

YIELD CURVE FITTING WITH TERM STRUCTURE MODELS: EMPIRICAL EVIDENCE FROM THE EURO MARKET

Javier F. Navas
Universidad Pablo de Olavide, Ctra. de Utrera, Km. 1, 41013
Sevilla, Spain. Phone: ¹+34 954349200. Fax: +34 954349339. E-mail:
jfernnav@dee.upo.es

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ABSTRACT

We study the fitting of the euro yield curve with the Longstaff and Schwartz (1992) (LS) two-factor general equilibrium model and the Schaefer and Schwartz (1984) (SS) two-factor arbitrage model of the term structure of interest rates. The Cox, Ingersoll, and Ross (1985b) (CIR) one-factor model is also studied as a reference. LS use the short-term interest rate and the volatility of the short-term interest rate as state variables, while SS use the spread between the short-term and the long-term interest rate and the long-term interest rate. Thus, the LS model should perform better (worse) than the SS model in pricing short-term (long-term) securities. Moreover, since the CIR model can be nested into the LS model, we expect the latter model to perform better than the former.

The results show that, as expected, the LS model is best adjusting to the short-term yields. Surprisingly, the CIR model is best fitting to long-term yields. In any case, the three models have difficulties matching both the entire yield curve and the term structure of volatilities.

JEL Classification: C21, C22, E43, G13

Keywords: term structure, yield curve, calibration

RESUMEN

En este trabajo estudiamos el ajuste de la curva de rendimientos de la zona euro con el modelo bifactorial de equilibrio general de Longstaff y Schwartz (1992) (LS) y con el modelo bifactorial de no arbitraje de Schaefer y Schwartz (1984) (SS). Como referencia, también utilizamos el modelo unifactorial de Cox, Ingersoll, y Ross (1985b) (CIR). LS usan como variables de estado el tipo de interés a corto plazo y su volatilidad, mientras que SS emplean el diferencial entre el tipo de interés a corto y el tipo a largo, y el propio tipo de interés a largo plazo. Por tanto, el modelo LS debería ser mejor (peor) que el modelo SS valorando activos a corto (largo) plazo. Además, dado que el modelo CIR se puede interpretar como un caso particular del modelo LS, deberíamos esperar que éste sea superior a aquél en la práctica.

Los resultados obtenidos muestran que, como era de esperar, el modelo LS es el mejor ajustando los rendimientos a corto plazo. Sin embargo, de forma sorprendente, el modelo CIR es el mejor ajustando los rendimientos a largo plazo. En cualquier caso, los tres modelos tienen dificultades para ajustarse a la estructura temporal de tipos de interés y a la estructura temporal de volatilidades al mismo tiempo.

Clasificación JEL: C21, C22, E43, G13

Palabras clave: estructura temporal de tipos de interés, curva de rendimientos, calibración

There are three approaches to price contingent claims when the evolution of interest rates is stochastic. The arbitrage approach derives a partial differential equation (PDE) for the value of any contingent claim by constructing portfolios of securities whose weights are chosen to make the rate of return on the portfolio non-stochastic. Then, to avoid the possibility of arbitrage profits, the rate of return on the portfolio is made equal to the instantaneous riskless rate of interest. Examples of this method are the one-factor models of Brennan and Schwartz (1977), Vasicek (1977) and Dothan (1978); and the two-factor models of Brennan and Schwartz (1979 & 1982), Schaefer and Schwartz (1984) (SS), and Moreno (2003).

The general equilibrium approach, developed by Cox, Ingersoll, and Ross (1981), (1985a) and (1985b), uses an intertemporal general equilibrium asset pricing model to study the term structure of interest rates. In their model, the pricing equations incorporate anticipations, risk aversion, investment alternatives, and preferences about the timing of consumption and asset prices and their stochastic properties are determined endogenously. The equilibrium price of any asset is given in terms of the underlying real variables in the economy and it is consistent with maximizing behavior and rational expectations. As an application, Cox, Ingersoll, and Ross (1985b) (CIR hereafter), develop a one-factor model of the term structure and use it to price bonds and bond options. In this model the term structure of interest rates at any point in time is given by the current level of the short-term interest rate, and the volatility of the interest rate process is proportional to the level of the short-term interest rate. A two-factor term structure model using the general equilibrium approach can be found in Longstaff and Schwartz (1992) (LS). They use the short-term interest rate and its instantaneous variance as state variables and they derive closed-form expressions for the price of discount bonds and discount bond options. The authors find empirical support for their model.

All these models imply term structures of interest rates that, generally, are not consistent with the market yield curve. Thus, the models will price incorrectly the underlying assets of most interest rate derivatives. As a consequence, a third kind of model, consistent with the current term structure of interest rates, is constructed. This is the approach followed by Ho and Lee (1986), using bond prices, and Heath, Jarrow, and Morton (1992), using forward interest rates. An equivalent technique is to extend non-consistent term structure models, using time-dependent parameters that are calibrated to make the model match the market yield curve [see, for example, Hull and White (1990a), Black, Derman, and Toy (1990), and Black and Karasinski (1991)].

Alternative derivative models can be compared in two ways. We can analyze the ability of their state variable processes to describe the evolution of interest rates. For example, Chan et al. (1992) use the Generalized Method of Moments of Hansen (1982) as a general framework to estimate and compare one-factor models of the short-term interest rate. They find that the most successful models in capturing the dynamics of interest rate are those that allow the volatility of interest rate changes to be highly sensitive to the level of the riskless rate². The comparison of two-factor models has received much less attention in the literature³, perhaps because it is difficult to develop a general econometric framework in which these models can be easily nested.

² A similar result is obtained by Navas (1999) in the Spanish market.

³ Some exceptions are Dai and Singleton (2000) and Bühler et al. (1999).

However, a model can be successful at describing the evolution of a particular interest rate but perform poorly when pricing contingent claims. This leads us to a second, and maybe more convenient, way of comparing alternative models: to study the performance of the models when pricing securities [see, for example, Bühler et al. (1999), Navas (1999), and Moraleda and Pelsser (2000)].

The purpose of this paper is to compare models of the term structure in terms of their ability to fit the term structure of interest rates in the euro market. Specifically, we compare the pricing of pure discount bonds with the two-factor general equilibrium model of Longstaff and Schwartz (1992) and the two-factor arbitrage approach of Schaefer and Schwartz (1984). As an additional reference, the single-factor CIR model is also used in the comparisons. We also study the term structure of volatilities implied by the models and compare them with the actual volatility curve.

The reason for concentrating on two-factor models is that, as LS argue, single-factor models imply that the instantaneous returns on bonds of all maturities are perfectly correlated, which is not supported empirically. Moreover, the reason for choosing these particular two-factor models is that they have closed-form solutions for bond prices^{4,5}.

The remainder of the paper is organized as follows. Section 1 presents the models under consideration. Section 2 describes the estimation and implementation of the models. Section 3 studies the performance of the models when fitting the yield and volatility curves. Finally, Section 4 summarizes the paper.

1. THE MODELS

Term structure models are developed assuming stochastic processes for the dynamics of one or more exogenous factors or state variables. Typically, single-factor models use different specifications of the short-term interest rate process and assume that the term structure of interest rates is a function of the level of the current riskless rate (and some parameters). Two-factor models assume that the term structure is a function of the current values of two state variables (frequently including the short-term interest rate).

Once the dynamics of the state variables are defined, the general equilibrium or the arbitrage approach can be used to derive a partial differential equation (PDE) for the value of any contingent claim. In this equation the coefficients of the variables are functions of the parameters of the processes describing the term structure and of the market prices of the risks associated with the state variables. Models based on the arbitrage approach make explicit assumptions about the functional forms of these market prices of risk, making it possible, in some cases, to reduce the number of unknown parameters to be estimated [see, for example, Brennan and Schwartz (1979) and Schaefer and Schwartz (1984)]. In these models the functional forms of the market prices of risk may not be consistent with market data. This problem is avoided when the

⁴ Although in the SS model the solution is only an approximation.

⁵ Balduzzi et al. (1996) and Moreno (2003) also provide closed-form expressions.

general equilibrium approach is used, since, in this case, the functional forms of the market prices of risk are obtained endogenously, ensuring that the risk premium is consistent with no arbitrage.

The pricing equation must be satisfied for the value of any security. Depending on the asset being priced, this PDE will have different boundary conditions and, therefore, different solutions. In general, closed-form solutions to the pricing equation are not known and numerical procedures are required [see Brennan and Schwartz (1978), Courtadon (1982), Geske and Shastri (1985), and Hull and White (1990b), among others].

1.1 The Longstaff and Schwartz (1992) model

The state variables in this model are the short-term interest rate, r , and the instantaneous variance of changes in the short-term interest rate, V . Using general equilibrium considerations, LS show that the dynamics of r and V are given by

$$dr = \left(\alpha\gamma + \beta\eta - \frac{\beta\delta - \alpha\xi}{\beta - \alpha} r - \frac{\xi - \delta}{\beta - \alpha} V \right) dt + \alpha \sqrt{\frac{\beta r - V}{\alpha(\beta - \alpha)}} dz_1 + \beta \sqrt{\frac{V - \alpha r}{\beta(\beta - \alpha)}} dz_2, \quad [1]$$

$$dV = \left(\alpha^2\gamma + \beta^2\eta - \frac{\alpha\beta(\delta - \xi)}{\beta - \alpha} r - \frac{\beta\xi - \alpha\delta}{\beta - \alpha} V \right) dt + \alpha^2 \sqrt{\frac{\beta r - V}{\alpha(\beta - \alpha)}} dz_1 + \beta^2 \sqrt{\frac{V - \alpha r}{\beta(\beta - \alpha)}} dz_2. \quad [2]$$

where α , β , γ , δ , η , and ξ are constants and z_1 and z_2 are uncorrelated standard Wiener processes.

