

Deciding what and when to seed: Mean reverting process and Real Options

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Abstract

This paper uses real option theory to assess the value of a piece of agricultural land that can be seeded with crops. We consider one- and two-factor models for the crop price and derive a partial differential equation for the value of the land. We also incorporate a crop rotation model to avoid the deterioration of the land when seeding the same crop consecutively over time.

Keywords: real options, agricultural land, irreversibility

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1 Introduction and literature review

This paper addresses the following question: how much should one pay for a piece of agricultural land? For simplicity, we assume that the land can only be used to seed crops. The land value should be closely tied to the present value market of the cash flows that can be obtained from it.

Several factors should be taken into account to perform an accurate valuation:

1. Characterization of the process followed by the crop price. Typically, some kind of mean reversion process is used.
2. There is some benefit from holding the crop (soya, corn, wheat, ...). The benefit will come from harvesting the crop and selling it in the market. This implies that we should include a convenience yield in the price process for the crops.
3. To maintain the quality of the land, it is of utmost importance to rotate the crops. In our analysis we will contemplate two different cases:
 - (a) We will allocate a percentage to each of the crops and we will rotate them every year in an ad hoc way. For instance, assuming we use three crops (soya, corn and wheat), we would allocate 33 percent of the land to each and we will rotate them on a triangular basis.
 - (b) Alternatively, we could look for the optimal way to decide these percentages, depending on current market prices. So, if the price of one of the crops is too high (in comparison with its historical mean), you may wish to seed only this crop for as long as the difference persists. But, as seeding consecutively the same crop damages severely the quality of the soil, this cannot be done for a long period. Therefore, there could be some “optimal” value for waiting until the appropriate time comes to seed a particular crop.
4. There is some value from owning the land, other than the proceeds of the land products (the present value of future cash flows from exploiting the land). The owner of the agricultural land has a certain “advantage” since it is usually very difficult to buy a land at a “fair” price. That is, there is some degree of irreversibility in the decision of selling the land. Notice that this irreversibility does not show up when holding other assets. While this shortage in agricultural land supply might be the result of some behavioral feature (like some emotional attachment to the asset or some rational decision to leave a bequest to the future generation), the consequence on pricing agricultural lands could

be important. This advantage can be viewed as a real option on the land: the owner has the option to sell the land at a price which is higher than the present value of future cash flows.

Thus, it should be clear that to value the land properly, we should contemplate the four factors just mentioned.

With respect to the first two, it is well known that some sort of mean reverting process provides a better description of the price path for many commodities. As noted by Schwartz (1997) in an equilibrium setting, we would expect that, when prices are relatively high, supply will increase as the higher cost producers of the commodity will enter into the market putting downward pressure on prices. Conversely, when prices are relatively low, the higher cost producers will exit the market putting upward pressure on prices.

In his paper, Schwartz (1997) proposes a framework to test whether commodities prices are mean reverting and to estimate the value of the corresponding model parameters. In particular, this author tests whether copper, oil, and gold prices follow mean-reverting processes using three models. His findings indicate that both copper and oil are mean-reverting but this was not the case of gold.

To incorporate mean reversion, Schwartz (1997) uses three processes: ¹

1. A one-factor model where the commodity spot price follows an Ornstein-Uhlenbeck (O-U) type of mean reversion.
2. A two-factor model that extends the previous one considering as a second factor the convenience yield, modelled as a mean-reverting process that can be correlated with the spot price.
3. An extension of the second model that allows for stochastic interest rates.

As shown later, in this paper we follow Schwartz (1997) and use different specifications to model the evolution of crop prices. In particular, we use both one- and two-factor models, where the convenience yield is assumed to be stochastic.

Once we have defined the process for the dynamics of crop prices, to value the land we need to consider the options embedded in owning the land and seeding the different crops optimally. These possibilities are generally known as real options. Real options were

¹These three models are analytically tractable and imply a linear relationship between the logarithm of futures prices and the underlying factor.

introduced in Finance by Brennan and Schwartz (1985) to evaluate the decision of extracting minerals. These authors showed that the Net Present Value method would lead to a non-optimal extraction of minerals because there is a value from waiting that is lost once the excavation begins.

