

Valuing the Option to Purchase an Asset at a Proportional Discount: A Correction

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This Draft: October 24th, 2007

Forthcoming: *The Quarterly Review of Economics and Finance*

JEL Classification: G13, J33

Keywords: option pricing, proportional-strike options, employee
compensation package

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Abstract

Gu (2002) introduces proportional-strike options to study a residential real estate program in China. Under this program, a state employee can buy her house at a fraction of the market price. The employee can also qualify for a subsidized mortgage. Given that the homeowner has the option, but not the obligation, of taking the subsidy, we show that the solution of the housing problem derived by Gu can be wrong. We provide a numerical example to illustrate the point.

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1 Introduction

Proportional-strike options are options with an exercise price that is a fraction of the market price of the underlying asset. The option is always in-the-money at maturity and its valuation is rather straightforward. Gu (2002) analyzes the case where the option is American, the underlying asset pays a continuous dividend and the exercise price is a deterministic function of time. He assumes that the strike declines exponentially over time at a constant rate and he derives analytical expressions for the optimal exercise date and option price.

As an application, Gu presents a housing savings program in China that allows state employees to buy a house at a proportion of the market price. This proportion declines with the seniority of the employee. Additionally, at the time of the purchase of the house, the employee can also benefit from a subsidized mortgage. Thus, the value of this program to the employee is the value of a proportional-strike option plus the value of the state subsidy. To find the optimal time to buy the house Gu maximizes the total value. However, for reasonable parameter values, the optimal exercise time may imply a negative subsidy, which is not economically feasible since there is no obligation to obtain the mortgage. In this note we explain when this happens and we show how to avoid this problem.

2 The Residential Housing Program in China

Let P_t be the market price of the house that the employee is entitled to. Under the savings program the employee can buy the house at a decreasing proportion $X_t = Ke^{-gt}$ of the market price, where K and g are a positive constants. The house provides a continuous dividend yield (in the form of rental rate) of q .

The value of this option at the time of the purchase of the house (T) is $(1 - X_T)P_T$, and its value today is given by $(1 - X_T)P_0e^{-qT}$, where P_0 is the current market price of the house.

At time T , the employee is also offered a subsidized mortgage (with maturity τ) at an interest rate (r_s) below the market rate (r_m). Gu shows that, at time T , this subsidy has a value of $\gamma(X_T P_T - S_T)$, where¹

$$\begin{aligned}\gamma &= 1 - \frac{r_s}{r_m} \left(\frac{1 - \left(1 + \frac{r_m}{12}\right)^{-12\tau}}{1 - \left(1 + \frac{r_s}{12}\right)^{-12\tau}} \right) \\ S_T &= C_0 e^{rT} \left(\frac{1 - e^{(w-r)(T+1)}}{1 - e^{w-r}} \right).\end{aligned}$$

Here, S_T denotes value of the savings of the employee to buy the house (see expression (8) in Gu's paper). The savings started at time 0 with a total contribution² of C_0 and grows at rate w . They have been capitalized at the risk-free interest rate r .

Hence, the value today of the housing program (V) is the value to buy the house at a discount plus the value of the subsidized mortgage (as explained in Gu (2002, equation (4'))):

$$V = \left(1 - Ke^{-gT}\right) P_0 e^{-qT} + \gamma \left[Ke^{-(g+q)T} P_0 - C_0 \left(\frac{1 - e^{(w-r)(T+1)}}{1 - e^{w-r}} \right) \right].$$

The employee's problem reduces to the calculation of the optimal time (T^*) at which she should buy the house. To find T^* , Gu differentiates V and solves (numerically) the equation (see equation (6') in his paper)

$$\left(q - (q + g)(1 - \gamma)Ke^{-gT} \right) e^{-qT} P_0 = \frac{\gamma C_0 (w - r) e^{(w-r)(T+1)}}{1 - e^{w-r}}. \quad (1)$$

¹Gu does not distinguish between the market mortgage rate (r_m) and the risk-free rate (r), which creates confusion.

