Working papers series

WP ECON 06.33

ON THE EQUIVALENCE BETWEEN COMPROMISE PROGRAMMING AND THE USE OF COMPOSITE COMPROMISE METRICS

Francisco J. André (U. Pablo de Olavide)
Carlos Romero (U. Politécnica de Madrid)

JEL Classification numbers: C61
Keywords: Compromise programming, composite metric, p-norms, economic optimization.

Department of Economics

ON THE EQUIVALENCE BETWEEN COMPROMISE PROGRAMMING AND THE USE OF COMPOSITE COMPROMISE METRICS[1].

Francisco J. André[2] and Carlos Romero[3]

Abstract. This paper analyzes the relationship between Compromise Programming and a close relative called Composite Programming that is based on the use of composite metrics. More specifically, it focuses on the possibility that the results of Compromise Programming are equivalent to those obtained with a particular case of Composite Programming in which a linear combination between the two bounds of the compromise set is established. Several situations, depending on the number of criteria involved and the mathematical structure of the efficient set, are studied. The most relevant result is obtained when two criteria are involved and the efficient set is continuously differentiable. In this case, it is possible to find a unique equivalent value of the control parameter in Composite Programming for each metric in Compromise Programming. It is remarked that this particular case is very relevant in many economic scenarios. On the other hand, it turns out that the equivalence between both approaches can not be extended to the case with more than two criteria.

Key Words. Compromise programming, composite metric, p-norms, economic optimization.
JEL code. C61

[1] Francisco J. André is grateful for financial support from the European Commission (research project EFIMAS, Proposal no. 502516) and the Spanish Ministry of Education and Science (projects SEJ2005-0508/Econ and SEJ2006-08416/ECON). The work of Carlos Romero was supported by the Spanish "Ministerio de Educación y Ciencia" under project SEJ2005-04392.
[2] Department of Economics. Universidad Pablo de Olavide. Carretera de Utrera, km.1 41008 Sevilla, Spain. E-mail: fjandgar@upo.es (corresponding autor)
[3] Departamento de Economía y Gestión Forestal, Escuela Tecnica Superior de Ingenieros de Montes, Universidad Politécnica de Madrid, Madrid, Spain. E-mail: carlos.romero@upm.es

1. Introduction

Compromise Programming (CP) is a Multiple Criteria Decision Making (MCDM) approach introduced by Yu and Zeleny in the seventies (Refs. 1-5) with many theoretical extensions and with applications in several fields. Its basic idea is to determine a subset of efficient solutions (called compromise set) that is nearest with respect to an ideal and infeasible point (called ideal point), for which all the criteria are optimized. The corresponding distance functions are introduced through a family of *p*-metrics. The basic structure of a CP model is the following:

$$\begin{aligned} &\min L_p = \left[\sum_{i=1}^{q} \left(w_i \ \left(f_i^* - f_i(x) \right) \Big/ \left(f_i^* - f_{i*} \right) \right)^p \right]^{1/p} = \left[\sum_{i=1}^{q} (w_i d_i)^p \right]^{1/p} \\ &s.t. \quad F\left[f_1(x), \ldots, f_i(x), \ldots, f_q(x) \right] = k \end{aligned} \qquad (1)$$

where x is the vector of decision variables; $f_i(x)$ is the mathematical expression for the i-th criterion ($i \in \{1,\ldots,q\}$); $f^* \equiv f_1^*(x),\ldots, f_i^*(x),\ldots, f_q^*(x)$ represents the vector of anchor values or ideal point, i.e., the optimum value for each attribute, without considering the achievement of the other attributes and $f_* \equiv f_{1*}(x),\ldots, f_{i*}(x),\ldots, f_{q*}(x)$ the vector of nadir values or anti-ideal point, i.e., the worst value of each criterion when the others are optimized; d_i stands for the degree of discrepancy for the i-th criterion (i.e., the normalized difference between the anchor value and the actual achievement of the i-th criterion); w_i is the weight or relative importance attached to the i-th criterion; $F\left[f_1(x),\ldots, f_i(x),\ldots, f_q(x) \right] = k$ is the mathematical expression of the opportunity set or the efficient set defined in the criteria space and p is the topological metric; i.e., a real number belonging to the closed interval $[0,\infty]$.

CP solutions enjoy useful economic and mathematical properties such as feasibility, Pareto optimality, uniqueness, asymmetry, etc; See Ref. 5, pp.71-74 for a rigorous analysis of these properties. Moreover, Yu and Freimer (Ref. 4) demonstrated that for the bi-criteria case metrics $p=1$ and $p=\infty$ define two bounds of the compromise set and the other best-compromise solutions fall between these two bounds. For more than two criteria this property in general does not hold. However, it was demonstrated (see Ref. 6) that under relatively general conditions (a convex feasible set limited by a differentiable hypersurface) usual in economic problems the boundness of the compromise set given by metrics $p=1$ and $p=\infty$ remains.

A composite form of CP based upon composite metrics was proposed in the eighties (see Refs. 7 and 8). A suitable particular case of this type of Composite Programming is obtained by minimizing a linear combination between the bounds corresponding to metrics $p=1$ and $p=\infty$ (e.g., Refs. 9 y 10) in the following way:

$$\begin{aligned} &\min \left\{ (1-\lambda) \cdot \max_{i=1,\dots,q} \{ w_i d_i \} + \lambda \sum_{i=1}^{q} w_i d_i \right\} \\ &s.t.: \quad F\left[f_1(x), \dots, f_i(x), \dots, f_q(x) \right] = k \end{aligned} \qquad (2)$$

Since the objective function in model (2) is not smooth, its minimization is usually performed by solving the following equivalent problem (see e.g., Ref. 9):

$$\begin{aligned} &\min \left\{ (1-\lambda) D + \lambda \sum_{i=1}^{q} w_i d_i \right\} \\ &s.t.: \quad w_i d_i \leq D \qquad i=1,\dots,q \\ &\qquad\quad F\left[f_1(x), \dots, f_i(x), \dots, f_q(x) \right] = k \end{aligned} \qquad (3)$$

where D represents the maximum degree of discrepancy. When $\lambda=1$, problem (3) collapses to the compromise problem with metric $p=1$, and for $\lambda=0$, (3) gives the compromise solution for metric $p=\infty$. For values of λ belonging to the open interval $(0,1)$ intermediate or composite solutions can be obtained if they exist. So seemingly the compromise set can be traced out or at least approximated through variations in the value of parameter λ. To undertake this task by resorting to model (2) or its smooth formulation (3) implies solving only linear programming problems, what represents an important computational advantage with respect to model (1). It should be noted that the basic parameter defining the CP model is the metric p, while the basic parameter defining model (3) is λ. The preferential meaning of p was clearly explained by Yu (Ref.1). Thus, this author demonstrates that p plays the role of a "balancing factor" between the "group utility" or average achievement of all the criteria (that is maximized for $p=1$) and the maximum discrepancy or individual regret (that is minimized for $p=\infty$).