The real option value is basically a premium over the expected net value of the project, reflecting the opportunity cost of investing now and foregoing the option to delay investment until more information about the future becomes available. Real options usually involve capital investment and natural resources, but have also been applied to real estate development decisions. The rationale behind using real options in real estate is very clear: there is uncertainty about future prices and there is some value in having the possibility of delaying a decision until some of the uncertainty is resolved.

Holding a farmland involves a real option because the farmer has the right (but not the obligation) to keep the land for farming or sell it to another for the same or other use.² Exercising this option means that the owner is willing to sell his land and hence close the door to all future opportunities that might be provided by the land being maintained.

A second type of option arises from the decision on which crop to seed. As mentioned before, the quality of the soil decreases quite fast if you seed the same crop consecutively. Then, seeding a certain kind of seed provides the option to seed a different one in a later stage.

So far, the literature of Agricultural Real Estate has priced the selling (put) option by subtracting the discounted cash flows provided by the asset from the market value of the asset assuming that the difference reflects the value of the Real Option (see Isgin and Forster, 2005). However, this procedure can lead to large errors. Since the data generating process of the underlying asset is (by definition) unknown, without a more systematic way to compute the real option value embedded in the land, the result will be of no interest. The explanation of this fact is that there are many factors (like, for instance, taxes or weather) whose inclusion in the analysis can affect largely the numerical solution of the pricing problem.

To the best of our knowledge, the rotation of crops has not been studied deeply in the Agricultural Real Estate literature. Up to this point, this literature has been interested only in timber's harvesting problem for rotation of the harvesting (see, for instance, Insley and Rollins, 2005). While this issue shares similar features with our land pricing, the presence of different prices for the different crops jointly with the possibility of introducing shocks in the production function might make the problem interesting enough to be studied separately.

²Among the different uses we could mention urban development, agropecuary uses and organic products.

The remainder of the paper is organized as follows. Section 2 presents different processes to model crop prices and Section 3 introduces the crop rotation model. Section 4 proposes a valuation technique to price the aforementioned options. Finally, Section 5 concludes the paper.

2 Land value and stochastic models for the crop price

This section describes the three different processes for the crop prices that will be compared to the usual benchmark case: the Geometric Brownian Motion, which is not mean-reverting.

We will assume that the land has a 30-years horizon of use and that, after this period, the land is totally depreciated.³ In this part we will also assume that the land is either seeded fully with one crop forever or seeded in equal proportions of the different crops rotating them every period in an “ad hoc” way.

When there is only one crop seeded and the land does not deteriorates over time, the discounted cash flow of owning the land is given by

$$NPV_0 = DCF(X_t) - K = X_t \sum_{j=1}^{30} Q_j e^{-cj} - C \sum_{j=1}^{30} Q_j e^{-rj} - K \quad (1)$$

where C indicates the production costs (assumed to be constant), Q_j represents the quantity of crop seeded in period j , and K is the strike price of the option to sell the land. This strike price can be interpreted as the opportunity cost of owing the land, that is, if the farmer owns the land in order to exploit it, he renounces to sell it at some given price K .⁴ Additionally, as we will assume that, ex ante, there should be no economic profits, this strike price will be set as the one that makes NPV_0 equal to zero. Then, equation (1) can be rewritten as

$$DCF(X_t) = X_t \beta_1 - \beta_2$$

where

$$\beta_1 = \sum_{j=1}^{30} Q_j e^{-cj}, \quad \beta_2 = C \sum_{j=1}^{30} Q_j e^{-rj} + K$$

Now, at the beginning of each period, the land owner has to make a decision regarding whether to sell or not the land: if the selling price is higher than the DCF, then the farmer

³This assumption might sound quite strong but is done only to assign some final day to the project. If we add more years, the relative weight of this additional period would be very low. Without loss of generality, we could certainly drop this assumption and value a perpetuity instead.

⁴As time goes by, the strike is supposed to decrease a constant amount (K/T) per period. In other words, at any time t ($< T$), we consider a value of $K(1 - \frac{t}{T})$.

will sell the land. On the other hand, if the selling price is lower than the DCF, then the farmer retains the land one more period. Since the realization of the new prices could lead to a higher DCF, the option leads the farmer to a better decision making.

To value the option to sell the land, we assume that the option lasts for the entire life of the project. We will also assume that the option can be exercised once every year. The motivation for this Bermuda feature is because the farmer only collects the crop once per year, so the uncertainty is solved at that moment.