If $P(r, V, \tau)$ represents the value of a pure discount bond with time to maturity τ , LS apply Theorem 3 of Cox, Ingersoll, and Ross (1985a) and derive a PDE for P , with boundary condition $P(r, V, 0) = 1$. Making the corresponding change of variables, the solution to this equation is given by

$$P(r, V, \tau) = A_{LS}^{2\eta}(\tau) B_{LS}^{2\eta}(\tau) \exp\{\pi\tau + C(\tau)r + D(\tau)V\} \quad [3]$$

where

$$\begin{aligned}
A_{LS}(\tau) &= \frac{2\phi}{(\delta + \phi)(\exp\{\phi\tau\} - 1) + 2\phi} \\
B_{LS}(\tau) &= \frac{2\psi}{(\nu + \psi)(\exp\{\psi\tau\} - 1) + 2\psi} \\
C(\tau) &= \frac{\alpha\phi(\exp\{\psi\tau\} - 1)B(\tau) - \beta\psi(\exp\{\phi\tau\} - 1)A(\tau)}{\phi\psi(\beta - \alpha)} \\
D(\tau) &= \frac{\psi(\exp\{\phi\tau\} - 1)A(\tau) - \phi(\exp\{\psi\tau\} - 1)B(\tau)}{\phi\psi(\beta - \alpha)}
\end{aligned}$$

and

$$\begin{aligned}
\nu &= \xi + \lambda_{LS} \\
\phi &= \sqrt{2\alpha + \delta^2} \\
\psi &= \sqrt{2\beta + \nu^2} \\
\pi &= \gamma(\delta + \phi) + \eta(\nu + \psi).
\end{aligned}$$

Thus, the discount bond price is a function of r , V , and τ , and depends on the parameters α , β , δ , γ , η , and ξ , as well as the market price of risk λ_{LS} (assumed to be constant through time). Finally, the yield to maturity can easily be obtained as follows

$$Y(r, V, \tau) = -\frac{\ln P(r, V, \tau)}{\tau}. \quad [4]$$

1.2 The Schaefer and Schwartz (1984) model

In this model the state variables are the spread between the short-term and the long-term interest rates, s , and the long term interest rate, l . The dynamics of these variables are assumed to be given by the following stochastic differential equations

$$ds = m(\mu - s)dt + \omega dz_1 \quad [5]$$

$$dl = \beta(s, l, t)dt + \sigma\sqrt{l}dz_2 \quad [6]$$

where m , μ , γ , and σ are constants and z_1 and z_2 are standard Wiener processes with instantaneous correlation ρ .

Thus, the spread is modeled as an Ornstein-Uhlenbeck process with constant volatility. However, the volatility of the long-term interest rate is assumed to be proportional to its level.

SS leave the drift of the long-interest rate process, $\beta(s, l, t)$ unspecified since any drift is compatible with their pricing equation. In this paper, we assume that there is mean reversion in long-term interest rates, so that this rate follows the square root process

$$dl = \kappa(\theta - l)dt + \sigma\sqrt{l}dz_2 \quad [7]$$

Assuming that the market price of spread risk, λ_{ss} , is constant and that the spread is uncorrelated with the long-term rate, i.e. $\rho = 0$ (consistent with the empirical evidence⁶, Schaefer and Schwartz (1984) show that the value of a default free bond, P , must satisfy the PDE

$$\frac{1}{2}\gamma^2 P_{ss} + \frac{1}{2}\sigma^2 l P_{ll} + P_s m(\hat{\mu} - s) + P_l(\sigma^2 - ls) - (l + s)P - P_\tau = 0, \quad [8]$$

with the boundary condition $P(s, l, 0) = 1$, where $\hat{\mu} = \mu - \lambda_{ss} \varpi / m$ and the subindexes for P represent partial derivatives with respect to the state variables.

An approximate solution to this equation is given by

$$P(s, l, \tau) = X(s, \tau) A_{ss}(\tau) \exp\{-B_{ss}(\tau)l\}, \quad [9]$$

where

$$\begin{aligned} X(s, \tau) &= \exp\left\{\frac{1}{m}(1 - \exp\{-m\tau\})(s_\infty - s) - \varpi_\infty - \frac{\varpi^2}{4m^3}(1 - \exp\{-m\tau\})^2\right\} \\ A_{ss}(\tau) &= \left[\frac{2u \exp\{(\hat{s} + u)\tau/2\}}{(\hat{s} + u)(\exp\{u\tau\} - 1) + 2u}\right]^2 \\ B_{ss}(\tau) &= \frac{2(\exp\{u\tau\} - 1)}{(\hat{s} + u)(\exp\{u\tau\} - 1) + 2u} \\ s_\infty &= \hat{\mu} - \frac{\varpi^2}{2m^2} \\ u &= \sqrt{\hat{s}^2 + 2\sigma^2}. \end{aligned}$$

Here, \hat{s} depends on the current values of s and l , as shown by SS, and is obtained numerically by equating messy nonlinear functions of \hat{s} . However, when the initial value of the spread (s_0) is equal to $\hat{\mu}$, \hat{s} is also equal to $\hat{\mu}$.

Using the bond price, we compute the yield to maturity as in expression [4].

⁶ See, for example, Ayres and Barry (1980) and Nelson and Schaefer (1983).

1.3 The Cox, Ingersoll, and Ross (1985b) model

In this model, the term structure of interest rates at time t is given by the short-term interest rate r which follows the square root process

$$dr = \omega(\varphi - r)dt + v\sqrt{r}dz, \quad [10]$$

where ω , φ , and v are positive constants and z is a Wiener process. In this model, the interest rate r is pulled towards its long-term mean φ at rate ω .

The price, $P(r, \tau)$, of any interest-rate contingent claim is the solution to the partial differential equation

$$\frac{1}{2}v^2(r)P_{rr} + (\omega(\varphi - r) - \lambda(r, t)v(r))P_r + P_\tau - rP = 0, \quad [11]$$

subject to appropriate terminal and boundary conditions. Here, $\lambda(r, t)$ is the market price of short-term interest rate risk and is supposed to be $\lambda(r, t) = \lambda_{CIR} \sqrt{r}/v$, where λ_{CIR} is a constant. As in the previous two-factor models, there will be positive risk premiums when λ_{CIR} is negative.

For a discount bond maturing in time τ , the terminal condition is $P(r, 0) = 1$, and the solution of [11] gives

$$P(r, \tau) = A_{CIR}(\tau) \exp\{-B_{CIR}(\tau)r\} \quad [12]$$

where

$$A_{CIR}(\tau) = \left(\frac{2\zeta \exp\{(\omega + \lambda_{CIR} + \zeta)\tau/2\}}{(\zeta + \omega + \lambda_{CIR})(\exp\{\zeta\tau\} - 1) + 2\zeta} \right)^{\frac{2\omega\varphi}{v^2}}$$

$$B_{CIR}(\tau) = \frac{2(\exp\{\zeta\tau\} - 1)}{(\zeta + \omega + \lambda_{CIR})(\exp\{\zeta\tau\} - 1) + 2\zeta}$$

$$\zeta = \sqrt{(\omega + \lambda_{CIR})^2 + 2v^2}.$$

2. ESTIMATION AND IMPLEMENTATION OF THE MODELS

2.1 The data

To estimate the models from historical data, we use daily one- and twelve-month Euribor rates and five-, ten-, twenty-, and thirty-year mid-market par swap rates⁷ for the period from January 1, 1999 through August 30, 2002 (948 observations).

Table 1 shows the means, standard deviations, and first three autocorrelations of the one-month and thirty-year rates and the spread between them. The unconditional means of the one-month, thirty-year and spread rates are 3.729%, 5.711% and -1.982%.

[Insert Table 1 about here]

The time series of the three rates during the sample period are shown in Figure 1.

[Insert Figure 1 about here]

2.2 Estimation of the Longstaff and Schwartz (1992) model

LS express the stochastic processes for r and V , given in expressions [1] and [2], as difference equations that can be rearranged to give the following econometric model

$$r_{t+1} - r_t = \alpha_0 + \alpha_1 r_t + \alpha_2 V_t + e_{t+1} \quad [13]$$

$$e_{t+1} \sim N(0, V_t)$$

$$V_t = \beta_0 + \beta_1 r_t + \beta_2 V_{t-1} + \beta_3 e_t^2 \quad [14]$$

This is a GARCH(1,1) model with conditional variance in the mean equation. In this model, the level of the interest rate appears as an additional regressor in both the mean and the variance equations.

The six stationary parameters of the LS model (α , β , γ , δ , η , and ξ) can be obtained from the estimated coefficients of the GARCH model. However, the relationship between the parameters and the coefficients is very complex. Consequently, Longstaff and Schwartz (1993) suggest an alternative way of computing the parameters of their continuous-time model. They estimate the GARCH model to obtain a time series of conditional variances that is used together with the original time series of interest rates to calculate α , β , γ , δ , η , and ξ as follows

⁷ I thank Juan Carlos García Céspedes and David García Martín for providing part of the data.

$$\alpha = \min\left(\frac{V_t}{r_t}\right)$$

$$\beta = \max\left(\frac{V_t}{r_t}\right)$$

$$\delta = \frac{\alpha(\alpha + \beta)(\beta E\{r_t\} - E\{V_t\})}{2(\beta^2 \text{Var}\{r_t\} - \text{Var}\{V_t\})}$$

$$\gamma = \frac{\delta(\beta E\{r_t\} - E\{V_t\})}{\alpha(\beta - \alpha)}$$

$$\xi = \frac{\beta(\alpha + \beta)(E\{V_t\} - \alpha E\{r_t\})}{2(\text{Var}\{V_t\} - \alpha^2 \text{Var}\{r_t\})}$$

$$\eta = \frac{\xi(E\{V_t\} - \alpha E\{r_t\})}{\beta(\beta - \alpha)}$$

where $E\{.\}$ and $\text{Var}\{.\}$ denote expected value and variance, respectively.

