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Multi-Utilitarian Bargaining Solutions

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Multi-Utilitarian Bargaining Solutions

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Abstract

This paper introduces and analyzes the class of multi-utilitarian solutions for cooperative bargaining problems. We show that generalized Gini solutions and inequality averse Choquet bargaining solutions are particular cases of this new multi-valued solution concept and provide a complete characterization of inequality averse multi-utilitarian solutions in which an invariance property consisting of a weakening of both the linear invariance axiom in Blackorby *et al.* (1994) and the restricted invariance axiom in Ok and Zhou (2000). Moreover, by relaxing the assumptions involved in the characterization, the class is extended to include equality averse multi-utilitarian solutions which are also studied in the paper.

Keywords: Axiomatic bargaining theory, multi-valued bargaining solutions, generalized Gini solutions, inequality averse Choquet solutions.

1 Introduction

A cooperative bargaining problem consists of a set of agents and a feasible set of utility payoffs. This paper deals with bargaining solutions. In cooperative bargaining theory it is commonly assumed that solutions are single-valued. However, in this paper more general solutions are permitted, they are assumed to be multi-valued, that is, choice correspondences that assign a connected subset of the feasible utility payoffs to each problem. Most solutions proposed in the literature share a common principle of rationality: they are consistent with the maximization of some ordering of the utility space. Following this principle, Blackorby *et al.* (1994) introduced and characterized generalized Gini bargaining solutions, which are rationalized by generalized Gini orderings of the utility vectors. These orderings can be represented by quasi-concave, increasing social welfare functions that are linear in some particular cones of $\mathbb{R}^n_+$.

Ok and Zhou (2000) provided a characterization of a wider family of bargaining solutions, known as Choquet bargaining solutions. In their approach, the authors substituted the Linear Invariance axiom of Blackorby *et al.* (1994) with a weaker one: Restricted Invariance. These Choquet solutions correspond to orderings that can be

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represented by social welfare functions derived from Choquet integrals with respect to a monotonic capacity (see Schmeidler, 1986). These functions are increasing and linear in every rank-ordered subset of $\mathbb{R}^n_+$, but not necessarily quasi-concave nor anonymous as in the generalized Gini solutions.

In the present paper we go a step further by introducing and characterizing a related class of bargaining solutions, the class of multi-utilitarian solutions. The solutions in this class are also rationalized by piecewise linear increasing social welfare functions defined on the agents' utility gains. They depend on a set of weighted utilitarian criteria and on the decision rule used to combine them.

If a conservative principle is to be applied to combine the different utilitarian criteria, then each outcome will be measured by the one that provides the minimum value. The bargaining solutions induced in this case are inequality averse and their analysis is the main goal of this paper.

Every inequality averse multi-utilitarian bargaining solution has an associated family of cones that covers $\mathbb{R}^n_+$ and is rationalized by an increasing social welfare function which is linear in every cone of the family. Since they can be represented as the minimum of a set of linear functions, the outcomes induced by these solutions can be computed by solving maximin optimization problems that have a linear formulation if the set of feasible utilities is a polyhedric set.

Our characterization is based on a weakening of the Restricted Invariance axiom, which we call Coregional Invariance. This new axiom states that if a problem is translated, and the Pareto frontier remains in the same cone or "region", then the solution is also translated.

As a consequence of our definition of inequality averse multi-utilitarian bargaining solutions, an alternative representation of both generalized Gini solutions and of inequality averse Choquet bargaining solutions is provided. We prove that they are particular cases of inequality averse multi-utilitarian solutions and therefore are induced by social welfare functions which can be described as the minimum of certain linear functions.

Furthermore, we consider the case where the feasible outcomes are measured by the criterion providing the highest value, and define a related class of solutions that are also induced by piecewise linear functions, equality averse multi-utilitarian bargaining solutions. For two-person bargaining problems, a joint characterization of these solutions and of inequality averse multi-utilitarian bargaining solutions is obtained when erasing a condition of compromisability from the set of axioms that characterizes inequality averse multi-utilitarian solutions. It is also shown that this result does not hold for bargaining problems with more than two agents.

The rest of the paper is organized as follows. In Section 2 previous concepts are presented and the notation is established. In Section 3 we introduce inequality averse multi-utilitarian solutions, study the connections with other bargaining solutions already existing in the literature and provide the axiomatization of these new solutions. Section 4 contains the analysis of equality averse multi-utilitarian solutions. Section 5 is devoted to some concluding remarks. Finally, Section 7 is an appendix with the proofs.

2 Previous concepts and notation

Let $\mathbb{R}$ ($\mathbb{R}_+$, $\mathbb{R}_{++}$) denote the set of all (non-negative, positive) real numbers and let $\mathbb{R}^n$ ($\mathbb{R}^n_+$, $\mathbb{R}^n_{++}$) be the n-fold Cartesian product of $\mathbb{R}$ ($\mathbb{R}_+$, $\mathbb{R}_{++}$). The origin of $\mathbb{R}^n$ is 0^n and 1^n is n-dimensional vector consisting of n ones. The set of all non-negative integers is denoted by $\mathbb{N}$ and $|H|$ represents the cardinality of the subset H in $\mathbb{N}$. We use conventional notation for comparison of vectors: $x \geq y$ means that $x_i \geq y_i$ for all $i = 1, 2, \ldots n$, $x > y$ indicates that $x \geq y$ and $x \neq y$ and $x \gg y$ means $x_i > y_i$ for all $i = 1, 2, \ldots, n$.