²Including the employer's matching contribution.

This solution is not correct since it can imply negative subsidies. This happens because as T increases the present value of the price paid for the house decreases while the present value of the savings increases. As a consequence, the subsidy decreases with T . In some cases, the optimal time obtained from equation (1) is large and the subsidy becomes negative and the employee would be better off without the subsidy than with the subsidy.

Thus, the correct way to solve the problem is to find the optimal time that maximizes

$$V = (1 - Ke^{-gT}) P_0 e^{-qT} + \max \left\{ 0; \gamma \left[Ke^{-(g+q)T} P_0 - C_0 \left(\frac{1 - e^{(w-r)(T+1)}}{1 - e^{w-r}} \right) \right] \right\}. \quad (2)$$

Note that this function is not differentiable, so that it has to be maximized numerically. We next present two examples to study the extent of the error.

3 Numerical Analysis

We first analyze the example proposed by Gu (2002). He considers the option to buy a house with a current market price of 170,000 yuan.³ The house can be purchased today at 67 percent of its value. This proportion decreases exponentially at a rate of 14 percent. The house pays a dividend yield of 6 percent. The risk-free rate is also 6 percent. The employee starts out her saving program with a total contribution today of 900. This amount increases at a rate of 7.5 percent per year. The employee is offered a subsidized 25-year loan at 5.742 percent interest rate, well below the market rate (6.940 percent).

With these data $\gamma = .1058$ and $T^* = 4.84$ years.⁴ The value of the option for the

³Gu indicates that this corresponds to a standard three-bedroom, 100-square-meter apartment.

⁴Notice that there are a number of errors in Gu's paper when analyzing this example. Gu obtains a value for γ of 0.182, which is incorrect. For this γ the optimal time is 4.15 years but the optimal value for the option is 91,031.40 yuan, and not 98,887 yuan as Gu claims. He compares this optimal

employee is 87,889.63 yuan, which includes the value of the subsidy (3,994.88 yuan). Obviously, this is greater than the value of the option if she purchased the house today 68,052.94 yuan (including a 11,952.94 yuan subsidy). As shown in Figure 1 the subsidy becomes negative for $T = 11.25$ years, so that in this case the optimal exercise time obtained from equation (1) is correct.

In Figure 2, we study the purchase of a smaller house with a market value of 120,000 yuan. The discount to buy the house is today 87 percent, and declines at a rate of 20 percent (that is, in five years the discount will be 32 percent). The house can provide a rental rate (net of costs) of 3 percent. The initial contribution to the housing savings account is 2,000 yuan and increases at 15 percent rate. The subsidized mortgage rate and the risk-free interest rate are the same as in the previous case.

[Insert Figure 1 about here]

If we maximize equation (1) as Gu suggests, the optimal exercise time is $T^* = 8.17$ years, and the value of the housing program for the employee is $V^* = 76,775.81$ yuan. However, as shown in Figure 2 this solution is not correct. It implies a negative subsidy of 1,195.11 yuan. Instead, if we maximize equation (2) we obtain an optimal exercise time $T^{**} = 9.49$, leading to an optimal value of the option $V^{**} = 78,499.97$ yuan, greater than before.

[Insert Figure 2 about here]

References

Gu, Y. A., 2002, Valuing the option to purchase an asset at a proportional discount, *The Journal of Financial Research* 25, 1, 99–109.

value with the value of the option if the employee exercised it today, but Gu includes the subsidy in the former case while not in the latter one.