Thus, the value of the land to the owner is $DCF(X_t) + V(X_t)$, where the term $V(X_t)$ denotes the value of the put option on the land. Clearly, these values rely on the process used to model the evolution of crop prices.

2.1 Benchmark process

We start the analysis by describing the benchmark process for the crop price: a risk-neutral Geometric Brownian Motion (GBM), that is, a process without mean reversion and a constant convenience yield:

$$dX_t = (r - \delta)X_t dt + \sigma X_t dW_t \tag{2}$$

where X_t is the price of one unit of the crop, r is the instantaneous risk-free interest rate, δ is the convenience yield, σ is the volatility of the crop return and dW_t is a standard Brownian process under the risk-neutral probability measure.

There are several important issues related to this benchmark model:

- The above stochastic process for the spot price implies that the convenience yield is constant. Thus, the model is not able to capture changes in the term structure of future prices. In fact, from an empirical point of view, the convenience yield can change through time.
- This model also implies that the volatility of all future returns is equal to the volatility of spot returns.
- Finally, this process assumes that the variance of the spot price grows linearly with time, whereas we expect some mean reversion in spot commodity prices.

2.2 One-factor process

We now first consider a one-factor model where the crop price follows an Inhomogeneous Geometric Brownian Motion (see Robel, 2001). This process is given by the expression

$$dX_t = \lambda(\bar{X} - X_t)dt + \sigma X_t dW_t \quad (3)$$

where λ is the speed of mean reversion, \bar{X} is the long-run crop price, σ is the instantaneous volatility and dW_t is a Wiener process. A full description of this process can be found in the Appendix. It is worth noting that now the convenience yield is no longer constant as it depends on the (stochastic) crop price.

As before, the land value for the owner is the sum of the DCF plus the option to sell the land. The DCF is the same as in equation (1), except that the DCF is now a function of the price given in equation (3), adapted to the risk-neutral version.

2.3 First two-factor model

The convenience yield, δ , is included now as an additional second factor. In this case, the convenience yield is assumed to follow an Ornstein-Uhlenbeck process. So, similar to Schwartz (1997), for the joint stochastic process, we have the following:

$$\begin{aligned} dX_t &= (\bar{X} - \delta_t)X_t dt + \sigma_1 X_t dW_1 \\ d\delta_t &= \kappa(\alpha - \delta_t)dt + \sigma_2 dW_2 \end{aligned}$$

where $dW_1 dW_2 = \rho dt$.

In this case, we allow for a stochastic convenience yield, instead of making it a function of the stochastic price. In short, we assume the convenience yield follows a Ornstein-Uhlenbeck stochastic process. To obtain the risk neutral process, we should remove the market price of risk, where here is called μ . After this, we get the following:

$$\begin{aligned} dX_t &= (r - \delta_t)X_t dt + \sigma_1 X_t dW_1^* \\ d\delta_t &= [\kappa(\alpha - \delta_t) - \mu]dt + \sigma_2 dW_2^* \end{aligned}$$

where $dW_1^* dW_2^* = \rho dt$.

2.4 Second two-factor model

The last model to be used is a two-factor model, where the convenience yield follows an Inhomogeneous Geometric Brownian Motion. Its expression is as follows:

$$\begin{aligned}dX_t &= (\bar{X} - \delta_t)X_t dt + \sigma_1 X_t dW_1 \\d\delta_t &= \lambda(\bar{\lambda} - \delta_t)dt + \sigma_2 \delta_t dW_2\end{aligned}$$

where $dW_1 dW_2 = \rho dt$.

Converting this process to a risk-neutral one, the parameter for the mean reversion changes too, so now we have that:

$$\begin{aligned}dX_t &= (r - \delta_t)X_t dt + \sigma_1 X_t dW_1^* \\d\delta_t &= [\lambda^*(\bar{\delta} - \delta_t) - \mu]dt + \sigma_2 \delta_t dW_2^*\end{aligned}$$

where $dW_1^* dW_2^* = \rho dt$.

3 The crop rotation model

We introduce now the possibility of rotating crops. Each year, the farmer will decide optimally which crops should be seed.

Now we have a path-dependent option.⁵ Here the land value depends on today's crop value, which will be a function of what was seed previously, because of the quality of the soil problem. Path dependency complicates considerably the (numerical) solution to the problem.