In Table 2 we report the estimates of the GARCH model. We see that the constant terms are not significant either in the mean or the variance equations (α_0 and β_0 , respectively). Moreover, the coefficient of the interest rate in the mean equation (α_1) is not significant. The rest of the coefficients are significant at the 0.01 level.

[Insert Table 2 about here]

In Table 3 we use our time series of interest rates and GARCH volatilities to calculate α , β , γ , δ , η , and ξ . The conditional volatilities are also needed to price bonds since the short-term volatility is one of the state variables in the LS model.

[Insert Table 3 about here]

2.3 Estimation of the Schaefer and Schwartz (1984) and CIR (1985b) models

In these models, the dynamics of the state variables can be represented as

$$dx = a(b - x) + cx^d dz, \quad [15]$$

where the relationship of a, b, c and d with the original parameters of the processes is given in the following table

	Spread	Long-term rate	Short-term rate
x	s	L	r
a	m	κ	ω
b	μ	θ	ϕ
c	γ	σ	ν
d	0	$\frac{1}{2}$	$\frac{1}{2}$

To estimate these models from historical data, we follow Nowman (1997) and use the Gaussian estimation method. We first express equation [15] in a discrete-time setting as

$$x_t = \exp\{-a\Delta t\}x_{t-1} + \theta(1 - \exp\{-a\Delta t\}) + \eta_t \quad (t = 1, 2 \dots T) \quad [16]$$

where Δt is the time interval and η_t satisfies the conditions

$$\begin{aligned} E(\eta_t) &= 0 \\ E(\eta_s \eta_t) &= 0, \quad s \neq t \\ E(\eta_t^2) &= \frac{c^2 x_{t-1}^{2d}}{2a} (1 - \exp\{-2a\Delta t\}). \end{aligned}$$

We then obtain the parameter estimates maximizing the Gaussian log-likelihood function of process [16] given by

$$L(a, b, c, d) = -\frac{1}{2} \ln 2\pi - \frac{1}{2} \ln E(\eta_t^2) - \frac{1}{2} \frac{[x_t - \exp\{-a\Delta t\}x_{t-1} - b(1 - \exp\{-a\Delta t\})]^2}{E(\eta_t^2)}. \quad [17]$$

This expression is the exact log-likelihood function for the spread process, but only an approximation of it for the long and short-rate processes [Brown and Schaefer (1996)].

In the SS model, the long-term interest rate process describes the dynamics of the yields of a riskless consol bond. Brennan and Schwartz (1982) approximate the consol rate, l , by the annualized yield to maturity on the highest-yielding U.S. Treasury Bond with a maturity exceeding 20 years; if no such bond is available in a particular month, they use the highest-yielding bond with a maturity of more than 15 years. In this paper, we use the 30-year mid-market par swap rate as a proxy for the consol rate in the euro market.

Table 4 shows the parameter estimates for the SS and CIR models (t-statistics are provided in parentheses). We see that the parameters are significant at the 0.01 level, with the exception of the mean reversion speed (a) in both the spread and the short-rate processes. Note that the long-term means of the long-rate, spread, and short-rate processes are close to the unconditional means reported in Table 5.

[Insert Tables 4 and 5 about here]

3. FITTING THE YIELD CURVE

To fit the interbank yield curve with the three models, we estimate the corresponding market prices of interest-rate risk, λ . We do so from daily cross-sections of one- and twelve-month Euribor rates and from five-, ten-, twenty-, and thirty-year

swap rates using the parameter estimates obtained from the time-series data. Each day, we minimize the sum of squared errors (SSE), i.e.

$$\min_{\lambda} \sum_{i=1}^6 (y_i - \hat{y}_i)^2 \quad [18]$$

where y_i and \hat{y}_i stand for market yield and theoretical yield, respectively, and $i=1, 2, \dots, 6$ represents the six maturities considered.

Figure 2 plots the estimates of λ in each model. We see that the parameter is unstable through time and ranges from -0.4104 to -0.0403 in the LS model, from -0.0675 to 0.0475 in the SS model and from -0.0595 to 0.0822 in the CIR model. Note that λ is negative in most cases, implying positive risk premiums.

[Insert Figure 2 about here]

To price bonds with the SS model, we assume that $\hat{s} \approx \hat{\mu}$, so that we can use the bond pricing formula provided by Schaefer and Schwartz (1984). As shown in Figure 3, this seems to be a good approximation. Moreover, it turns out that bond prices are not very sensitive to the value of \hat{s} .

[Insert Figure 3 about here]

We now obtain the predicted yields of the LS, SS, and CIR models by computing discount bond prices from equations [3], [9], and [12], respectively.

To compare the models, we use the mean absolute percentage error (MAPE) measure, defined as

$$MAPE = \frac{1}{n} \sum_{i=1}^n \frac{|y_i - \hat{y}_i|}{y_i}, \quad [19]$$

where n is the number of observations (948).

Given the characteristics of the models, we would expect two results. First, LS nest the single-factor CIR model within their two-factor model of the term structure. Thus, the LS model is likely to be more successful at describing the evolution of short rates than the CIR model and should be superior in pricing contingent claims. Second, LS use the short-term interest rate and its volatility as state variables to describe the term structure of interest rates, while SS use the long-term rate and the spread between the short-term and the long-term rates. Consequently, the LS model should also be superior to the SS model in pricing short-term contingent claims.

Table 6 reports the MAPE for each model for different maturities. For the one-month yield we see that, as expected, the pricing error of the LS model is substantially smaller than the others (0.01% versus 0.16% for the SS model and 0.13% for the CIR model). Nonetheless, the three models fit the short end of the yield curve very well. Note that the pricing error of the one-factor CIR model is smaller than that of the SS

two-factor model, which is not surprising given that in the CIR model the volatility of the short-term rate depends on the level of the riskless rate, which seems to be supported by the empirical evidence.

[Insert Table 6 about here]

For one-year yields, the error increases substantially in all instances. Surprisingly, the yield error is greater in the LS model (9.51%) than in the CIR model (8.95%). The smallest error is found in the SS model (8.80%).

The MAPEs for five- and ten-year maturities continue to increase. In the latter case, the MAPEs are 27.71%, 16.49% and 17.66% for the LS, SS, and CIR models, respectively. Again, the CIR model outperforms the LS model.

For long maturities (20 and 30 years), the error decreases. Interestingly, the one-factor CIR model outperforms the other two-factor models (the MAPEs are 9.04%, 8.98%, and 8.81% for the LS, SS, and CIR models, respectively).

The performance of the models adjusting the one-, five-, ten-, twenty-, and thirty-year rates is shown in Figures 4 to 8. We see that the fitting of the SS and CIR models is very similar and better than that of the LS model.

[Insert Figures 4 to 8 about here]

3.1 Linear yield prediction

To further compare the performance of the models, we test them for unbiased linear prediction of the actual one-month, ten-year, and thirty-year interbank rates.

For each model, the following linear regression is estimated

$$\hat{y}_i = \alpha_0 + \alpha_1 y_i + \varepsilon_i \quad [20]$$

$$\varepsilon_i \sim N(0, \zeta),$$

where ζ is constant.

If the estimated intercept term, α_0 , is not significantly different from 0, and the coefficient of the actual bond yields, α_1 , is not significantly different from 1, we cannot reject the hypothesis that the computed bond yield is a linear unbiased predictor of the actual bond yield. The R^2 of the regressions provides an additional measure of the goodness of fit of the models.

Table 7 provides the results of the regressions of the computed yields on the actual ones. For one-month rates (rows 2-7), the intercept term is significantly different from 0 and the slope is significantly different from 1 (at the .01 level) for the three models. Therefore, the hypothesis that the model is an unbiased linear predictor is

rejected in the three cases. This should not be discouraging since the R^2 of the regressions are almost equal to 1, implying that the biases are very small.

[Insert Table 7 about here]

In rows 8-13 we present the results for ten-year interbank rates. We reject again, at the .01 level, that the LS, SS and CIR models are unbiased linear predictors of ten-year rates since the slope is significantly different from 1 in the three models. Moreover, the goodness of fit of the models, as measured by R^2 , is quite poor (0.25, 0.40, and 0.35, respectively).

Finally, in rows 14-19 we see that for thirty-year swap rates we reject, at the .01 level, that the three models are unbiased linear predictors. Note that the R^2 of the LS model (0.51) is much lower than that of the SS and CIR models (0.76 and 0.78, respectively).

In general, these findings are consistent with those given by the MAPE measure.

4. FITTING THE VOLATILITY CURVE

To price interest-rate derivatives accurately, a pricing model should be consistent not only with the term structure of interest rates but also with the term structure of volatilities. In this section we study the ability of the three models to match the market volatility curve.