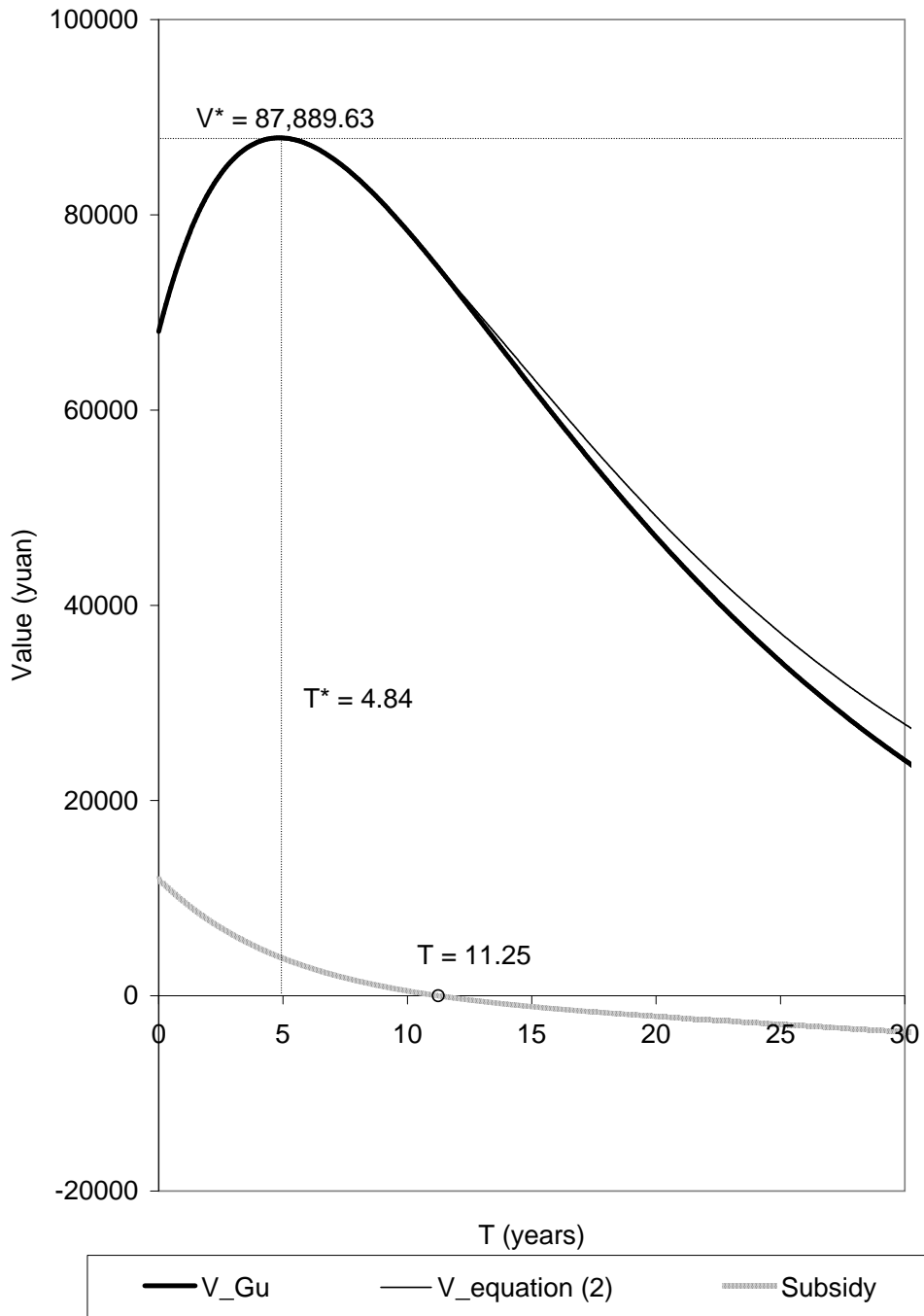


Figure 1: Value of the option to buy the house at a proportional discount and value of the subsidized mortgage for the example in Gu (2002). The parameter values are: $P_0 = 170,000$ yuan, $K = 0.67$, $C_0 = 900$ yuan, $r = 0.06$, $r_s = 0.05742$, $r_m = 0.0694$, $q = 0.06$, $g = 0.14$, $w = 0.075$, $\tau = 25$.