In areas where rotation is fairly short, the seeding decision proscribed by this problem will be expected to be quite different from the previous dummy decision model where no rotation is allowed.

The price of the commodities are assumed to follow a known stochastic process. The land value is estimated assuming that the seeding will be determined optimally in the future no matter what the price path turns out to be.

⁵ A path-dependent option is one whose value depends on the history of an underlying state variable and not just on its final value. Financial instruments with path-dependency features include, for instance, Asian options or barrier options.

The crop age, that is, the number of periods that the same crop has been seeded, α , will be given as $\alpha = t - t_h$, where t_h is the starting period from when the seed has been seeded consecutively. We will assume that the volume produced is a (continuous) decreasing function of the number of periods that the same seed is used consecutively, being zero a lower bound.⁶ So the state variables for the dynamic problem are the crop age and the prices. Another possibility would be to make the quantity of crop obtained from the agricultural land, Q , dependent on the α -value of other crops such that the productivity of the soil is path-dependent.

So the decision of which crop to seed can be formulated as a dynamic problem, where the owner of the farm must decide in each period whether it would be better to stop with the crop that is being used or continue with it. Calling $Q(\alpha)$ the volume obtained after α periods seeding the same crop consecutively, the Bellman equation for this situation would be

$$\begin{aligned} & V(X_t^{(1)}, X_t^{(2)}, \alpha^{(1)}, \alpha^{(2)}) \\ = & \max[(X_t^{(1)} - C)Q(\alpha^{(1)}) + (1 + r\Delta t)^{-1}E[V(X_{t+\Delta t}^{(1)}, X_{t+\Delta t}^{(2)}, \alpha^{(1)} + \Delta\alpha^{(1)}, \alpha^{(2)})]] \\ & ; \quad (X_t^{(2)} - C)Q(\alpha^{(2)}) + (1 + r\Delta t)^{-1}E[V(X_{t+\Delta t}^{(1)}, X_{t+\Delta t}^{(2)}, \alpha^{(2)} + \Delta\alpha, \alpha^{(1)})]] \end{aligned}$$

The first term in this equation represents the return, conditional on seeding the crop number (1), obtained if continue seeding the current crop. It includes the net revenue from the current crop and the value of the land if tomorrow we would seed the same crop. The second term is the value of the land if we seed crop number (2).

4 Valuation of the put option and analysis of the optimal rotation scheme

The option to sell the land is of Bermuda-type, which means that at each exercise time, the land-owner must compare the profits for exercising the option against the profits for exercising it at a later stage (this is the continuation value of the option).

Longstaff and Schwartz (2001) proposed a method named Least-Squares Monte Carlo (LSM) to evaluate this continuation value using OLS regressions to provide a measure of the expected value one period later. The expected value is just the estimated value. As the

⁶This is a shortcut, since actually it is usually not recommended to seed more than three periods the same crop, which means that the actual function should be discontinuous.

process is iterated backwards in time starting at the expiration period, the right exercise moment can be obtained and, then, the option price can be computed. The regressions can be done through a linear combination of orthogonal basis such as, for instance, Chebichev polynomials.⁷

For the rotation problem, the question is somehow more complicated. First of all, the Bellman equation involves too many states variables. Then, to find a numerical solution we should reduce the number of variables. For instance, as only relative prices matter, we could get rid of one state variables by expressing everything in terms of the price ratio or differences in prices. If we assume that the relative age of the crop is the relevant variable, we can get rid of another state variable. This assumption is not very strong since, if you have seed the same crop for too many periods, we will have to seed another one to let the soil recover from its exhaustive use. We could also set that, whenever we decide to change the crop, the variable α goes to zero.

With these changes, we can try to find a numerical solution to the dynamic programming problem. Since an explicit solution to the real option might be too complicated to be obtained, we can rely on numerical methods for its solution. In particular, we can use the Least Squares Monte Carlo technique proposed in Longstaff and Schwartz (2001) or the Crank- Nicholson method for solving the second order PDE's that is obtained.