The term structure of instantaneous volatilities in the LS model is given by

$$\sigma_{LS}(\tau) = \frac{\sigma_{P(r,V,\tau)}}{\tau P(r,V,\tau)}$$

where $\sigma_{P(r,V,\tau)}$ is the instantaneous volatility of bond returns and can be computed from the following expression

$$\begin{aligned} \sigma_{P(r,V,\tau)}^2 = & r \left[\frac{\alpha\beta\psi^2(\exp\{\phi\tau\} - 1)^2 A_{LS}^2 \tau - \alpha\beta\phi^2(\exp\{\psi\tau\} - 1)^2 B_{LS}^2 \tau}{\phi^2\psi^2(\beta - \alpha)} \right] \\ & + V \left[\frac{-\alpha\psi^2(\exp\{\phi\tau\} - 1)^2 A_{LS}^2 \tau - \alpha\beta\phi^2(\exp\{\psi\tau\} - 1)^2 B_{LS}^2 \tau}{\phi^2\psi^2(\beta - \alpha)} \right] \end{aligned}$$

see Rebonato (1998) for more details.

For the CIR model, it is easy to see that the term structure of volatilities is given by

$$\sigma_{CIR}(\tau) = \frac{v\sqrt{r}}{\tau} B_{CIR}(\tau).$$

Finally, we obtain the term structure of volatilities of the instantaneous interest rate in the SS model using the volatility curves of long rates (computed as in the CIR model for the short rate) and the spread, given by

$$\sigma_s(\tau) = \frac{\overline{\sigma}}{m\tau}(1 - \exp\{-m\tau\}).$$

In Figure 9 we plot the actual volatility⁸ of one-month Euribor rates and the spot rate volatilities implied by the three models. We see that the computed volatilities are very similar and that the GARCH estimates in the LS model are the only ones that somehow follow the peaks in actual volatility.

[Insert Figure 9 about here]

Figure 10 depicts the term structure of volatilities implied by the three models compared with the actual volatility curve on August 1, 2002⁹. As expected, the LS model matches the short-term (one-month) volatility but the three models are not consistent with the complete market curve, which will lead to option pricing errors. Similar graphics are obtained for different days.

[Insert Figure 10 about here]

5. CONCLUSIONS

This paper compares the Longstaff and Schwartz (1992) general equilibrium model and the Schaefer and Schwartz (1984) arbitrage model of the term structure of interest rates. The two models differ in: (1) the considerations they use to derive the partial differential equation for pricing assets; (2) LS use two short-term state variables, while SS use one short-term and one long-term variable; and (3) LS develop an exact solution, while SS offer an approximate analytical solution to the PDE.

Two results are expected. On the one hand, since the CIR model can be nested into the LS model, the LS model should perform better than the CIR model in pricing both short-term and long-term bonds. On the other hand, the LS model should perform better (worse) than the SS model in pricing short-term (long-term) securities, since it uses two short-term state variables.

Our empirical results support the second conjecture but not the first. For one-month rates, the three models perform very well with mean absolute percentage errors (MAPE) smaller than 0.2%. The LS model performs best with a MAPE of only 0.01%. The performance of the three models decreases considerably for medium- and long-term yields (with MAPEs often well above 10%). Surprisingly, for all maturities (except one-month), the LS model performs the worst.

⁸ Measured as the absolute value of the day-to-day change in the one-month Euribor rate.

⁹ As before, actual volatilities are computed as the absolute value of day-to-day changes in interest rates.

The models studied can be forced to fit the long end of the yield curve by estimating the market prices of risk from daily cross-sections of one-month and thirty-year rates. The three models are able to achieve an almost perfect fit to both short and long rates but the pricing errors for other maturities remain large.

Thus, it seems that, to fit the entire yield curve, generalized versions of these models must be used, as suggested by Longstaff and Schwartz (1993) and Hull and White (1990a). Note that this does not imply that the models will price interest-rate options accurately, since they should also be consistent with the term structure of volatilities.

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TABLE 1: DESCRIPTIVE STATISTICS.

	r	l	s
Mean	0.03729	0.05711	-0.01982
Standard Deviation	0.00775	0.00368	0.00688
ρ_1	0.99903	0.99348	0.99676
ρ_2	0.99779	0.98770	0.99343
ρ_3	0.99651	0.98115	0.98979

Summary statistics for one-month Euribor rate (r), thirty-year mid-market par swap rate (l) and the spread between them (s) from January 1, 1999 through August 30, 2002. Daily data is used (948 observations). The autocorrelation coefficient of order i is denoted as ρ_i .

TABLE 2: GARCH ESTIMATES OF THE LONGSTAFF AND SCHWARTZ (1992)
TWO-FACTOR MODEL.

	α_0	α_1	α_2	β_0	β_1	β_2	β_3	Log-Likelihood
Value	-6.30E-05	-5.26E-04	550.1319	4.27E-09	1.05E-06	0.6000	0.14999	6185.17
z-statistic	-0.6582	-0.2137	3.3851	0.8685	7.4120	14.1005	10.3325	
p-value	0.5104	0.8308	0.0007	0.3851	0.0000	0.0000	0.0000	

The continuous-time Longstaff and Schwartz (1992) model is approximated by the following GARCH(1,1) model with conditional variance in the mean equation:

$$r_{t+1} - r_t = \alpha_0 + \alpha_1 r_t + \alpha_2 V_t + e_{t+1},$$

$$e_{t+1} \sim N(0, V_t),$$

$$V_t = \beta_0 + \beta_1 r_t + \beta_2 V_{t-1} + \beta_3 e_t^2.$$

The parameters are obtained from the maximum likelihood estimates of the GARCH model. The data are daily one-month Euribor rates during the period from January 1, 1999 to August 30, 2002 (948 observations).

TABLE 3: STATIONARY PARAMETERS OF THE LONGSTAFF AND SCHWARTZ (1992) TWO FACTOR MODEL.

Parameter	MLE Estimate
α	2.6696E-06
β	8.3633E-05
γ	12.3990
δ	9.0490E-04
η	0.0418
ξ	4.8949E-03

The parameters are obtained from the maximum likelihood estimates of the parameters of a GARCH model for daily changes in the one-month Euribor rate. The data are daily one-month Euribor rates during the period from January 1, 1999 to August 30, 2002 (948 observations).

TABLE 4: GAUSSIAN ESTIMATES OF THE SCHAEFER AND SCHWARTZ (1984) (SS) AND THE COX, INGERSOLL AND ROSS (1985B) (CIR) TWO-FACTOR MODELS.

		a	b	c	d	Log-Likelihood
SS	Long-term process	0.007839 (2.1253)	0.057319 (32.4720)	0.001760 (43.3449)	0.5	6023.3808
	Spread process	0.003352 (1.2775)	-0.020362 (-3.7762)	0.000555 (43.9116)	0.0	5756.7405
CIR	Short-rate process	0.001086 (0.7264)	0.038042 (3.5189)	0.001798 (43.5242)	0.5	6211.1200

The data are daily one-month Euribor rates and thirty-year mid-market par swap rates, during the period from January 1, 1999 to August 30, 2002 (948 observations). It is assumed that, in the Schaefer and Schwartz model, the long-term interest rate follows a square root process. The continuous-time model is $dx = a(b-x) dt + cx^d dz$, where $x = l$ and $x = s$ for the long-term and spread processes, respectively, in the Schaefer and Schwartz model and $x = r$ in the CIR model. Gaussian estimates (expressed in yearly basis) with t-statistics in parentheses are presented for each model.

TABLE 5: MEAN ABSOLUTE PERCENTAGE ERRORS --MAPE-- (IN %) FOR DIFFERENT MATURITIES USING CROSS-SECTIONS OF 1- AND 12-MONTH EURIBOR RATES AND 5-, 10-, 20- AND 30-YEAR MID-MARKET PAR SWAP RATES.

Model	Maturity					
	1 Month	1 Year	5 Years	10 Years	20 Years	30 Years
LS	0.01	9.51	21.34	27.71	20.64	9.04
SS	0.16	8.80	15.27	16.49	7.46	8.98
CIR	0.13	8.95	16.02	17.66	8.80	8.81

Comparison of Longstaff and Schwartz (1992) two-factor general equilibrium model, Schaefer and Schwartz (1984) two-factor model and Cox Ingersoll and Ross (1985) one-factor model. The data are daily one- and twelve-month Euribor rates and five-, ten-, twenty- and thirty-year mid-market par swap rates during the period from January 1, 1999 to August 30, 2002 (948 observations). The comparisons are made in terms of the mean absolute percentage error (MAPE) defined as $\frac{1}{n} \sum_{i=1}^n \frac{|y_i - \hat{y}_i|}{y_i}$, where y_i is the actual market yield in day i and \hat{y}_i is the computed yield in day i .

TABLE 6: MEAN ABSOLUTE PERCENTAGE ERRORS --MAPE-- (IN %) FOR DIFFERENT MATURITIES USING CROSS-SECTIONS OF ONE-MONTH EURIBOR RATES AND THIRTY-YEAR MID-MARKET SWAP PAR RATES.

Model	Maturity					
	1 Month	1 Year	5 Years	10 Years	20 Years	30 Years
LS	0.01	9.51	21.41	28.01	24.83	0.00
SS	0.13	8.94	16.37	18.81	12.56	0.00
CIR	0.12	9.05	16.92	19.64	13.46	0.00

Comparison of Longstaff and Schwartz (1992) two-factor general equilibrium model, Schaefer and Schwartz (1984) two-factor model, and Cox Ingersoll and Ross (1985) one-factor model. The data are daily one- and twelve-month Euribor rates and five-, ten-, twenty- and thirty-year mid-market par swap rates during the period from January 1, 1999 to August 30, 2002 (948 observations). The comparisons are made in terms of the mean absolute percentage error (MAPE) defined as $\frac{1}{n} \sum_{i=1}^n \frac{|y_i - \hat{y}_i|}{y_i}$, where y_i is the actual market yield in day i and \hat{y}_i is the computed yield in day i .