Let $co(A)$ denote the convex hull of the set A in $\mathbb{R}^n$, $cch(A)$ denote the convex and comprehensive hull of A and $int(A)$ denote the relative interior of A.

By xy we denote the scalar product of the vectors x, $y \in \mathbb{R}^n$, that is, $xy = \sum_{i=1}^n x_i y_i$. Let Δ^n denote the n-dimensional simplex, $\Delta^n = \left\{\lambda \in \mathbb{R}^n_+ \mid \lambda \cdot 1^n = 1\right\}$. A polyhedric subset $\Lambda \subseteq \Delta^n$ is the intersection of Δ^n and a finite set of half-spaces in $\mathbb{R}^n$, Let $ext(\Lambda)$ denote the set of its extreme points, that is, $ext(\Lambda) = \{\lambda \in \Lambda \mid \nexists\ \lambda_1, \lambda_2 \in \Lambda, \alpha \in (0,1)$ such that $\lambda = \alpha\lambda_1 + (1-\alpha)\lambda_2\}$.

$N = \{1, 2, \ldots, n\}$ is a set of n agents and 2^N is the set of all possible subsets of N. Let $\pi : N \to N$ denote a permutation function, in which $\pi(i) = j$ means that the i-th component of the n-dimensional permuted vector is j. Let x_π be the π-permutation of vector $x \in \mathbb{R}^n$, $x_\pi = (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$. The set of all possible permutation functions is denoted by Π.

An *n-person bargaining problem* can be described by a set of feasible utility vectors, $S \subseteq \mathbb{R}^n$, where we assume that S is convex, compact, and comprehensive (relative to $\mathbb{R}^n_+$), that is, when $x \in S$, then $y \in S$ for all $0 \leq y \leq x$, and that there exists $x \in S \cap \mathbb{R}^n_{++}$. The set of n-person bargaining problems is denoted by Σ.

A *bargaining solution* is a correspondence $F : \Sigma \to \mathbb{R}^n_+$ such that $F(S) \subseteq S$ for all $S \in \Sigma$. Notice that F is allowed to yield multiple outcomes to a bargaining problem.

Some classic solutions for cooperative bargaining which are of interest in this paper, are the egalitarian solution and the weighted utilitarian solutions.

The *egalitarian solution*, E, (Kalai (1977)) is for each $S \in \Sigma$, $E(S) = \{x \in S \,|\, x_i = x_j$ for all $i, j = 1, 2, \ldots, n,$ and there is no $y \in S$ such that $y \gg x\}$.

The *weighted utilitarian solution*, U_λ, (Myerson (1977)) is for each $S \in \Sigma$, $U_\lambda(S) = \{x \in S \,|\, \sum_{i=1}^n \lambda_i x_i \geq \sum_{i=1}^n \lambda_i y_i$ for all $y \in S\}$ with $\lambda \in \mathbb{R}^n_{++}$. A particular case is the *utilitarian solution*, U, obtained for $\lambda = 1^n$.

The solutions addressed in this paper extend the class of utilitarian solutions to the case where several weighted utilitarian criteria are taken into account at the same time.

3 Inequality averse multi-utilitarian solutions

A social welfare function is said to be inequality averse if it is quasi-concave on $\mathbb{R}^n_+$. We now introduce a class of solutions induced by concave social welfare functions which we call inequality averse multi-utilitarian solutions. The rationale behind these solutions is a mix of the egalitarian principle and the utilitarian principle, so that they are inequality averse but include weighted utilitarian criteria to achieve compromise results.

In a situation where several weighted utilitarian criteria have to be taken into account, a way of finding a compromise between egalitarism and weighted utilitarism is the adoption of a conservative position by measuring the outcomes by a concave social welfare function described by the minimum of the different utilitarian functions.

Definition 3.1. Given the polyhedron $\Lambda \subseteq int(\Delta^n)$, with extreme points λ^i, $i = 1, 2, \ldots, k$, the *inequality averse multi-utilitarian bargaining solution*, F_Λ, is for each $S \in \Sigma$,

$$F_\Lambda(S) = \arg\max_{x \in S} m_\Lambda(x)$$

where $m_\Lambda(x) = \min\left\{\lambda^1 \cdot x, \lambda^2 \cdot x, \ldots, \lambda^k \cdot x\right\}$.