On the other hand, control parameter λ can be interpreted in a rather similar way, as a trade-off or marginal rate of substitution between "group utility" (i.e., minimum of $\sum_{i=1}^{q} w_i d_i$) and the "utility of the criterion most displaced with respect to the solution obtained" (i.e., minimum of D).

This paper investigates the connection between model (1) and model (2)-(3), and more specifically it establishes links between metric p and control parameter λ. In short, we wonder if there is a functional relationship such as $\lambda = g\left(p\right)$ or, in other words, is it always possible to find a value of λ which allows to obtain the same solution obtained for a certain value of metric p?

In section 2 we prove that, like (1), the solution of (2)-(3) is always Pareto-efficient. In section 3 we analyze the connection under both approaches when there are two criteria and we show that, given a value of p, it is always possible to find at least one value of λ in (2)-(3) which provides the same solution. If the compromise set is continuously differentiable and not linear, then there is a function providing a unique value of λ for each p whereas if the compromise set is linear or piece-wise linear, then there is an interval correspondence between both parameters. Finally, in section 4 we show that this connection can not be extended to the case with more than 2 criteria.

2. A Preliminary Result

We start with a preliminary result which has interest in order to validate theoretically formulations (2) or (3). In fact, we are going to demonstrate the Pareto-efficient character of solutions provided by models (2) or (3).

The proof of this result is rather simple. In fact, assume that the solution of (2) is $f \equiv \left(f_1,\ldots, f_q \right)$ and it is inefficient. This means that there is another feasible solution $f\,' \equiv \left(f_1',\ldots, f_q' \right)$ which Pareto-dominates f, i.e., $d_i\,' \le d_i \quad \forall\, i = 1,\ldots, q$, and there is some $j \in \{1,\ldots,q\}$ such that $d_j\,' < d_j$. This implies $L_\infty\,' \le L_\infty$ and $L_1' < L_1$ which, in turn, implies $\lambda\, L_1' + (1\text{-}\,\lambda\,)\, D' < \lambda\, L_1 + (1\text{-}\,\lambda\,)\, D$, and this result contradicts the hypothesis that f is the solution to problem (2) or (3). We conclude that, if f is the solution of (2)-(3) it must by Pareto-efficient. Note that, in general, the solution of (2)-(3) for a given value of λ (in terms of the criteria) is not a linear combination of the solutions for $\lambda = 1$ and $\lambda = 0$.

3. Problems with two Criteria

3.1 First case: a differentiable and non-linear efficient boundary

Assume $q = 2$ and the set of efficient solutions represented by $F\left[f_1(x), f_2(x) \right] = k$, which defines implicitly f_2 as a strictly concave of function of f_1 over the compromise set.

In this case, the Lagrangean first order conditions are straightforwardly obtained from (1) as follows:

$$-\left[pw_1^p \big/ \left(f_1^* - f_{1*} \right) \right] \cdot \left[\left(f_1^* - f_1(x) \right) \big/ \left(f_1^* - f_{1*} \right) \right]^{p-1} + \mu \cdot \partial F \big/ \partial f_1(x) = 0 \tag{4}$$

$$-\left[pw_2^p \big/ \left(f_2^* - f_{2*} \right) \right] \cdot \left[\left(f_2^* - f_2(x) \right) \big/ \left(f_2^* - f_{2*} \right) \right]^{p-1} + \mu \cdot \partial F \big/ \partial f_2(x) = 0 \tag{5}$$

where μ is the Lagrange multiplier associated to the constraint of problem (1). Reordering (4) and (5) we get the following tangency condition:

$$\frac{\partial F \big/ \partial x_1}{\partial F \big/ \partial x_2} = \frac{w_1^p \left(f_2^* - f_{2*} \right)}{w_2^p \left(f_1^* - f_{1*} \right)} \left(\frac{\left(f_2^* - f_{2*} \right)\left(f_1^* - f_1(x) \right)}{\left(f_1^* - f_{1*} \right)\left(f_2^* - f_2(x) \right)} \right)^{p-1} \equiv \frac{w_1^p \left(f_2^* - f_{2*} \right)}{w_2^p \left(f_1^* - f_{1*} \right)} \left(\frac{d_1}{d_2} \right)^{p-1} \tag{6}$$

Figure 1 illustrates graphically this problem: the objective is to find a point in the boundary such that an iso-*p*-distance curve is tangent to the efficient boundary. Figure 2 illustrates the solutions for $p = 1$ and $p = \infty$. These solutions, according to Yu´s theorem, bound the compromise set.

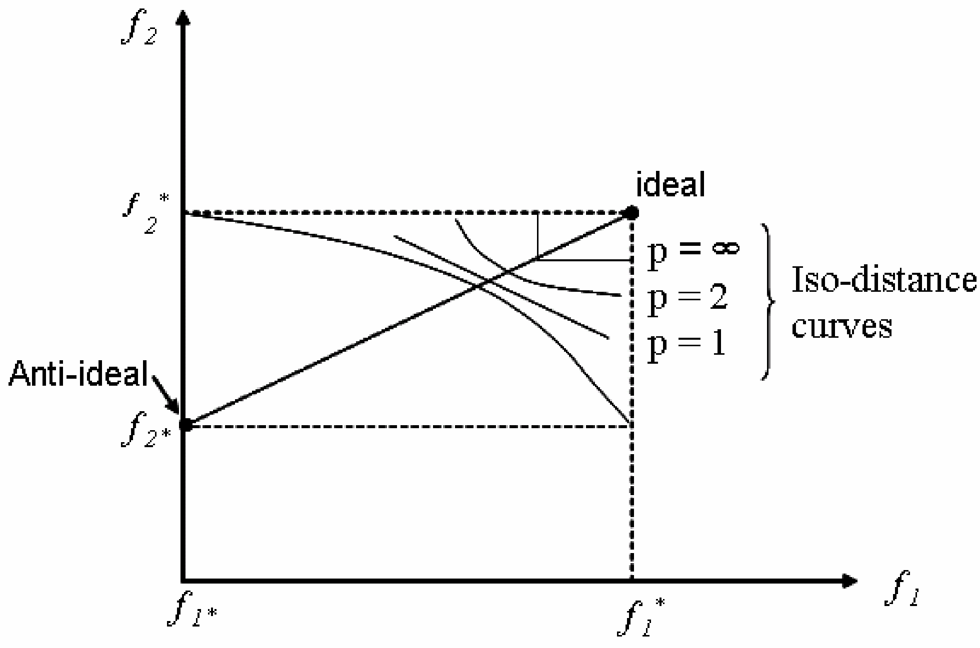

FIGURE 1. Several iso-distance curves for a Compromise Programming problem.