TABLE 7: TEST FOR UNBIASED LINEAR PREDICTION OF MARKET YIELDS.

Maturity	Model	α_0	α_1	R^2
1 month	Longstaff-Schwartz	7.43E-06* (7.66)	1.0** (-3.65)	1.0
	Schaefer-Schwartz	0.0001* (85.47)	0.9983** (-47.80)	1.0
	CIR	8.83E-05* (87.94)	0.9999** (-42.02)	1.0
10 years	Longstaff-Schwartz	-0.0024 (-1.03)	0.7683** (-5.34)	0.2492
	Schaefer-Schwartz	0.0029 (1.79)	0.7795** (-7.18)	0.4053
	CIR	0.0022 (1.25)	0.7797** (-6.42)	0.3529
30 years	Longstaff-Schwartz	0.0072* (4.07)	0.9639 (-1.16)	0.5067
	Schaefer-Schwartz	-0.0018 (-1.54)	1.1216** (5.96)	0.7615
	CIR	-0.0009 (-0.89)	1.1053** (5.48)	0.7784

Comparison of Longstaff and Schwartz (1992) two-factor general equilibrium model, Schaefer and Schwartz (1984) two-factor model, and Cox Ingersoll and Ross (1985) one-factor model. Each model is tested for unbiased linear predictor of the actual one-month Euribor rate and ten- and thirty-year mid-market par swap rates from January 1, 1999 to August 30, 2002 (948 observations). For each model, the following regression is estimated:

$$\hat{y}_i = \alpha_0 + \alpha_1 y_i + \varepsilon_i$$

$$\varepsilon_i \sim N(0, \zeta)$$

where y_i is the actual market yield in day i , \hat{y}_i is the computed yield in day i and ζ is constant. Daily cross-sections of six different yields are used to estimate the market prices of risk. The t-values are in parentheses in columns 3 and 4, respectively.

* Significantly different from 0 at the .01 level

** Significantly different from 1 at the .01 level

FIGURE 1: SHORT-TERM RATE, LONG-TERM RATE, AND SPREAD FROM JANUARY 1, 1999 THROUGH AUGUST 30, 2002.

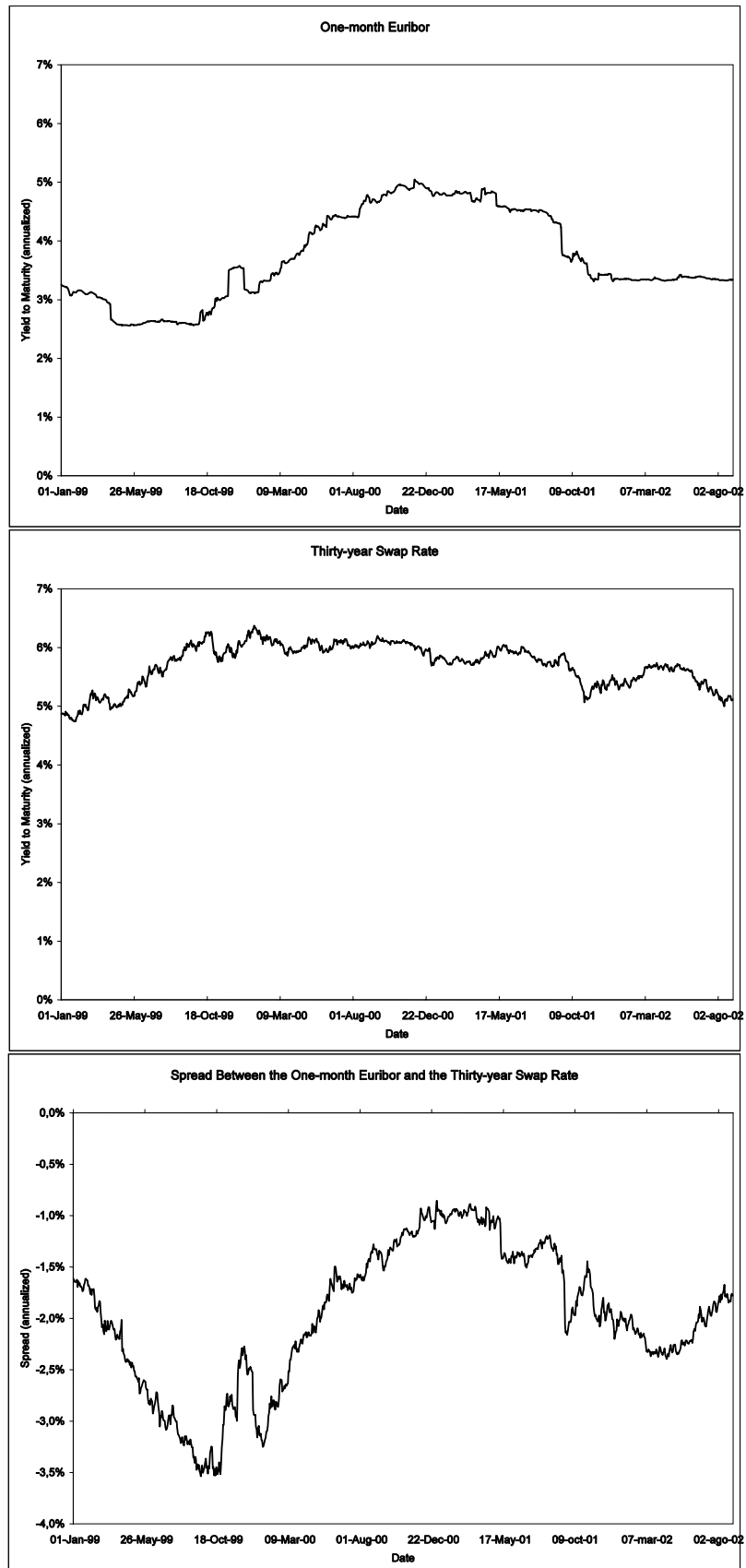


FIGURE 2: ESTIMATION OF THE MARKET PRICE OF RISK λ

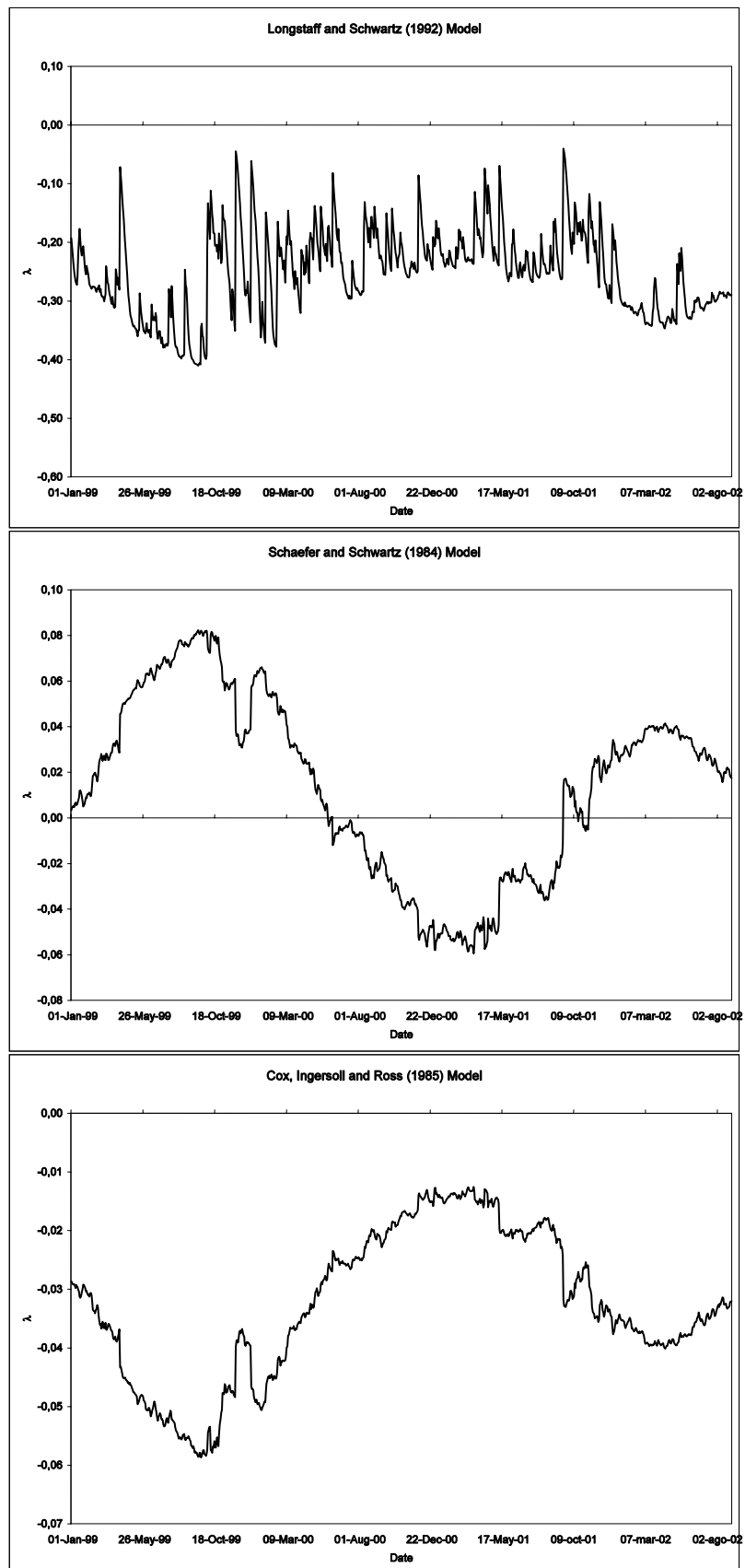


FIGURE 3: SPREAD BETWEEN THE SHORT RATE AND THE LONG RATE VERSUS $\hat{\mu}$ IN THE SCHAEFER AND SCHWARTZ (1984) MODEL.