For a vector $x \in \mathbb{R}^n_+$, as a consequence of the linearity of the scalar product, if $m_\Lambda(x)$ attains its minimum at $\hat{\lambda} \in ext(\Lambda)$, then the real number $m_\Lambda(x) = \hat{\lambda} \cdot x$ represents the minimum level attainable by all the functions $f(x) = \lambda \cdot x$, $\lambda \in \Lambda$, across the axis determined by 1^n. In this sense, the inequality averse multi-utilitarian bargaining solution F_Λ can be interpreted as a compromise solution between the egalitarian solution and the λ-utilitarian solutions for $\lambda \in \Lambda$.

As a particular case, when Λ is a singleton then F_Λ is the corresponding weighted utilitarian solution. Note also that the limit case, where $\Lambda = \Delta^n$, yields the egalitarian solution.

Example 3.2. Figure 1 illustrates the results provided by the solution F_Λ for two different bargaining problems. In Case I the result coincides with that obtained with the weighted utilitarian solution, U_λ, for a vector of weights $\lambda \in \Lambda$. The result in Case II coincides with the egalitarian result.

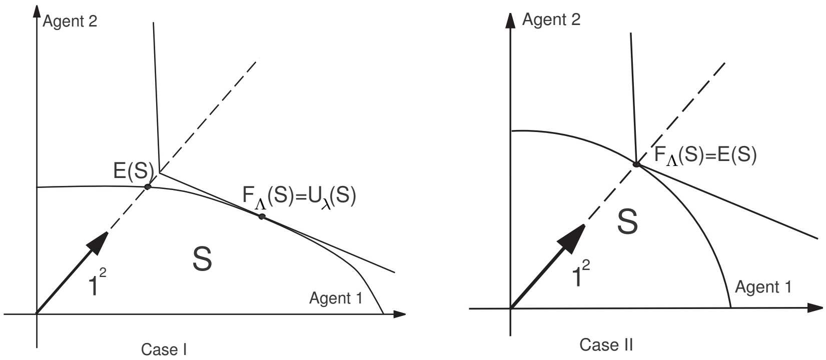

Figure 1: Inequality averse multi-utilitarian solution.

A setting where inequality averse multi-utilitarian solutions could be of interest for the analysis of cooperative bargaining problems is the following: There are several arbitrators or social planners working to help the players to cooperate, each using a different weighted utilitarian criteria to measure the feasible outcomes. The corresponding inequality averse multi-utilitarian solution selects those results in which the minimum level between the criteria of all the arbitrators and also between all possible convex combinations of these criteria, is maximized.

The outcomes provided by an inequality averse multi-utilitarian bargaining solution can be computed by solving the optimization problem (3.1):

$$\left.\begin{array}{lll} max & t & \\ s.t: & \lambda^i \cdot x \geq t & i=1,2,\ldots,k \\ & t \geq 0 & \\ & x \in S & \end{array}\right\} \tag{3.1}$$

When $S \in \Sigma$ is a polyhedric set, then (3.1) is a linear programming program.

Associated to each inequality averse multi-utilitarian bargaining solution there is a family of cones or regions, $\{C_\Lambda(\hat{\lambda})\}_{\hat{\lambda}\in ext(\Lambda)}$, in the orthant $\mathbb{R}^n_+$, where

$$C_\Lambda(\hat{\lambda}) = \left\{ x \in \mathbb{R}^n_+ \text{ such that } m_\Lambda(x) = \hat{\lambda}\cdot x \right\}.$$

In each of these cones the function m_Λ performs as a linear increasing function, and therefore, inequality averse multi-utilitarian bargaining solutions are rationalized by a monotone function which is piecewise linear.

In the next result, some properties of the families of cones induced by these solutions are established.

Proposition 3.3. *Given the polyhedron $\Lambda \subseteq int(\Delta^n)$, the family of cones $\{C_\Lambda(\hat{\lambda})\}_{\hat{\lambda}\in ext(\Lambda)}$ verifies the following properties:*

1. *For each $\hat{\lambda} \in ext(\Lambda)$, $C_\Lambda(\hat{\lambda})$ is the intersection of a finite set of half-spaces determined by hyperplanes containing the set $\{t1^n,\ t \in \mathbb{R}_+\}$.*

2. *For each $\hat{\lambda} \in ext(\Lambda)$, $int\left(C_\Lambda(\hat{\lambda})\right) \neq \emptyset$.*

3. *$\mathbb{R}^n_+ = \bigcup_{\hat{\lambda}\in ext(\Lambda)} C_\Lambda(\hat{\lambda})$.*

4. *If $\hat{\lambda}^1$, $\hat{\lambda}^2 \in ext(\Lambda)$, $\hat{\lambda}^1 \neq \hat{\lambda}^2$, then $int\left(C_\Lambda(\hat{\lambda}^1) \cap C_\Lambda(\hat{\lambda}^2)\right) = \emptyset$.*

Note that different polyhedrons may generate the same family of cones. For instance all the polyhedrons generated by the permutations of a positive vector generate the rank ordered division of $\mathbb{R}^n_+$. Consider the equivalence relation $\mathcal{R}$ defined in the set of polyhedrons included in Δ^n as $\Lambda\mathcal{R}\Lambda'$ if the family of cones generated by m_Λ is the same as the family of cones generated by $m_{\Lambda'}$. This equivalence relation induces classes of equivalence in the set of polyhedrons contained in $\mathbb{R}^n_+$.