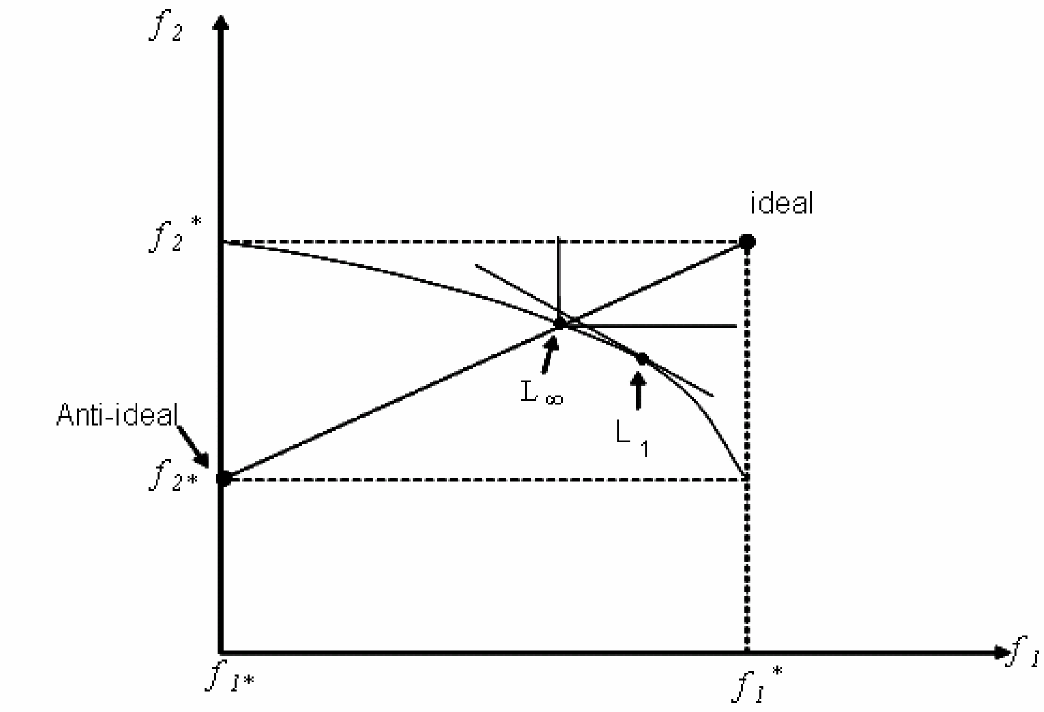

FIGURE 2. Illustration of the Compromise Programming solutions for metrics L_1 and L_∞

Without loss of generality, we can write problem (3) for the bi-criteria case in such a way that, in the solution, the maximum (weighted) degree of discrepancy corresponds to the second criterion (i.e., $w_2 d_2 \geq w_1 d_1$). Thus, we have the following equivalent optimization problem:

$$\min \ \lambda \left(w_1 d_1 + w_2 d_2\right) + \left(1\text{-}\lambda\right) w_2 d_2 = \lambda \ w_1 \frac{\left(f_1^* - f_1(x)\right)}{\left(f_1^* - f_{1*}\right)} + w_2 \frac{\left(f_2^* - f_2(x)\right)}{\left(f_2^* - f_{2*}\right)} \qquad (7)$$

$$s.t: \qquad F\left[f_1(x), f_2(x)\right] = k$$

From (7) it is straightforward to interpret parameter λ as the slope of the iso-distance or iso-utility curves (in terms of $w_2 d_2$ with respect to $w_1 d_1$). Figure 3 illustrates the shape of these iso-distance or iso-utility curves for different values of λ.