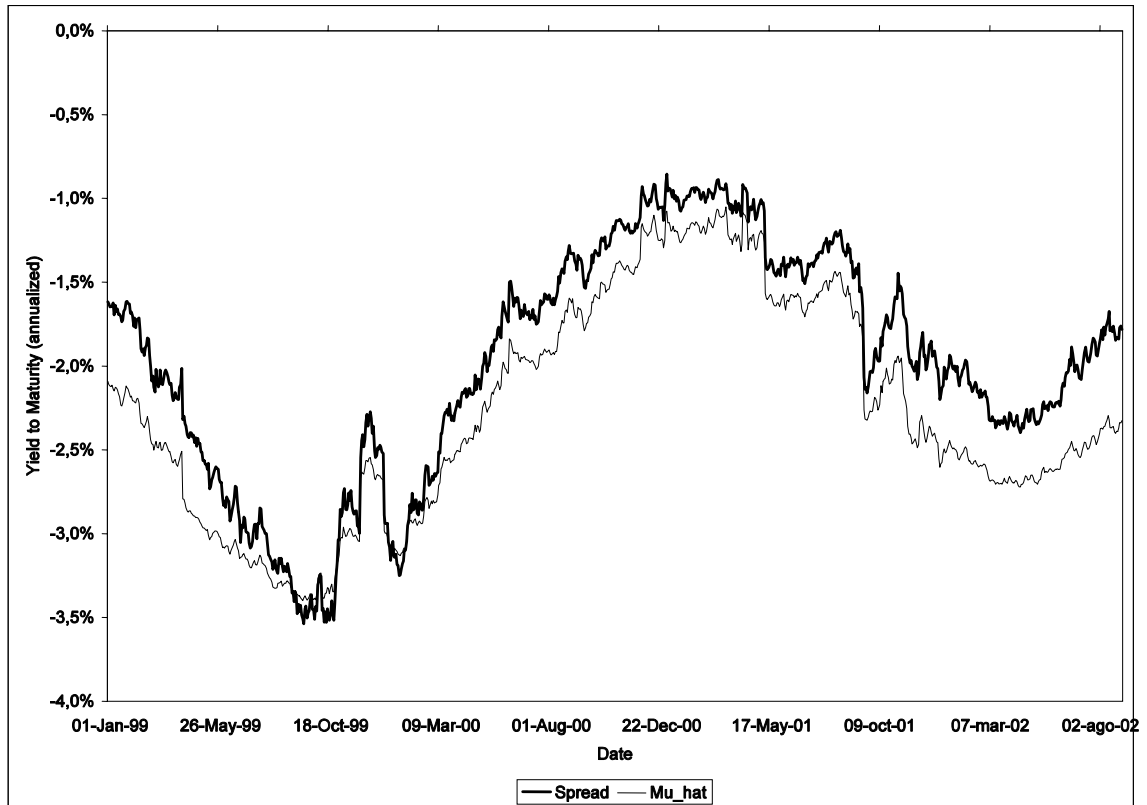


FIGURE 4: ONE-YEAR YIELDS.

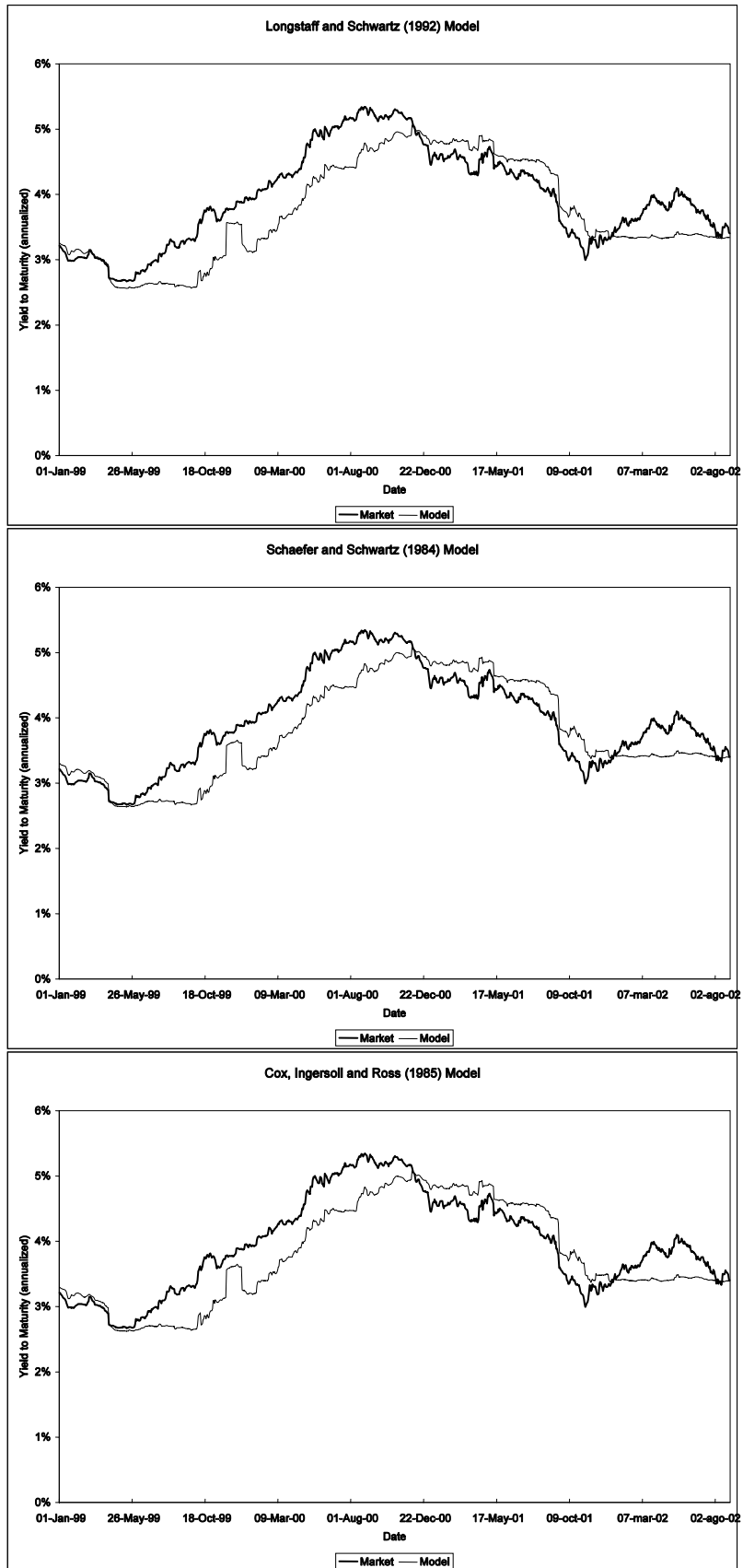


FIGURE 5: FIVE-YEAR YIELDS.

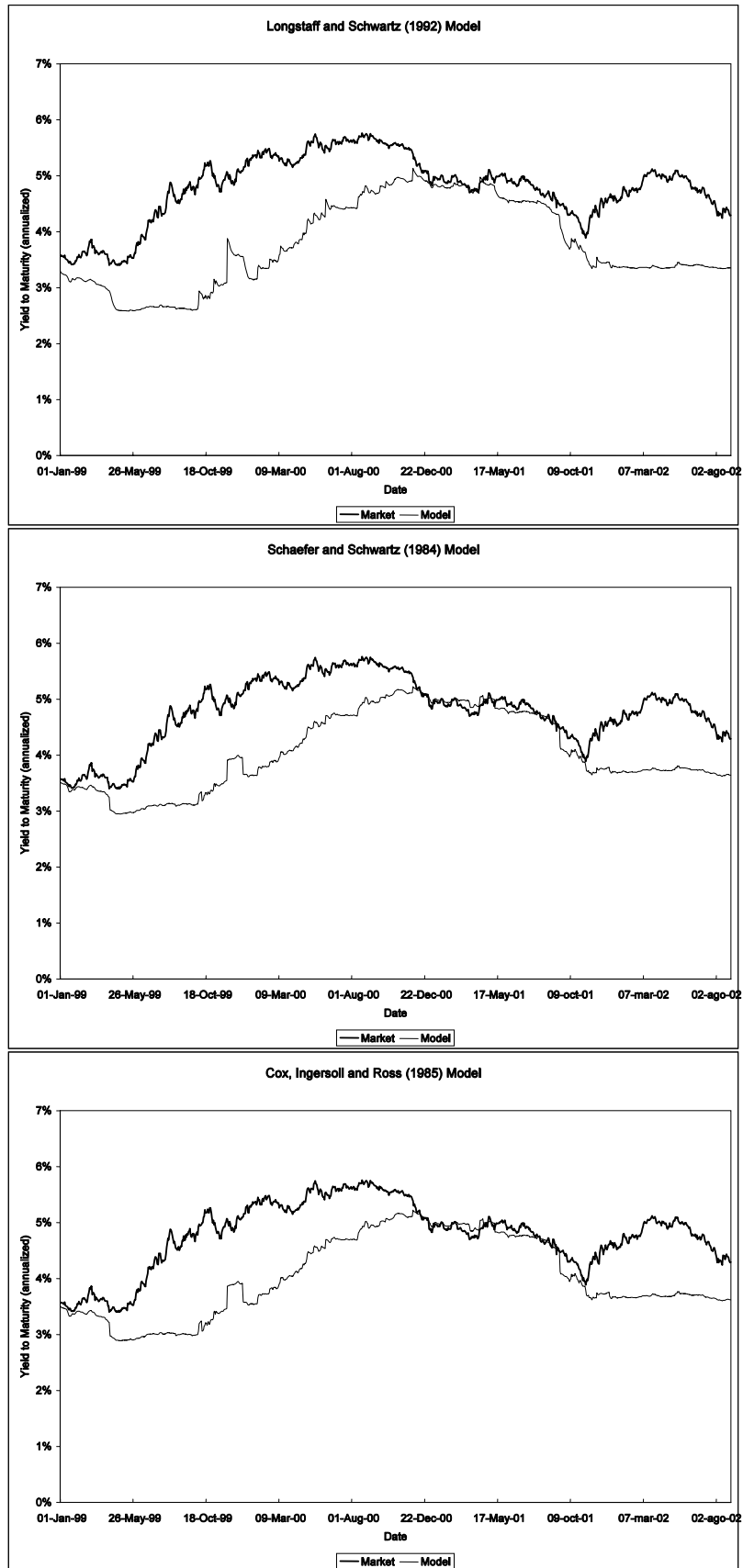


FIGURE 6: TEN-YEAR YIELDS.