For two-person bargaining problems, inequality averse multi-utilitarian solutions can only induce two different families of cones or regions in $\mathbb{R}^2_+$. If Λ is a singleton, there is a unique cone that consists of the whole set $\mathbb{R}^2_+$. If Λ has two extreme points,

λ^1, λ^2, then the division $\{C_\Lambda(\hat{\lambda}^1), C_\Lambda(\hat{\lambda}^2)\}$ does not depend on Λ. In this case $C_\Lambda(\hat{\lambda}^1) = \{x \in \mathbb{R}^2_+ \mid x_1 - x_2 \geq 0\}$, and $C_\Lambda(\hat{\lambda}^2) = \{x \in \mathbb{R}^2_+ \mid x_1 - x_2 \leq 0\}$. Therefore, for two-person bargaining problems, two equivalence classes exist $\{\Omega_1, \Omega_2\}$, where $\Omega_1 = \{\Lambda \subseteq \Delta^n$ with a unique extreme point$\}$ and $\Omega_2 = \{\Lambda \subseteq \Delta^n$ with two extreme points$\}$. Nevertheless, for $n \geq 3$, the number of cones in the family $\{C_\Lambda(\hat{\lambda})\}_{\hat{\lambda} \in ext(\Lambda)}$ coincides with the number of extreme points of the polyhedron Λ.

Example 3.4. For a three-person bargaining problem, consider $\Lambda \subset int(\Lambda^3)$ with extreme points $\lambda^1 = (3/5, 1/5, 1/5)$, $\lambda^2 = (1/10, 4/5, 1/10)$, $\lambda^3 = (1/5, 1/5, 3/5)$. Figure 2 represents the family of cones induced in $\mathbb{R}^3_+$.

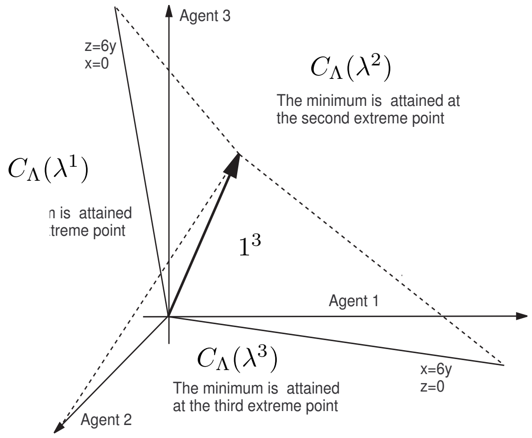

Figure 2: Family of cones associated with F_Λ.

Proposition 3.3 allows us to define the concept of coregionalism for each class of equivalence induced by the equivalence relation $\mathcal{R}$ in the set of polyhedrons included in Δ^n. Let Ω be a class of equivalence.

Definition 3.5. $x, y \in \mathbb{R}^n_+$ are Ω-coregional if for any $\Lambda \in \Omega$, $m_\Lambda(x) = \lambda^* x$ and $m_\Lambda(y) = \lambda^* y$ for a fixed $\lambda^* \in ext(\Lambda)$. The subsets T_1, $T_2 \subset \mathbb{R}^n_+$ are Ω-coregional if x and y are Ω-coregional for all $x \in T_1$ and $y \in T_2$.

Note that the property of comonotonicity defined in Ok and Zhou (2000) is a particular case of coregionalism when the corresponding equivalence class is formed by the polyhedrons generated by all the permutations of a positive n-dimensional vector (as we will show in the following subsection, other polyhedrons exist that generate the same rank ordered division of $\mathbb{R}^n_+$).

3.1 Connections with other Bargaining Solutions

We now analyze the relationship between the class of inequality averse multi-utilitarian bargaining solutions and two interesting classes of solutions which already exist in the literature: generalized Gini bargaining solutions and Choquet bargaining solutions.

Generalized Gini bargaining solutions were introduced by Blackorby et al. (1994). They are defined as follows:

For $a \in \mathbb{R}^n_{++}$, such that $a_1 \geq a_2 \geq \ldots \geq a_n > 0$, $g_a : \mathbb{R}^n \to \mathbb{R}$ represents the generalized Gini ordering:

$$g_a(x) = \sum_{i=1}^{n} a_i x_{(i)},$$

where $(x_{(1)}, x_{(2)}, \ldots, x_{(n)})$ denotes a rank-ordered permutation of $x \in \mathbb{R}^n$, that is, $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$.
The *generalized Gini solution*, G_a is, for each $S \in \Sigma$,

$$G_a(S) = \arg\max_{x \in S} g_a(x).$$

The following example in $\mathbb{R}^2$ illustrates this solution concept.

Example 3.6. Figure 3 represents the compromise outcome provided by the generalized Gini bargaining solution for $g_a(x) = 2/3x_{(1)} + 1/3x_{(2)}$, together with the outcomes produced by the egalitarian and the utilitarian solution.