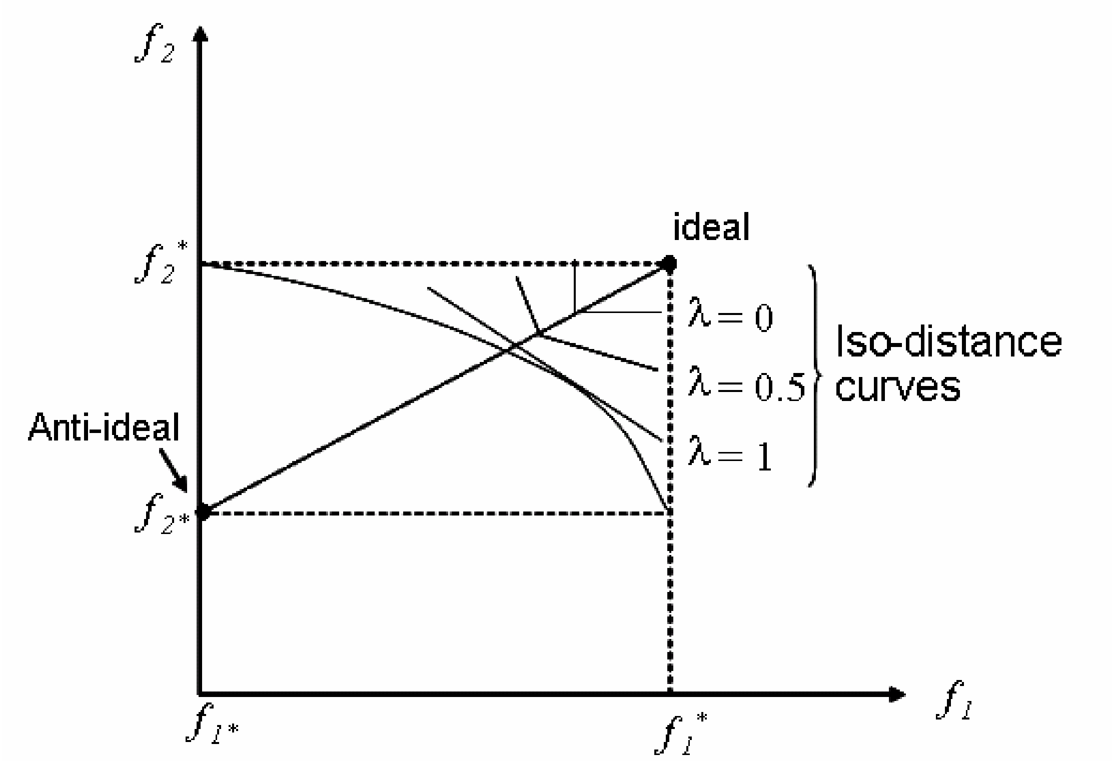

FIGURE 3. Several iso-distance curves for Composite Programming with different λ s.

The Lagrangean conditions of problem (7) are the following:

$$-\lambda w_1 \Big/ \left(f_1^* - f_{1*} \right) + \mu \cdot \partial F / \partial f_1 (x) = 0 \tag{8}$$

$$-w_2 \Big/ \left(f_2^* - f_{2*} \right) + \mu \cdot \partial F / \partial f_2 (x) = 0 \tag{9}$$

and rearranging (8) and (9) we get the following tangency condition:

$$\frac{\partial F / \partial f_1(x)}{\partial F / \partial f_2(x)} = \lambda w_1 \left(f_2^* - f_{2*} \right) \Big/ \left[w_2 \left(f_1^* - f_{1*} \right) \right] \tag{10}$$

For problems (1) and (2) in order to have the same solution, given p, we must find a value of λ such that the solution satisfies simultaneously both set of first order conditions. Equivalently, we need that the tangency conditions (6) and (10) hold at the same time.

Equalizing (6) and (10) and rearranging terms, we get the following relationship between p and λ:

$$\lambda = \left({w_1 d_1} \middle/ {w_2 d_2} \right)^{p-1} \tag{11}$$

or, more generally,

$$\lambda = \left(min\left\{ w_1 d_1, w_2 d_2 \right\} \middle/ max\left\{ w_1 d_1, w_2 d_2 \right\} \right)^{p-1} \tag{12}$$

Since problem (3) is usually computationally easier to solve than (1), in practice it can be more useful to solve (3) for a given λ and then find the equivalent value of p to sort out which metric exactly corresponds to the solution found. This can be done by obtaining the inverse function in (11) or in (12). Thus, we get the following inverse function:

$$p = 1 + log_a(\lambda) \tag{13}$$

where $log_a(\lambda)$ denotes the logarithm of λ to the basis a and $a \equiv w_1 d_1 / w_2 d_2$ or, more generally, $a \equiv min\{w_1 d_1, w_2 d_2\} / max\{w_1 d_1, w_2 d_2\}$.

In this case we get the following conclusions: given a value of p (λ), it is possible to find a unique λ (p) such that the slope of the iso-distance or iso-utility curves are the same in the optimum and, therefore, both problems have the same solution. Nevertheless, it is not possible to find a general a-priori relationship between both parameters, since this relationship depends on the value of the degrees of discrepancy (and, hence, on the value of the criteria) in the solution and this solution, in general, depends on the structure of the problem.

Example 1

To check this relationship, assume that the efficient boundary is given by the following equation:

$$F\left(f_1(x), f_2(x)\right) = \left(f_1(x) - 50\right)^2 \big/ 10000 + \left(f_2(x)\right)^2 \big/ 400 = 10 \tag{14}$$

being $f_1(x) = x_1$ and $f_2(x) = x_2$. Moreover, the additional constraint $f_2(x) \leq 50$ is considered. For simplicity, assume also $w_1 = w_2$. The ideal point is (366.2278, 50) and the anti-ideal point is (243.6492, 0). The following table shows the solutions of problem (1) for different values of p, the associated value of λ according to equation (12), and the solution of problem (3) using this value of λ.

Solution problem (1) (with *p* metric)					**Solution problem (2) (with λ)**		
P	$f_1(x)$	$f_2(x)$	d_1	d_2	λ	$f_1(x)$	$f_2(x)$
1	333.93	27.84	0.26345	0.44313	1	333.93	27.84
2	325.63	31.00	0.33123	0.37993	0.8718	325.63	31.00
3	324.27	31.48	0.34232	0.37036	0.8543	324.27	31.48
4	323.71	31.67	0.34684	0.36652	0.84741	323.71	31.67
5	323.41	31.78	0.34929	0.36445	0.84372	323.41	31.78
6	323.22	31.84	0.35082	0.36315	0.84143	323.22	31.84
10	322.87	31.96	0.35369	0.36074	0.83718	322.87	31.96
15	322.71	32.02	0.35504	0.35962	0.83563	322.74	32.01
∞	322.40	32.12	0.35754	0.35754	0	322.40	32.12

By construction, the solution for $\lambda = 1$ (0) is always identical to that with $p = 1$ (∞). For low values of p, the analytic relationship (11) exactly holds in the example. For larger values of p, d_1 and d_2 get closer and their ratio tends to 1. The calculation of λ then gets computationally more imprecise, since it involves raising a number very close to one to a very high power (note that 1^∞ is a mathematical indeterminacy). Nevertheless, in the example, the results are very exact for $p \leq 15$ and, for $p > 15$, the solution is virtually identical to the solution corresponding to metric $p = \infty$.

3.2 Second case: linear or piece-wise linear efficient boundary

If the efficient boundary (and therefore, the compromise set) is linear or piece-wise linear we have the following situation: the iso-distance or iso-utility curves of problem (1) are still non-linear so that problem (1) can have both corner solutions and (unique) interior solutions. On the other hand, the iso-distance curves of problem (2) are piece-wise linear and we could only get corner solutions (if the slope of the iso-distance curves is different from that of the efficient set) or multiple solutions (if the slope of the iso-distance curves is exactly the same as that of the compromise set). Hence, it is not possible in general to find a unique and precise value of λ for each value of p, and vice versa. Typically, it will be possible to find an interval correspondence between metric *p* and control parameter λ. This situation is clarified in what follows with the help of two examples. The first example illustrates the situation when the compromise set is linear, and the second one when the compromise set is piece-wise linear. Finally, we summarize the general results for this case.