We consider that, whenever you decide to change the crop, you forgo the dividend or payoff in that period. Then, you will never exercise the option at time T and change to another crop. We can assume that there is a constant rate of depreciation from seeding the same crop consecutively, as a way to match the decrease in the productivity of the soil. Therefore, the relevant backward decision process is the following:

- At time T , it is never optimal to exercise the option. Therefore, you get the market price of whatever was seeded in time $T - 1$.
- At time $T - 1$, we should compare $P_1 Q(\alpha) + \beta E[(P_1 + \Delta t) Q(\alpha + \Delta t) | I_t]$ against $\beta E[(P_2 + \Delta t) Q(0) | I_t]$.

If the first value is higher than the second one, we do not exercise the option at this period. Then, at time $T - 2$, we have to compare $P_1 Q(\alpha) + \beta E[(P_1 + \Delta t) Q(\alpha + \Delta t) + \beta(P_1 + 2\Delta t) Q(\alpha + 2\Delta t) | I_t]$ versus $\beta E[(P_2 + \Delta t) Q(0) + \beta(P_2 + 2\Delta t) Q(\Delta t) | I_t]$ and so on.

Since it is pretty likely that, for a high α , there will not be a price path such that the

⁷Moreno and Navas (2003) proved that the results do not depend on the chosen basis in most cases.

option is not exercised, we will consider that the relevant period for the option is that when the option has some value.

Thus, the option will be “*in the money*” whenever the termination value is higher than the continuation one.

5 Conclusions

We know from Financial Economics that the price of an asset should be tied to its stream of dividends, payoffs or cash-flows. However, when some sort of irreversibility arises, the market value of the asset is typically greater than the net present value of future cash flows. This happens because the irreversibility really represents an embedded real options that is valuable to the holder of the asset.

Given that once a piece of land is sold it is generally difficult to buy it back at a fair price, owning a farmland incorporates an embedded put option for the owner that must be considered when computing its value. In this paper, we have studied the valuation problem of the land using alternative processes for the prices of the crops that can be seeded. We have also analyzed the case in which we rotate the crops to maximize production.

The valuation problem turns out to be very complex, and numerical techniques must be used. We have shown how to set up the Bellman equations to solve the dynamic problem for the land price.

For further research, several things can be done. The accuracy of the model could likely be increased if we include jumps in the productivity of the soil as well as improvements in the production function. Another way to foster the model could be to incorporate new variables into the analysis, such as weather conditions, soil productivity, or governments policies regarding taxes.

Appendix

- The first moment for the Inhomogeneous Geometric Brownian Motion process (3) can be computed in the following way:

$$E(dX_t) = \lambda(\bar{X} - E(X_t))dt$$

Using the linearity of the expectation, we arrive at the following formula

$$\frac{dE(X_t)}{dt} = \lambda(\bar{X} - E(X_t))$$

Then,

$$e^{\lambda t} \left[\frac{dE(X_t)}{dt} + \lambda E(X_t) \right] = e^{\lambda t} \lambda \bar{X}$$

Integrating this equation, we can find the first moment:

$$\int_0^t e^{\lambda t} \left[\frac{dE(X_t)}{dt} + \lambda E(X_t) \right] dt = \int_0^t e^{\lambda t} \lambda \bar{X} dt$$

Then, we get

$$E(X_t)e^{\lambda t} - E[X_0] = \bar{X}e^{\lambda t} - \bar{X}$$

or, equivalently

$$E(X_t) = \bar{X} + (X_0 - \bar{X})e^{-\lambda t}$$

As expected, as time goes to infinity, $E(X_\infty) = \bar{X}$, corroborating that \bar{X} is the long-term value at which the variable X converges to.

- For the second moment, defining $f(X_t) = X_t^2$ and applying Itô's lemma, we obtain

$$\frac{dE(X_t^2)}{dt} = 2\lambda\bar{X}E(X_t) + (\sigma^2 - 2\lambda)E(X_t^2)$$

Substituting for $E(X_t)$, we get

$$\frac{dE(X_t^2)}{dt} + (2\lambda - \sigma^2)E(X_t^2) = 2\lambda\bar{X}(\bar{X} + (X_0 - \bar{X})e^{-\lambda t})$$

Using $(2\lambda - \sigma^2)t$ as the integrating factor, we have

$$\int_0^t e^{(2\lambda - \sigma^2)t} \left[\frac{dE(X_t^2)}{dt} + (2\lambda - \sigma^2)E(X_t^2) \right] dt = \int_0^t e^{(2\lambda - \sigma^2)t} [2\lambda\bar{X}(\bar{X} + (X_0 - \bar{X})e^{-\lambda t})] dt$$