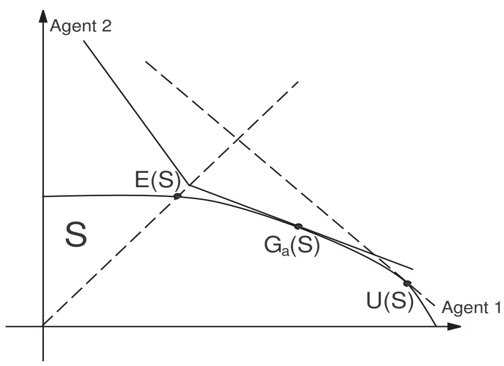

Figure 3: Egalitarian, utilitarian and generalized Gini solution.

Notice that the generalized Gini ordering $g_a(x)$ can be alternatively written as $g_a(x) = \min\{2/3\,x_1 + 1/3\,x_2,\, 1/3\,x_1 + 2/3\,x_2\}$. This expression coincides with m_Λ for the inequality averse multi-utilitarian solution F_Λ, where $\Lambda = co(\hat{\lambda}^1, \hat{\lambda}^2) \subset \Lambda^2$, whose extreme points are $\hat{\lambda}^1 = (2/3,\, 1/3)$ and $\hat{\lambda}^2 = (1/3,\, 2/3)$.

This result is general as stated in Theorem 3.7, where we prove that generalized Gini solutions are particular cases of inequality averse multi-utilitarian bargaining solutions. They are induced by the polyhedrons generated by the permutations of vectors with positive components.

Consider $a = (a_1, a_2, \ldots, a_n) \in \Delta^n$, $a_1 \geq a_2 \geq \ldots \geq a_n > 0$. Denote by Λ_a the polyhedron generated by the permutations of the vector a, $\Lambda_a = co(\{a_\pi, \pi \in \Pi\})$, and let $\alpha^1, \alpha^2, \ldots, \alpha^k$ be its extreme points. Note that, in general, $k \leq n!$, and the inequality is strict when some components of a coincide.

Theorem 3.7. *The generalized Gini solution G_a is, for each $S \in \Sigma$,*

$$G_a(S) = F_{\Lambda_a}(S) = arg\max_{x \in S} m_{\Lambda_a}(x).$$

It follows that the class of generalized Gini solutions is the subclass of inequality averse multi-utilitarian solutions obtained by considering the polyhedron generated by the set of permutations of positive vectors.

Notice that if $a_1 = a_2 = \ldots = a_n$, then $\Lambda_a = \{a\}$ and the corresponding generalized Gini solution is the utilitarian solution, $G_a = F_{\Lambda_a} = U$. Furthermore, when a_1 approaches 1 and the remaining components $a_2, a_3, \ldots, a_n$ tend to 0, then Λ_a approaches the n-dimensional simplex, Δ^n, and G_a provides an outcome close to the egalitarian result.

A wider class of bargaining solutions related to inequality averse multi-utilitarian solutions is the class of Choquet bargaining solutions, introduced by Ok and Zhou (2000).
Let v be a monotonic ($v(K) < v(L)$ for all $K \subset L$) real-valued set function on 2^N with $v(\emptyset) = 0$, called a *capacity* on N (we suppose that v is 1-normalized, that is, $v(N) = 1$). A *Choquet social welfare function* is defined as:

$$W_v(x) = \sum_{i=1}^{n} \left[v\left(\{(i), (i+1), \ldots, (n)\}\right) - v(\{(i+1), \ldots, (n)\})\right] x_{(i)},$$

where $(x_{(1)}, x_{(2)}, \ldots, x_{(n)})$ denotes the rank-ordered permutation of $x \in \mathbb{R}^n$ such that $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$ and, by convention, $\{(n+1), (n)\} \equiv \emptyset$.
The *Choquet bargaining solution*, $C_v(S)$, is for each $S \in \Sigma$,

$$C_v(S) = \arg\max_{x \in S} W_v(x).$$

It is worth noting that the normative properties of a Choquet solution, C_v, are completely embodied in v. Hence, by focusing on certain subclasses of monotonic capacities, some particularly interesting subclasses of Choquet bargaining solutions are obtained.

Inequality averse Choquet bargaining solutions are obtained from quasi-concave Choquet social welfare functions, which are associated to convex capacities, that is, capacities with the property that $v(K \cup \{i\}) - v(K) \leq v(L \cup \{i\}) - v(L)$, for all $K \subset L \subseteq N$ and $i \notin L$.

Note that if the marginal contribution of any agent to any coalition is a positive constant, that is to say, $v(T) - v(S) = t \in \mathbb{R}_{++}$ for all T, $S \subseteq N$, $|T| = |S| + 1$, then the Choquet bargaining solution is the utilitarian, $C_v = U$.

An *anonymous* Choquet social welfare function ($W_v(x) = W_v(x_\pi)$ for all $\pi \in \Pi$) can be obtained by considering *symmetric* capacities ($v(K) = v_k$ for all $K \subseteq N$ with $|K| = k$) and the resulting Choquet solution is then called a *symmetric* Choquet bargaining solution.