Example 2: a linear compromise set

Assume a MCDM problem in which the feasible set is defined in the criteria by the following constraints:

$$\begin{aligned} &f_1(x) \leq 100; \\ &f_2(x) \leq 100; \\ &4 f_1(x) + f_2(x) \leq 400 \end{aligned} \qquad (15)$$

being $f_1(x) = x_1$ and $f_2(x) = x_2$. Assume also $w_1 = w_2$.

The ideal point is (100, 100) and the anti-ideal point (0,0). The efficient set is the linear segment which connects the points (75,100) and (100,0) as it is shown in Figure 4. When problem (1) is solved for different values of p the solution shifts smoothly from the L_1 solution (75, 100) to the L_2 solution (80, 80) and the compromise set is the linear segment between these two bounds. Figure 4 shows the feasible set and the solutions of problem (1) for some values of metric p. The situation is rather different for problem (2). Figure 5 shows the solutions for $\lambda = 0$, $\lambda = 1$ and $\lambda = 0.25$. It is easy to see that, for any $\lambda < 0.25$ we get the solution corresponding to L_1 For any $\lambda > 0.25$ we get the L_∞ solution and for $\lambda = 0.25$, the slope of the iso-distance curve is the same as that of the compromise set and we get multiple solutions, since any point of the compromise set provides exactly the same value of the objective function of (2). Then we conclude that it is not possible to find a unique value of λ for each value of p and vice versa, but there is an interval correspondence between both parameters. The following table shows the solution for different values of λ and the value(s) of p which corresponds to each λ:

λ	*f*	f_2	*p*
[0, 0.25)	75.00	100.00	1
0.25	$(f_1, f_2) \in [L_1, L_\infty]$		$[1, \infty]$
(0.25, 1]	80.00	80.00	∞

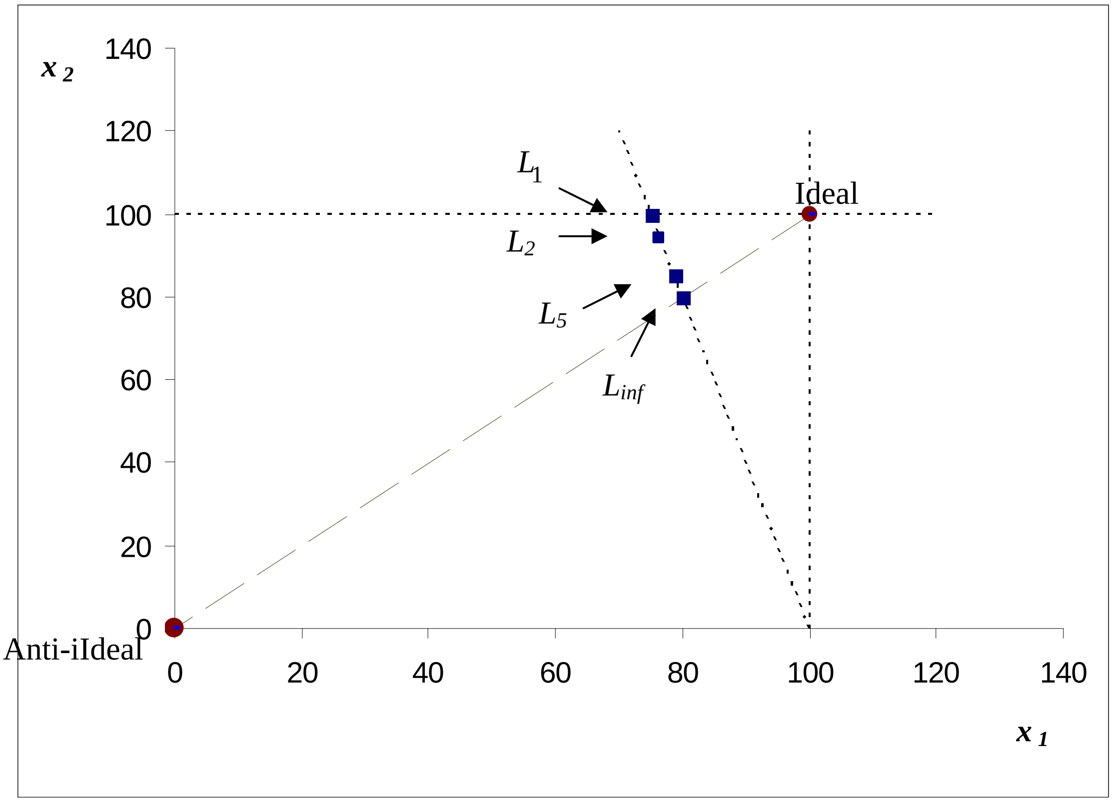

FIGURE 4. Example 2. Compromise Programming solutions for several values of p .