Solving the integral, we obtain

$$E(X_t^2)e^{(2\lambda - \sigma^2)t} - X_0^2 = \int_0^t 2\lambda\bar{X}^2 e^{(2\lambda - \sigma^2)t} dt + \int_0^t 2\lambda\bar{X} (X_0 - \bar{X})e^{(\lambda - \sigma^2)t} dt$$

Then,

$$e^{(2\lambda-\sigma^2)t}E(X_t^2) = \frac{2\lambda\bar{X}^2}{2\lambda-\sigma^2}(e^{(2\lambda-\sigma^2)t}-1) + \frac{2\lambda\bar{X}(X_0-\bar{X})}{\lambda-\sigma^2}(e^{(\lambda-\sigma^2)t}-1) + X_0^2$$

Provided that $(2\lambda-\sigma^2)(\lambda-\sigma^2) \neq 0$,⁸ we obtain

$$E(X_t^2) = \frac{2\lambda\bar{X}^2}{2\lambda-\sigma^2}(1-e^{(\sigma^2-2\lambda)t}) + \frac{2\lambda\bar{X}(X_0-\bar{X})}{\lambda-\sigma^2}(e^{-\lambda t}-e^{(\sigma^2-2\lambda)t}) + X_0^2e^{(\sigma^2-2\lambda)t}$$

As $Var(X_t^2) = E[(X_t - E(X_t))^2]$, the second non-central moment can be computed as

$$\begin{aligned} Var(X_t) &= e^{(\sigma^2-2\lambda)t} \left(X_0^2 + \frac{2\lambda\bar{X}^2}{\sigma^2-2\lambda} + \frac{2\lambda\bar{X}(X_0-\bar{X})}{\sigma^2-\lambda} \right) \\ &+ e^{-\lambda t} \left(\frac{2\lambda\bar{X}(X_0-\bar{X})}{\lambda-\sigma^2} - 2\bar{X}(X_0-\bar{X}) \right) \\ &+ -e^{-2\lambda t}(X_0-\bar{X})^2 + \frac{2\lambda\bar{X}^2}{2\lambda-\sigma^2} - \bar{X}^2 \end{aligned}$$

So $Var(X_\infty) = \frac{2\lambda\bar{X}^2}{2\lambda-\sigma^2} - \bar{X}^2 = \frac{\sigma^2}{2\lambda-\sigma^2}\bar{X}^2$ Then, this variance converges to a finite value if and only if the speed of mean reversion is high enough ($\lambda > \sigma^2/2$).

The explicit solution to the Inhomogeneous Geometric Brownian Motion is:

$$X_t = e^{-(\lambda+\frac{\sigma^2}{2})t+\sigma W_t} \left[X_0 + \lambda\bar{X} \int_0^t e^{(\lambda+\frac{\sigma^2}{2})s-\sigma W_s} ds \right]$$

Now, since the above process is not risk-neutral, we should replace the drift with the growth rate in a risk-neutral world. This is achieved by discounting the risk premium to obtain the correspondent risk-neutral process⁹ The risk premium is $\rho\sigma\frac{r_m-r}{\sigma_m}X_t = \rho\sigma\phi X_t$

Then, the risk-neutral version is:

$$d\widehat{X}_t = [\lambda(\bar{X} - \widehat{X}_t) - \rho\sigma\phi X_t]dt + \sigma\widehat{X}_t dZ_t$$

Proceeding similarly to the previous computations, the first two (risk-neutral) moments are easily computed.

⁸We actually have two more possibilities: If $\sigma^2 - 2\lambda = 0$, then $E(X_t^2) = -(2k\bar{X}^2)t + 2(X_0 - \bar{X})(e^{-\lambda t} - 1) + X_0^2$. Alternatively, if $\lambda = \sigma^2$, then $E(X_t^2) = 2\bar{X}^2(1 - e^{-\lambda t}) + 2\bar{X}(X_0 - \bar{X})te^{-\lambda t} + X_0^2e^{-\lambda t}$.

⁹The growth rate in a risk-neutral world is given by $r - \delta$, where δ is the convenience yield.

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