Ok and Zhou (2000) showed that for monotonic, symmetric, and convex capacities, the corresponding Choquet solution is the generalized Gini solution and therefore, the concept of inequality averse Choquet bargaining solution is the extension of the generalized Gini solution obtained by erasing the condition of anonymity of the social welfare function.

Our next result, Theorem 3.9, shows that an alternative way of writing inequality averse Choquet bargaining solutions is obtained as a consequence of the convexity of the capacities. In this representation it is easy to see that inequality averse Choquet bargaining solutions are particular cases of inequality averse multi-utilitarian bargaining solutions. To prove the result we rely on the following well-known characterization of convexity of capacities (see, for instance Curiel (1997)). Let $c_i^{\pi}(v)$ denote the marginal contribution of agent i to the coalition of his π-predecessors, $P(\pi, i) = \{j \in N \mid \pi^{-1}(j) < \pi^{-1}(i)\}$, that is, $c_i^{\pi}(v) = v(P(\pi, i) \cup \{i\}) - v(P(\pi, i))$.

Lemma 3.8. *v is a convex capacity if and only if $\min_{\pi \in \Pi} \sum_{i \in T} c_i^{\pi}(v) = v(T)$, for all $T \subseteq N$.*

For $x \in \mathbb{R}^n_+$, let $\pi_x \in \Pi$ denote a permutation that arranges the components of vector $x \in \mathbb{R}^n_+$ in decreasing order, that is, if $x_{\pi(i)} \geq x_{\pi(i+1)}$, for all $i = 1, \ldots, n-1$. It is easy to verify that the social welfare function, W_v, can be rewritten in terms of the permutations π_x as follows:

$$W_v(x) = \sum_{i=1}^{n} \left[v(P(\pi_x, i) \cup \{i\}) - v(P(\pi_x, i)) \right] x_i = \sum_{i=1}^{n} c_i^{\pi_x}(v) x_i.$$

Notice that the coefficient of x_i in $W_v(x)$ represents the marginal contribution of agent i to the coalition $P(\pi_x, i)$ with respect to the capacity v. This coalition is included in the set of agents that obtains an outcome greater than or equal to that obtained by agent i in the allocation $x \in S$.

Consider the vectors c^{π} whose components are the marginal contributions of each agent to the coalition of their predecessors according to each permutation $\pi \in \Pi$. Consider also the Weber set (Weber, 1988) associated with v, that is, the polyhedron, $\Lambda_v^{\Pi} \subseteq int(\Delta^n)$, generated by c^{π}, $\pi \in \Pi$. Theorem 3.9 establishes that inequality averse Choquet bargaining solutions are inequality averse multi-utilitarian bargaining solutions with respect to the Weber set associated with v.

Theorem 3.9. *If v is a monotonic and convex capacity on N, then the inequality averse Choquet bargaining solution $C_v(S)$ is for all $S \in \Sigma$,*

$$C_v(S) = F_{\Lambda_v^{\Pi}}(S) = arg \max_{x \in S} m_{\Lambda_v^{\Pi}}(x).$$

3.2 Axiomatization

This section aims to examine the axiomatic basis of the class of inequality averse multi-utilitarian bargaining solutions. The axioms involved in the characterization are stated below.

For each $S \in \Sigma$, we denote by $PO(S)$ the set of all strongly Pareto optimal points of S, $PO(S) = \{x \in S \mid \text{there exists no } y \in S,\, y > x\}$.

Strong Pareto Optimality (SPO): For all $S \in \Sigma$, $F(S) \subseteq PO(S)$.

Arrow's Choice Axiom (ACA) (Arrow (1959)): For all $S, T \in \Sigma$, if $T \subseteq S$ and $F(S) \cap T \neq \emptyset$, then $F(T) = F(S) \cap T$.

Arrow's Choice Axiom requires that choices be consistent with respect to contraction of the choice set. This axiom reduces to the property of *Nash's independence of irrelevant alternatives* (Nash (1950)) in the case of single-valued bargaining solutions.

Continuity Axiom (CON): For any $S \in \Sigma$ and $x \in S$, if there exists a sequence S_m such that (i) $|F(S_m)| = 1$ for all m; (ii) $S_m \to S$ (in Hausdorff topology); (iii) $F(S_m) \to \{x\}$, then $x \in F(S)$.

An invariance property will play a central role in characterizing inequality averse multi-utilitarian bargaining solutions.

Coregional Invariance Axiom (Co-INV): There exists a class of equivalence, Ω, such that, for all $S \in \Sigma$, $x \in F(S)$ and $y \in \mathbb{R}^n$, with $cch(S + y) \in \Sigma$, if $PO(S)$ and $PO(S + y)$ are Ω-coregional, then $x + y \in F(cch(S + y))$.

A final axiom is established to impose a "fairness" requirement that can be interpreted as a kind of connectedness.

Compromisability Axiom (COM): For any $S \in \Sigma$, $|F(S)| \neq 2$.

Compromisability was introduced by Ok and Zhou (2000) and is a weakening of the Connectedness axiom of Blackorby et al. (1994).