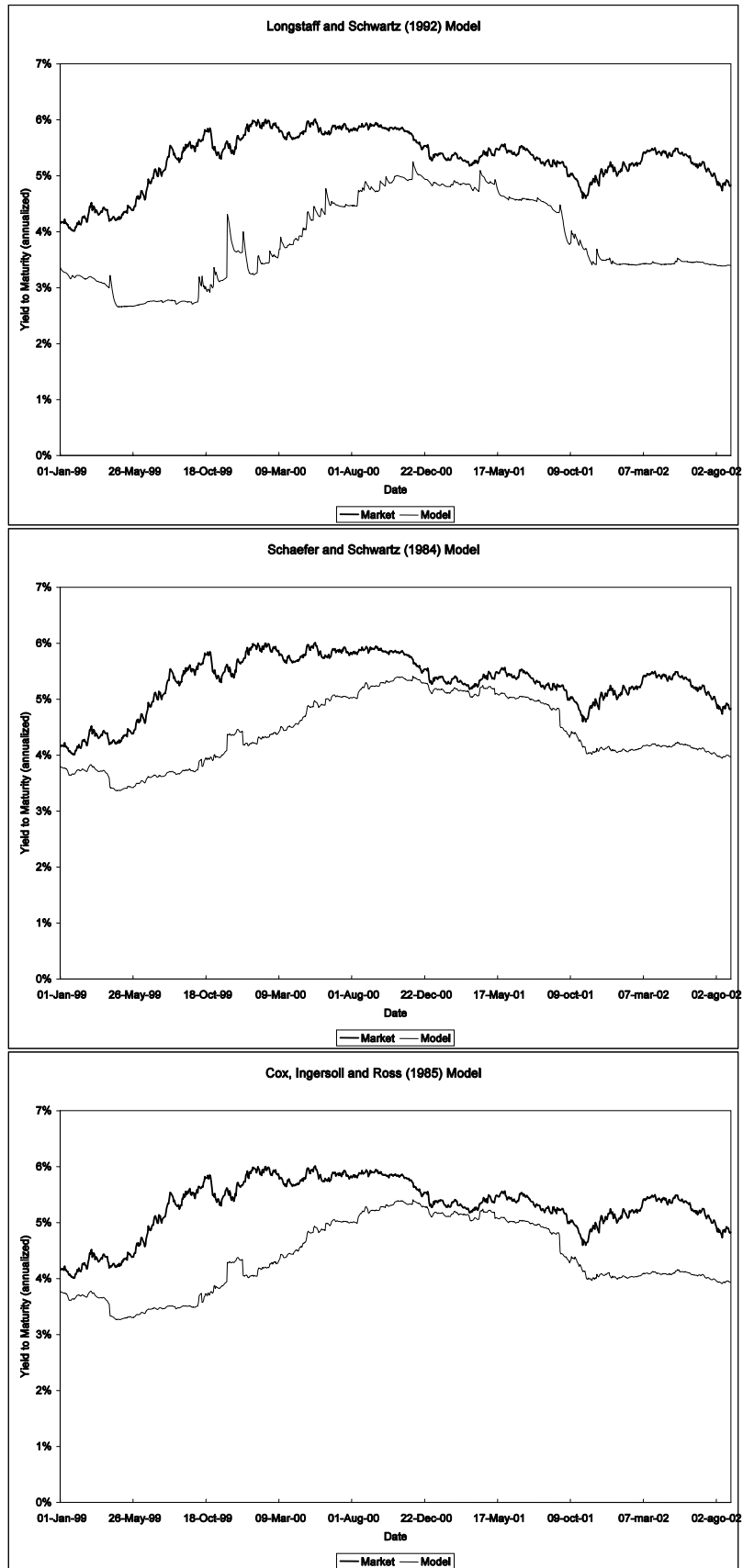


FIGURE 7: TWENTY-YEAR YIELDS.

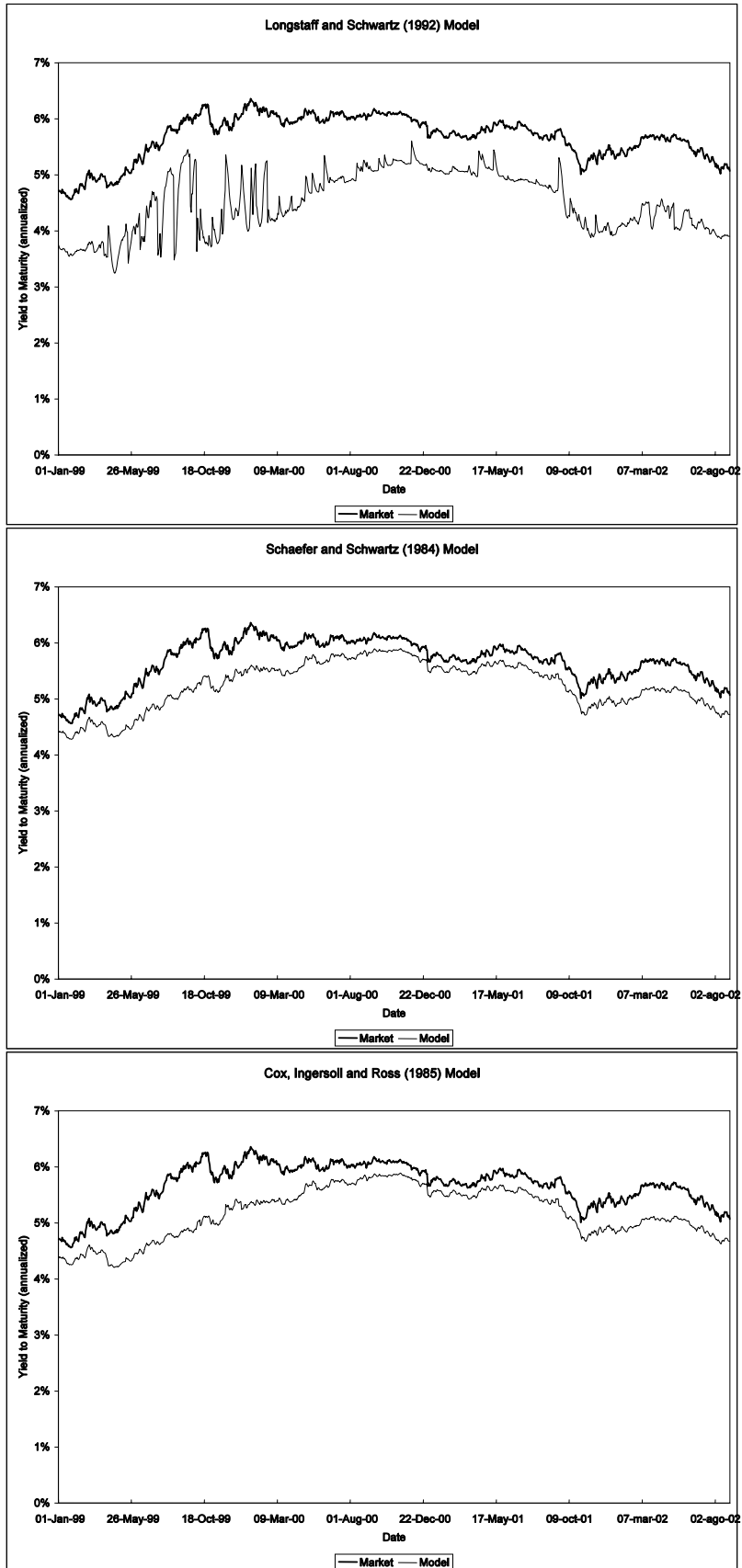


FIGURE 8: THIRTY-YEAR YIELDS.