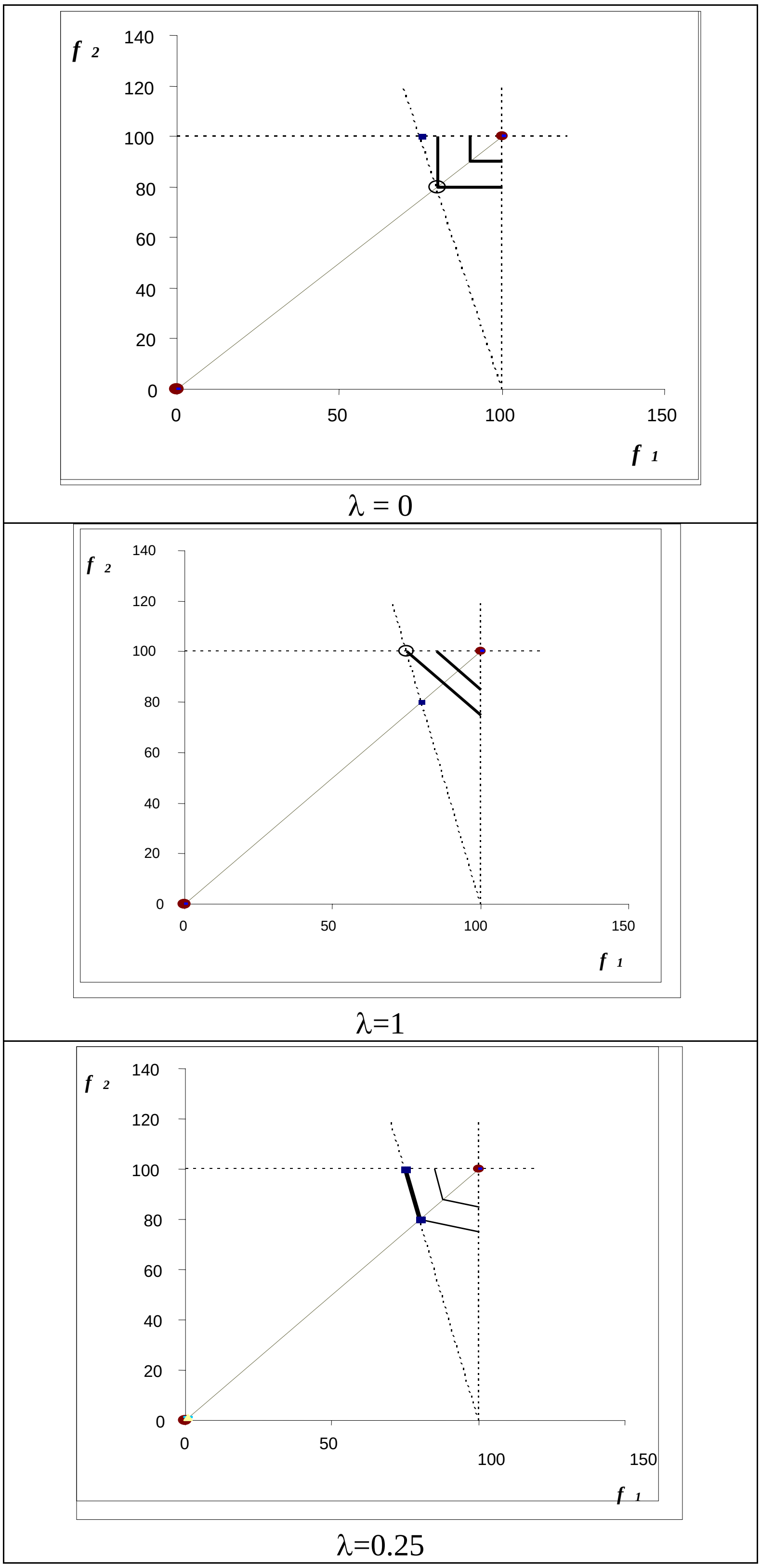

FIGURE 5. Example 2. Solutions for different values of λ.

Example 3: a piece-wise linear compromise set

Assume the following feasible set defined in the criteria space:

$$\begin{aligned} &f_1(x) \le 120; \\ &f_2(x) \le 100; \\ &f_1(x) + 5f_2(x) \le 500; \\ &f_1(x) + 2.7f_2(x) \le 320 \end{aligned} \qquad (16)$$

being $f_1(x) = x_1$ and $f_2(x) = x_2$. Assume also $w_1 = w_2$.

As it is shown in Figure 6 the compromise set is piece-wise linear and it is delimited by points ABC, where A = (120, 74.07), corresponding to the L_1 solution, B = (108.70, 78.26) and C = (96.77, 80.65), corresponding to the L_∞ solution.

When problem (1) is solved for $p \in (1, 1.97)$ the solution shifts smoothly in the interval (A,B). For $p \in [1.97, 2.71)$ the solution is always B. Finally, for $p \in [2.71, \infty)$ the solution changes smoothly in the interval (B ,C). Regarding problem (1), for $\lambda \in [0, 0.24)$ we get C as a unique solution. For $\lambda = 0.24$ any point in the segment [B,C] is a solution. For $\lambda \in (0.24, 0.44)$, we get B as a unique solution. If $\lambda = 0.44$, any point in the segment [A,B] is a solution. Finally, for $\lambda \in (0.44, 1]$, we get A as a unique solution. The following table shows the solution for different values of λ and the equivalent value(s) of p.

λ	***Solution***	***Type***	***p***	***Solution***	***Type***
[0, 0.24)	C	Unique	∞	C	Unique
0.24	$[B,C]$	Multiple	[2.71, ∞)	(B,C)	Unique
(0.24, 044)	B	Unique	[1.97, 2.71)	B	Unique
0.44	[A,B]	Multiple	(1, 1.97)	(A,B)	Unique
(0.44,1]	A	Unique	1	A	Unique

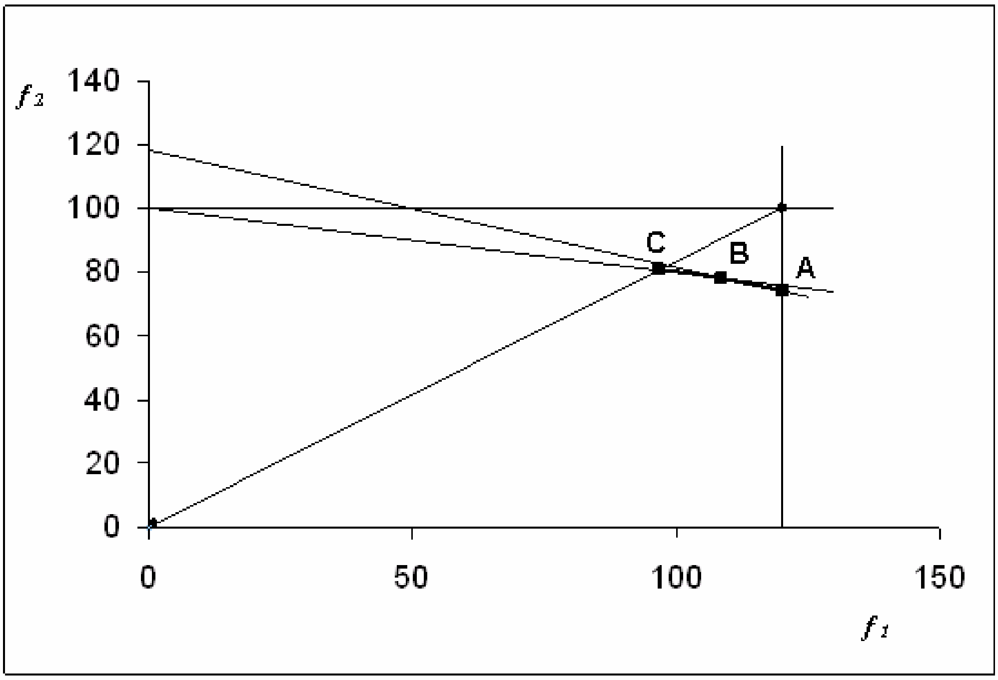

FIGURE 6. Example 3. Feasible set and compromise set.