The main result of this section is the following characterization of inequality averse multi-utilitarian solutions.

Theorem 3.10. *A bargaining solution, F, satisfies SPO, ACA, CON, Co-INV and COM if and only if there exists $\Lambda \in int(\Delta^n)$ such that $F = F_\Lambda$.*

Note that, if the following axiom of *Anonymity* is added to those axioms that characterize inequality averse multi-utilitarian solutions, then the class of generalized Gini solutions is obtained.

Anonymity Axiom (A): For all $S \in \Sigma$ and $x \in F(S)$, $x_\pi \in F(\{y_\pi \,|\, y \in S\})$ for any $\pi \in \Pi$.

4 Equality averse multi-utilitarian solutions

When the condition of compromisability is erased from the set of axioms in Theorem 3.10, a wider class of bargaining solutions emerges. These solutions are still induced by piecewise linear functions, but not necessarily concave. We will now focus on the subclass of those solutions rationalized by convex social welfare functions. They are also based on weighted utilitarian criteria, but in this case they are not inequality averse and they yield results biased towards extreme utilitarian payoffs. We will call them *equality averse multi-utilitarian bargaining solutions.*

Definition 4.1. Given the polyhedron $\Lambda \subseteq int(\Delta^n)$, with extreme points λ^i, $i = 1, 2, \ldots, k$, the *equality averse multi-utilitarian bargaining solution*, $\bar{F}_\Lambda$, is for each $S \in \Sigma$,

$$\bar{F}_\Lambda(S) = \arg\max_{x \in S} M^p_\Lambda(x), \text{ where } M^p_\Lambda(x) = \max\left\{\lambda^1 \cdot x, \lambda^2 \cdot x, \ldots, \lambda^k \cdot x\right\}.$$

Figure 5 illustrates the result obtained from an equality averse multi-utilitarian bargaining solution together with the outcomes produced by the egalitarian and the utilitarian solutions.

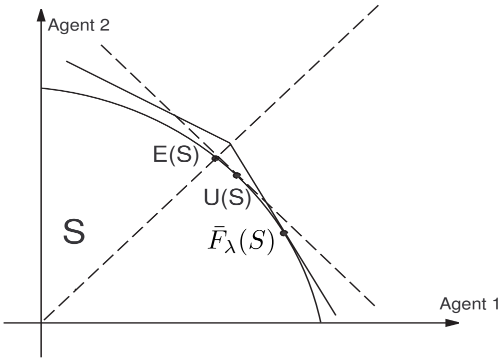

Figure 5: Equality averse multi-utilitarian solution.

The outcomes provided by these solutions can be computed by solving k optimization problems. For $j = 1, \ldots, k$, denote the optimal value of the j-th problem as t_j^*:

$$\left.\begin{array}{rl} t_j^* = max & \lambda^j \cdot x \\ s.t: & x \in S. \end{array}\right\}$$

Let $t_r^* = Max_j\{t_j^*\}$, then the outcomes provided by the equality averse multi-utilitarian bargaining solution, $\bar{F}_\Lambda(S)$, are the set of optimal solutions of the r-th problem.

The following result establishes that for two-person bargaining problems ($n = 2$) the axioms SPO, ACA, CON, and Co-INV jointly characterize both inequality averse multi-utilitarian bargaining solutions and equality averse multi-utilitarian bargaining solutions.

Theorem 4.2. *For two-person bargaining problems, a bargaining solution, F, satisfies SPO, ACA, CON and Co-INV if and only if there exists $\Lambda \in int(\Delta^n)$ such that either, $F = F_\Lambda$ or $F = \bar{F}_\Lambda$.*

It is worth pointing out that for $n \geq 3$, there exist piecewise linear increasing functions (generated by extreme points of polyhedrons $\Lambda \subset \Delta^n$) that are neither concave nor convex, and therefore the bargaining solutions induced are neither inequality averse multi-utilitarian bargaining solutions nor equality averse multi-utilitarian bargaining solutions, as is shown in the following example.

Example 4.3. Consider a polyhedron, $\Lambda \subset \Lambda^3$, generated by the extreme points in Figure 6.

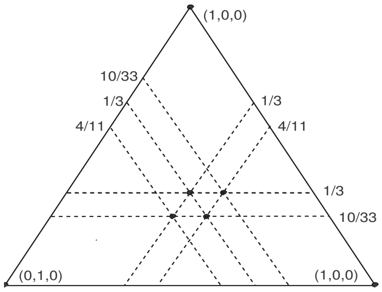

Figure 6: The vectors λ^i, $i = 1, 2, 3, 4$ that generate the polyhedron Λ.