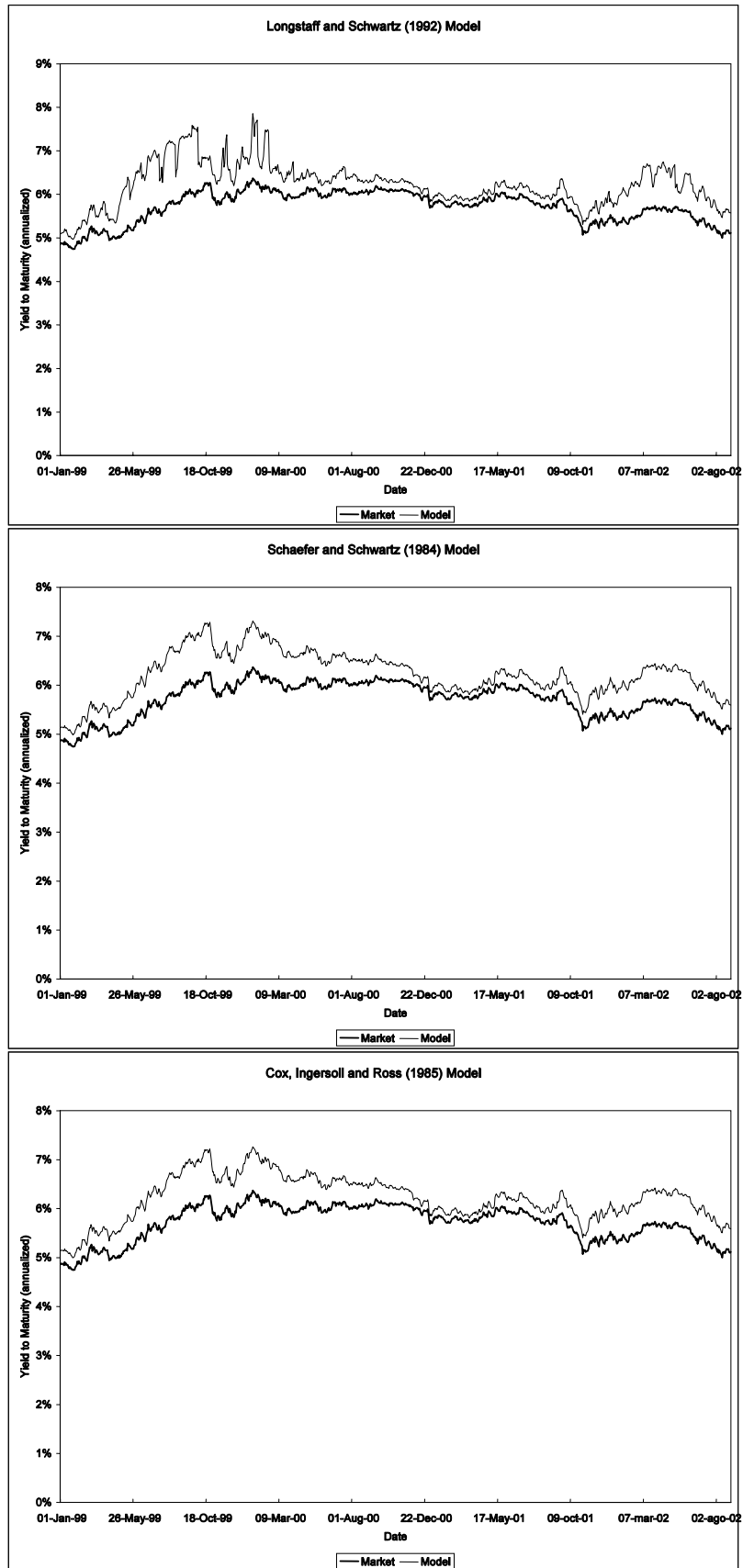
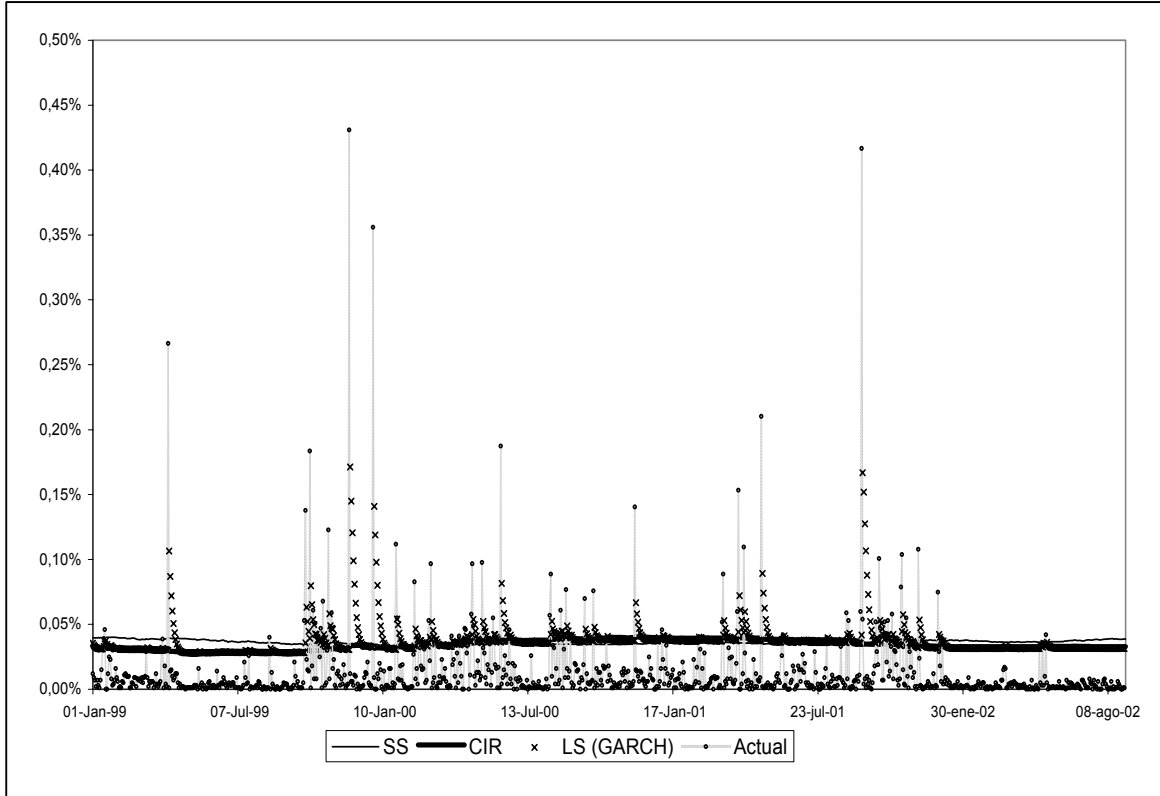
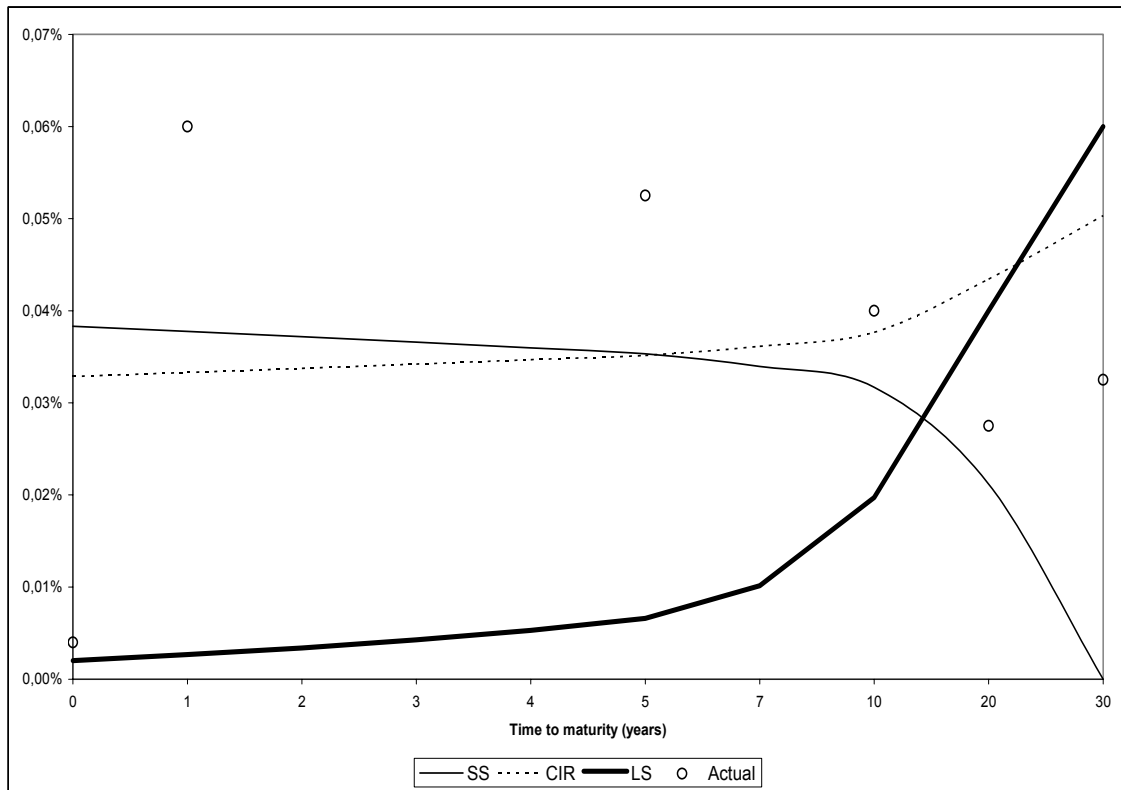


FIGURE 9: VOLATILITY OF ONE-MONTH EURIBOR RATES.



The actual volatility is measured as the absolute value of the day-to-day change in one-month Euribor rates. The LS volatility is the square root of the conditional variance implied by the estimates of the GARCH model.

FIGURE 10: TERM STRUCTURE OF VOLATILITIES ON AUGUST 1, 2002.



The actual volatility is measured as the absolute value of the day-to-day change in interest rates.