General results for the two criteria case and a piece-wise linear compromise set

Given a MCDM problem with two criteria, assume the compromise set is continuous, concave towards the origin, and piece-wise linear with a finite number I-1 of linear segments connecting I extreme points. The problem is written in such a way that, in all the points of the compromise set, it holds $d_2 \geq d_1$ (the maximum degree of discrepancy corresponds to the second attribute). Let $P^i \equiv \left(f_1^i, f_2^i \right)$ denote the i-th extreme point in terms of the attributes, where the points are numbered in such a way that P^1 is the L_1 solution, P^2 is the next corner point which is closest to the L_1 solution and so on. Finally, P^I is the L_∞ solution.

Define

$$S^i = \left| w_2 \left(d_2^{i+1} - d_2^i \right) \Big/ \left[w_1 \left(d_1^{i+1} - d_1^i \right) \right] \right| \qquad i = 1, \dots, I-1 \qquad (17)$$

$$p^i \equiv \left\{ p \in [1,\infty] / P^i = \arg \min_{(f_1,f_2)} L_p \right\} \qquad i = 1,\ldots,I \tag{18}$$

meaning that S^i is the slope (in terms of w_2d_2 with respect to w_1d_1) of the i-*th* segment, i.e., the segment connecting points P^i and P^{i+1}, and $|\bullet|$ denotes absolute value. As the compromise set is concave towards the origin, we know $S^i > S^{i+1} \;\forall\, i$. In turn, p^i is the set of L_p metrics supporting point P^i as a solution, i.e., those values of p such that P^i is the solution to the problem of minimizing L_p. By definition, we know that $1 \in p^1$ and $\infty \in p^I$.

First, we conclude that, if $\lambda = S^i$ holds, then

$$\arg \min_{(f_1,f_2)} \left\{ (1-\lambda) \cdot \max_{i=1,\ldots,q} (w_i d_i) + \lambda \sum_{i=1}^{q} w_i d_i \right\} = \overline{P^i P^{i+1}} \tag{19}$$

where $\overline{P^i P^{i+1}}$ is the segment connecting points P^i and P^{i+1}. In words, since λ is equal to the slope of segment $\overline{P^i P^{i+1}}$, any point in this segment is a solution of problem (2)-(3) with this specific value of λ.

On the other hand, if it holds that $\lambda \in \left(S^{i+1}, S^i\right)$, then

$$\arg \min_{(f_1,f_2)} \left\{ (1-\lambda) \cdot \max_{i=1,\ldots,q} (w_i d_i) + \lambda \sum_{i=1}^{q} w_i d_i \right\} = P^{i+1} \tag{20}$$

and we conclude that solving problem (2)-(3) with this value of λ gives a unique solution, which is the same as that obtained when solving problem (1) with any value of p in the set p^{i+1}. We come up with the relationship between λ and p given in the following table:

λ	***Solution***	***Type***	***p***
$\left[0, S^{I-1}\right)$	L_∞	Unique	p^I
…	…	…	…
S^{i+1}	$\overline{P^{i+2}, P^{i+1}}$	Multiple	$\left[max\ p^{i+1}, min\ p^{i+2}\right]$
$\left(S^{i+1}, S^i\right)$	P^{i+1}	Unique	p^{i+1}
S^i	$\overline{P^i, P^{i+1}}$	Multiple	$\left[max\ p^i, min\ p^{i+1}\right]$
…			
$\left(S^1, 1\right]$	L_1	Unique	1

4. General Scenario with q Criteria

From section 3, we conclude that, with two criteria, the best possible situation (for the sake of relating problems (1) and (2)-(3)) happens when the compromise set is continuously differentiable and not linear, because in this case there is a unique relationship between p and λ.

In this section, we will demonstrate that the above result cannot be extended to a general scenario involving q criteria. The first order conditions for problem (1) are now the following:

$$-\left(pw_i^p / f_i^* - f_{i*}\right) \cdot \left[\left(f_i^* - f_i(x)\right) \Big/ \left(f_i^* - f_{i*}\right)\right]^{p-1} + \mu \cdot \partial F / \partial f_i(x) = 0 \quad i = 1, \ldots, q \quad (21)$$

and rearranging terms, we get the following tangency condition:

$$\frac{\partial F / \partial f_i(x)}{\partial F / \partial f_j(x)} = \frac{w_i^p\ (f_j^* - f_{j*})}{w_j^p\ (f_i^* - f_{i*})} \left(\frac{\left(f_j^* - f_{j*}\right)\left(f_i^* - f_i(x)\right)}{\left(f_i^* - f_{i*}\right)\left(f_j^* - f_j(x)\right)} \right)^{p-1} \qquad i, j \in \{1, \ldots q\} \qquad (22)$$

In order to obtain the first order conditions for problem (3) we are going to assume without loss of generality that $\max\left\{d_1,\ldots d_q\right\}=d_q$, that is, the maximum deviation or degree of discrepancy corresponds to attribute q. Then the problem can be expressed in the following way:

$$\min \quad \sum_{i=1}^{q} w_i d_i \; + \; (1\text{-}\,\lambda)\,)d_q = \lambda \sum_{i=1}^{q\text{-}1} w_i \frac{\left(f_i^* - f_i(x)\right)}{\left(f_i^* - f_{i^*}\right)} \; + \; w_q \frac{\left(f_q^* - f_q(x)\right)}{\left(f_q^* - f_{q^*}\right)} \tag{23}$$
$$s.t: \qquad F\left[\,f_1(x),\ldots,f_q(x)\right]=k$$

The first order conditions are

$$w_i\left[-\lambda\Big/\left(f_i^* - f_{i^*}\right)+\mu\,\partial F\big/\partial f_i\left(x\right)\right]=0 \quad i=1,\ldots q-1 \tag{24}$$

$$w_q\left[-1\Big/\left(f_q^* - f_{q^*}\right)+\mu\cdot\partial F\big/\partial f_q\left(x\right)\right]=0 \tag{25}$$

and rearranging terms, we get the following tangency conditions:

$$\frac{\partial F\big/\partial f_i(x)}{\partial F\big/\partial f_q(x)}=\lambda w_i\left(f_q^* - f_{q^*}\right)\Big/\left[w_q\left(f_i^* - f_{i^*}\right)\right] \quad i=1,\ldots,q-1 \tag{26}$$

$$\frac{\partial F\big/\partial f_i(x)}{\partial F\big/\partial f_j(x)}=w_i\left(f_j^* - f_{j^*}\right)\Big/\left[w_j\left(f_i^* - f_{i^*}\right)\right] \quad i,j=1,\ldots,q-1 \tag{27}$$