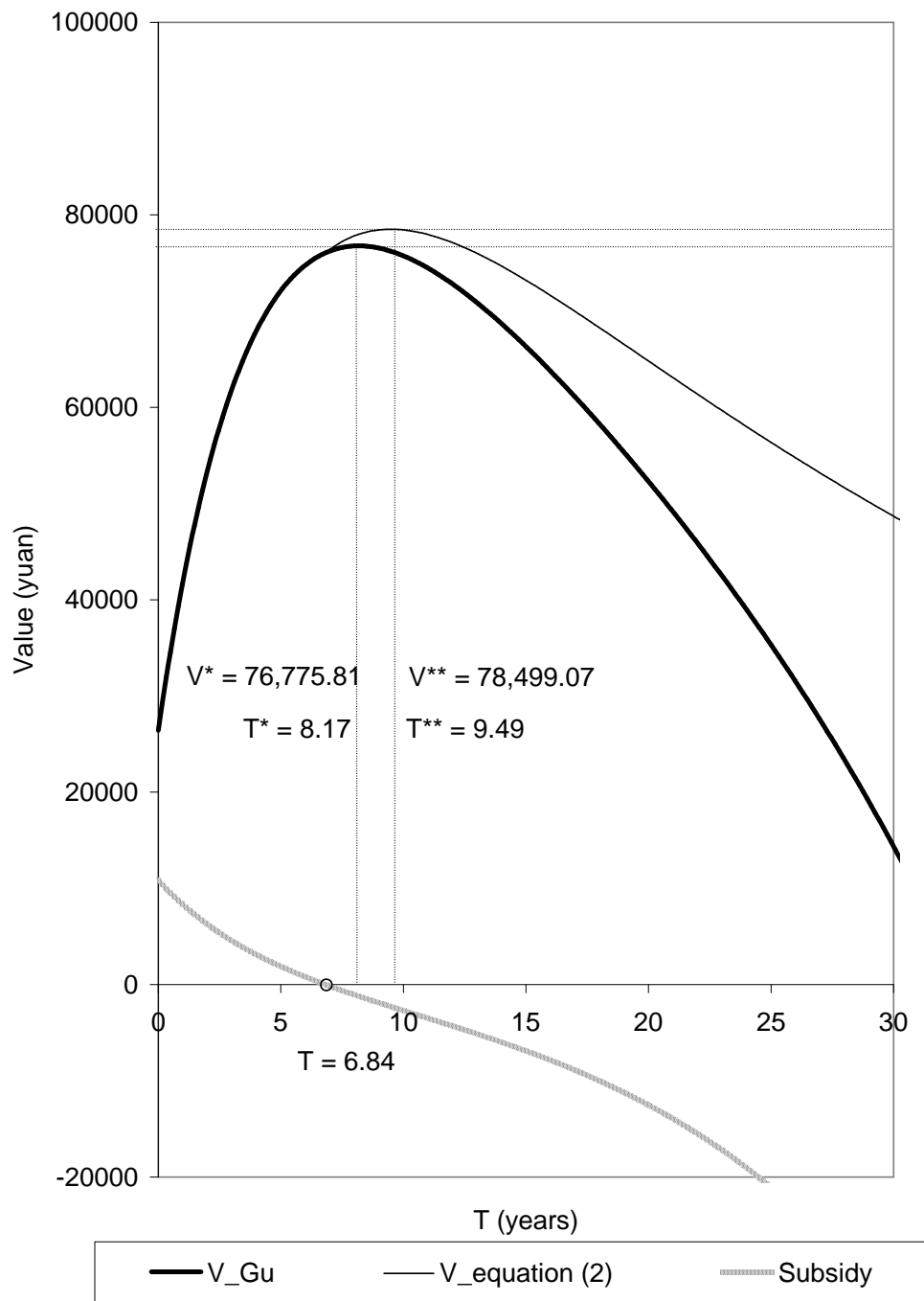


Figure 2: Value of the option to buy the house at a proportional discount and value of the subsidized for the following parameter values: $P_0 = 120,000$ yuan, $K = 0.87$, $C_0 = 2,000$ yuan, $r = 0.06$, $r_s = 0.05742$, $r_m = 0.0694$, $q = 0.03$, $g = 0.20$, $w = 0.15$, $\tau = 25$.