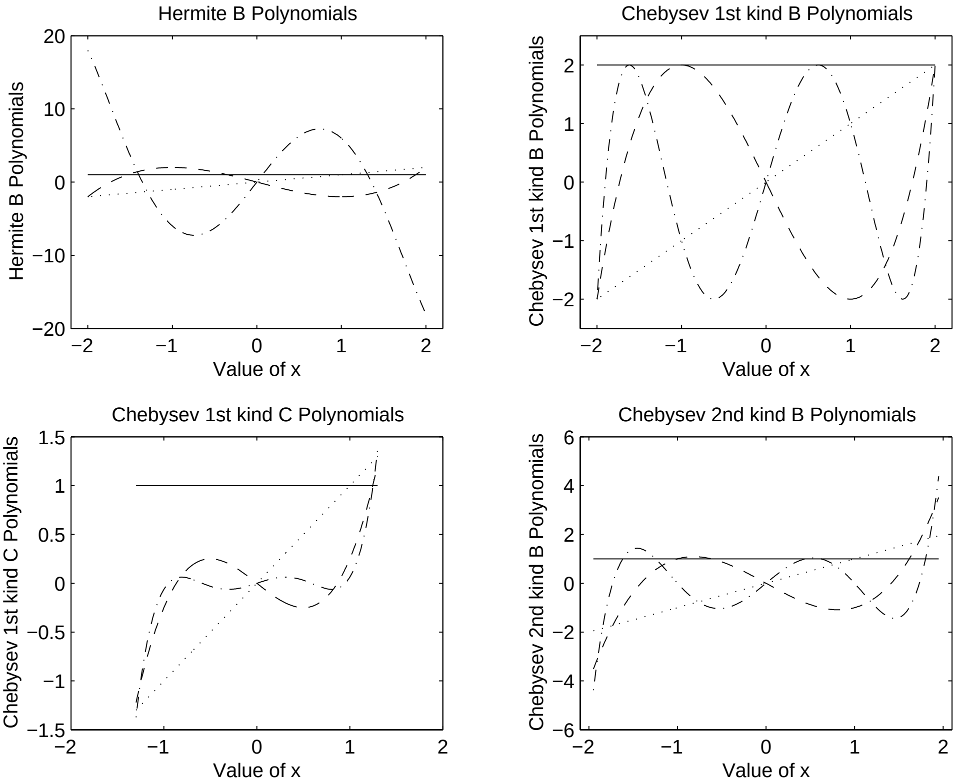


Figure 1: **Plots of polynomials**. This graph includes, from left to right, top to bottom, the polynomials $H_{e_n}(x)$, $C_n(x)$, $T_n^*(x)$ and $S_n(x)$ for degrees 0 (solid line), 1 (dotted line), 3 (dashed line), and 5 (dashed-dotted line).

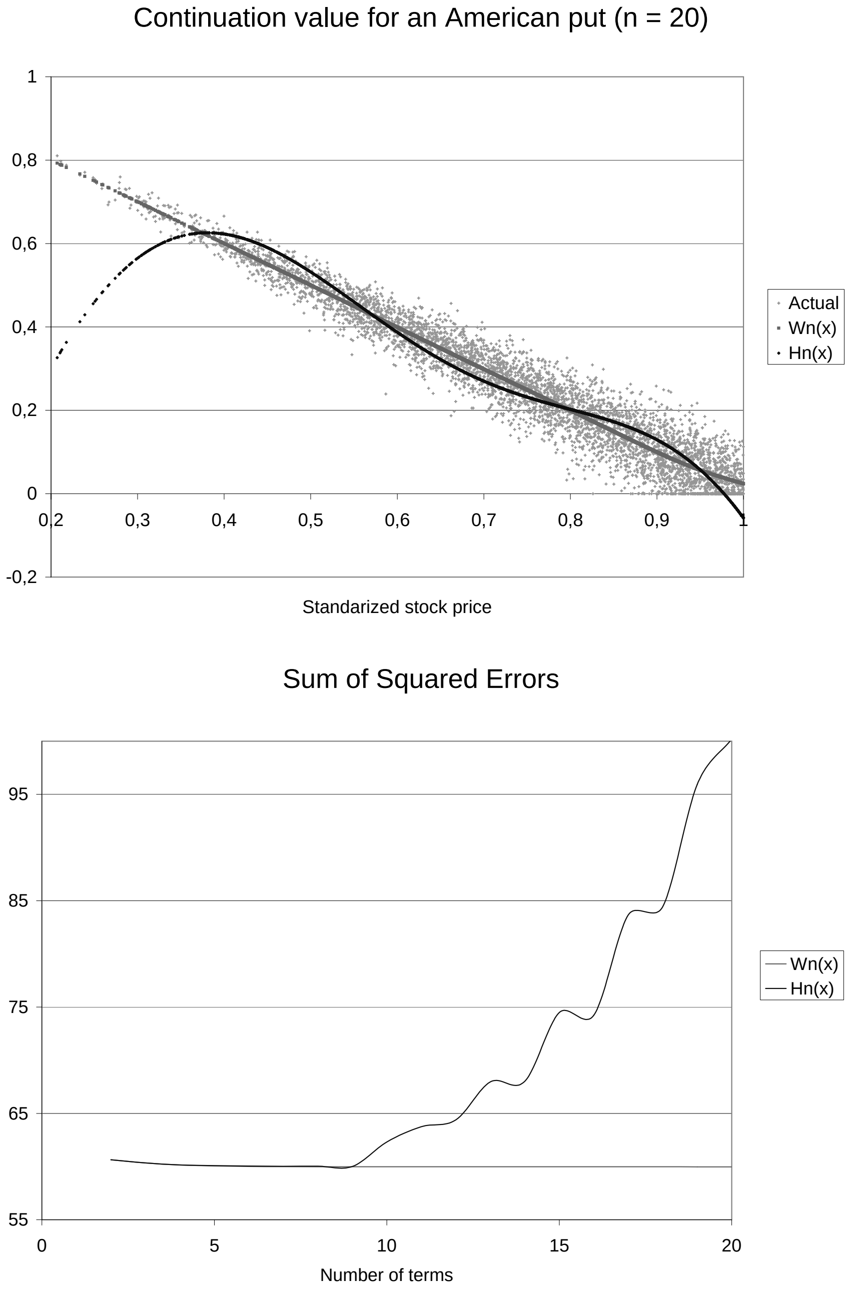
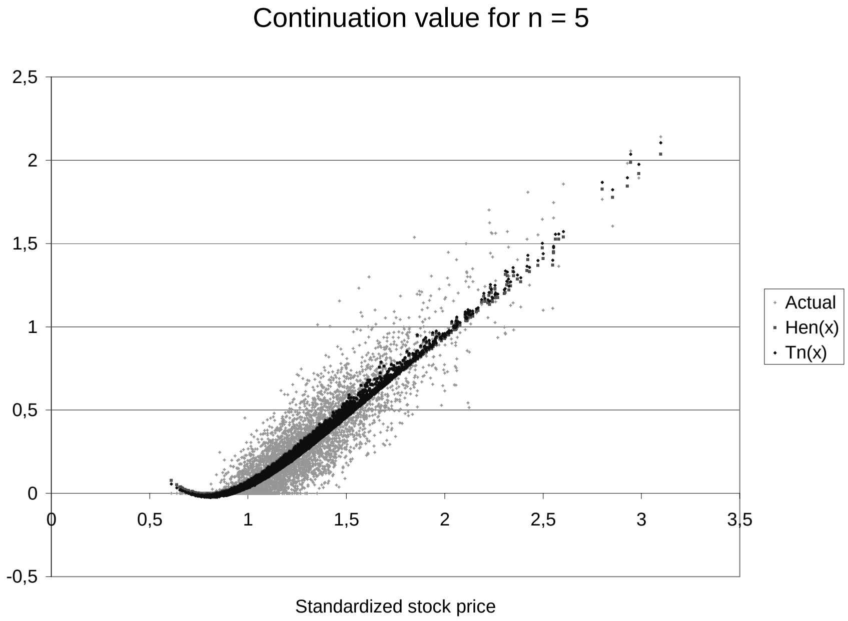