Note that these conditions are different depending on if they include or not attribute q. Consider two criteria different from the q-th one. For a point to be a solution of both problems at the same time, it must satisfy simultaneously (22) and (27). Using both conditions, we have:

$$\frac{w_i^p(\,f_j^* - f_{j^*})}{w_j^p(\,f_i^* - f_{i^*})}\left(\frac{d_i}{d_j}\right)^{p-1} = w_i\left(f_j^* - f_{j^*}\right)\Big/\left[w_j\left(f_i^* - f_{i^*}\right)\right]\Rightarrow\left(\frac{w_i d_i}{w_j d_j}\right)^{p-1}=1 \tag{28}$$

and this implies $w_i d_i = w_j d_j$ which, in general, does not hold in an arbitrary point of the compromise set.

This result has the following interpretation. In order to obtain a precise mapping between metric p and the control parameter λ (that is, an equivalence of solutions between problems (1) and (2)-(3)), it is necessary that the slope of the iso-distance curves involved by problem (2) coincide with the respective slopes in the optimum. When there are two criteria, there is only one relevant slope that can be univocally replicated by modifying the value of λ. However, when there are more than criteria the number of possible slopes coincide with the number of "paired" criteria. Hence, with a single value of the parameter is not possible to replicate so many slopes. Thus, in problem (2) when the criterion considered is not the q-th one, then the slopes of the objective function are always equal to 1 (in terms of the weighted discrepancies), and when one of the two criteria is the q -th the slope is equal to λ. In short, given different values to control parameter λ, it will be possible to replicate only those solutions such that $d_i = d_j \quad \forall i, j = 1, \ldots, q$.

5. Conclusions

Model (2)-(3) represents a particular case of a composite form of CP, which is usually taken as a surrogate of classic CP (model (1)). However, although both problems are rather similar in spirit, they are not always fully equivalent. In fact, it has been demonstrated in this paper that the similarities are actually very strong only in the bi-criteria case, when the efficient set is given by a continuously differentiable and not linear function (such that one of the attributes is implicitly defined as a strictly concave function of the other one). In this case, the compromise set can be traced out by changing the value of metric p or by changing the value of control parameter λ. Moreover, for this particular case there is a rather stable relationship between both parameters in the sense that every value of p has a unique equivalent value of λ and vice versa (see expressions (11)-(13)). However, when the differentiability of the efficient set disappears or the number of criteria involved is more than two, then the close relationship between the two approaches almost vanishes.

With two criteria and a linear or piece-wise linear efficient set, it is still possible to find some relationship between p and λ, and for every point of the compromise set, there is always at least one value of λ such that this point is obtained as a solution of (2)-(3). Nevertheless, it is not possible to find a function that links each value of p with a unique value of λ and vice versa. On the other hand, with more than two criteria, in general, by solving problem (2)-(3) it is not possible to trace out the compromise set. Since we have proven that problem (2)-(3) always provides efficient solutions, we can conclude that, by solving (1) we obtain the compromise set, which is a subset of the efficient set, and by solving (2)-(3) we get a bundle of solutions that is a different subset of the efficient set.

It is important to notice that the result obtained for the bi-criteria case under the conditions of continuity and differentiability of the efficient set are specially relevant in economics. In fact, this type of mathematical properties are usually assumed in economics and besides there are many relevant economic problems defined in a bi-criteria space. Some examples are the following. The Markowitz approach in portfolio selection, the Philips curve in macroeconomic policies, the Baumol sales-revenue hypothesis in firm theory, the determination of the optimal level of externality in environmental economics, the trade-off curve between efficiency and equity in welfare economics, etc (see Ref. 11). Therefore, the empirical interest of this result is clearly reinforced.

References

1. YU, P.L., *A Class of Solutions for Group Decision Problems*, Management Science, Vol. 19, pp.936-946, 1973.
2. ZELENY, M., *Compromise Programming, Multiple-Criteria Decision Making*, Edited by J.L. Cochrane and M. Zeleny, University of South Carolina Press, Columbia, South Carolina, pp. 262-301, 1973.
3. ZELENY, M., *A Concept of Compromise Solutions and the Method of the Displaced Ideal*. Computers and Operations Research, Vol. 1, pp. 479-496, 1974.
4. FREIMER. M., and YU, P.L. *Some New Results on Compromise Solutions for Group Decision Problems*, Management science, Vol. 22, pp.688-693, 1976.
5. YU, P.L., *Multiple-Criteria decision Making: Concepts, Techniques, and Extensions,* Plenum Press, New York, New York, 1985.
6. BLASCO, F., CUCHILLO-IBÁÑEZ, E., MORÓN, M.A., and ROMERO, C., *On the Monotonicity of the Compromise Set in Multicriteria Problems*, Journal of Optimization Theory and Applications, Vol. 102, pp. 69-82, 1999.
7. MICHALOWSKY, W., *The Choice of Final Compromise Solution in Multiple Criteria Linear Programming Problem, Organizations: Multiple Agents with Multiple Criteria,* Edited by J. N. Morse, Springer-Verlag, Berlin, pp. 233-238, 1981.
8. BÁRDOSSY, A., and BOGÁRDI, I., *Network Design for the Spatial Estimation of Environmental Variables*, Applied Mathematics and Computation, Vol. 12, pp. 339-365, 1983.
9. NAKAYAMA, H., *Trade-off Analysis Using Parametric Optimization Techniques,* European Journal of Operational Research, Vol. 60, pp. 87-98, 1992.

10. ANDRÉ, F.J., CARDENETE, M.A and ROMERO, C., *Using Compromise Programming for Macroeconomic Policy Making in a General Equilibrium Framework: Theory and Applications to the Spanish Economy*. Submitted to the Journal of the Operational Research Society.

11. BALLESTERO, E., and ROMERO,.C., *Multiple-Criteria Decision Making and its Application to Economic Problems*, Kluwer Academic Publishers, Boston, Massachusetts, 1998.