Figure 2: **Least-squares regressions for an American put option one exercise date before maturity.** The parameters of the option are: $\sigma = 0.5$, $r = 0.06$, $T = 1$ year, $S_0 = 40$, $E = 30$. We approximate the American option considering 70 exercise dates. We use 100,000 (50,000 plus 50,000 antithetic) simulations and 20 terms of the polynomials $W_n(x)$ and $H_n(x)$ as basis functions. $n \geq 0$ denotes the degree of the polynomial. We depict 5,000 actual (simulated) points.



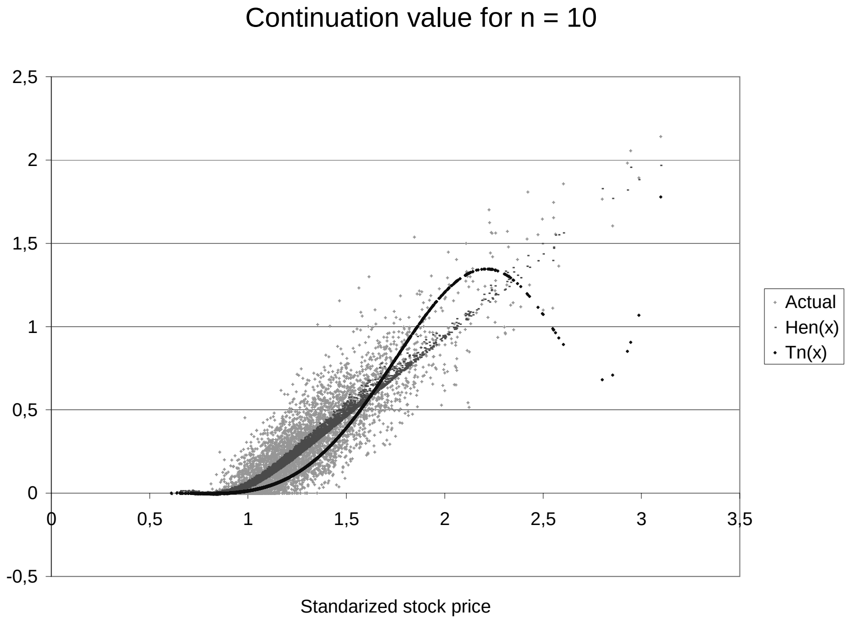


Figure 3: **Continuation value of a Bermuda option on the maximum of five assets one date before maturity.** The parameters are: $\sigma = 0.2$ (for the five assets), $r = 0.05$, dividend yield $= 0.1$, $T = 3$ years, $E = 100$, there are three exercise times per year, and the initial asset price is 100 for the five assets. We use 100,000 (50,000 plus 50,000 antithetic) simulations and the following basis functions: a constant, the second to the fifth maximums and their squares, the four products of consecutive pairs of maximums, the product of the five assets, and zero to ten terms of the polynomials $H_{e_n}(x)$ and $T_n(x)$. $n \geq 0$ denotes the degree of the polynomial. We depict 5,000 actual (simulated) points.