Consider also a piecewise linear increasing function $W : \mathbb{R}^3 \to \mathbb{R}$ defined in each cone as follows:

$ext(\Lambda)$	Linear function	Cone or region
$\lambda^1 = (4/11, 1/3, 10/33)$	$W_1(x) = 4/11x + 1/3y + 10/33z$	$C_\Lambda(\lambda^1) = \{x, \mid y \leq z,\ x \geq y\}$
$\lambda^2 = (4/11, 10/33, 1/3)$	$W_2(x) = 4/11x + 10/33y + 1/3z$	$C_\Lambda(\lambda^1) = \{x, \mid y \geq z,\ x \geq y\}$
$\lambda^3 = (1/3, 4/11, 10/33)$	$W_3(x) = 1/3x + 4/11y + 10/33z$	$C_\Lambda(\lambda^1) = \{x, \mid y \leq z,\ x \leq y\}$
$\lambda^4 = (1/3, 1/3, 1/3)$	$W_4(x) = 1/3x + 1/3y + 1/3z$	$C_\Lambda(\lambda^1) = \{x, \mid y \geq z,\ x \leq y\}$

Figure 7 represents the level 1 curve of W. Since $W(3/2, 3/2, 0) = 1 = W(3/2, 0, 3/2)$ and $W(3/2, 3/4, 3/4) = 45/44 > 1$, W is not convex. Since $W(3/2, 0, 3/2) = 1 = W(0, 3/2, 3/2)$ and $W(3/4, 3/4, 3/2) = 43/44 < 1$, W is not concave. Therefore $\arg\max_{x \in S} W(x)$ provides solutions that are neither inequality averse multi-utilitarian bargaining solutions nor equality averse multi-utilitarian bargaining solutions.

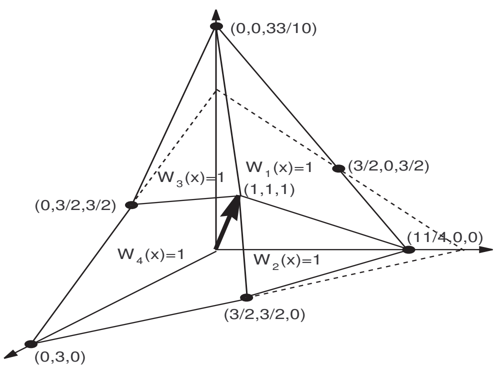

Figure 7: Curve of level 1 of W(x).

Inequality averse multi-utilitarian bargaining solutions and equality averse multi-utilitarian bargaining solutions are rationalized by m_Λ and M_Λ respectively. Both functions perform as linear functions in each cone induced by Λ as represented in Figure 8.

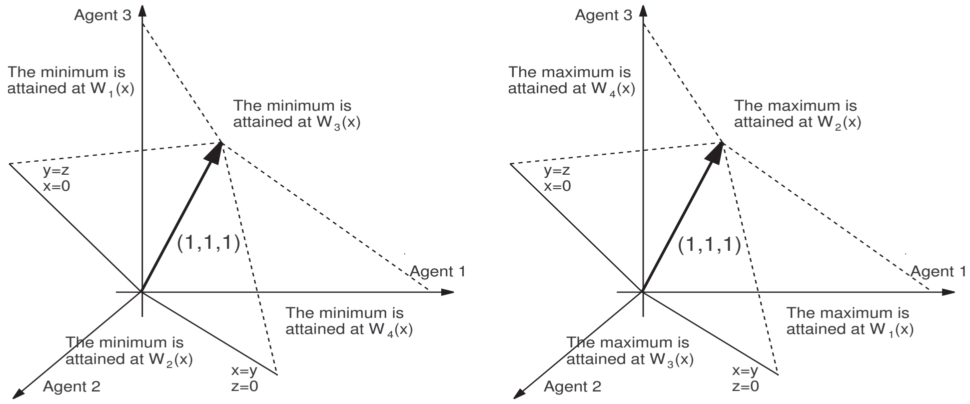

Figure 8: Family of cones, $\{C_\Lambda(\lambda^i)\}_{i=1,2,3,4}$, induced by m_Λ and M_Λ.

Figure 9 represents the level-1 curves for m_Λ and M_Λ respectively:

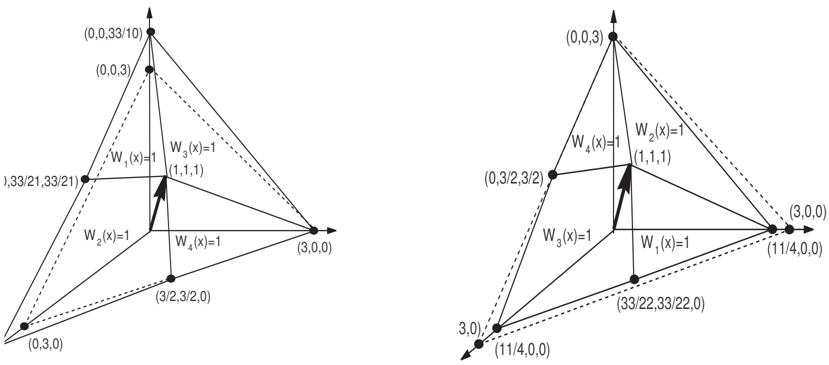

Figure 9: Level 1 curves for m_Λ and M_Λ.

5 Concluding remarks

Inequality averse multi-utilitarian and equality averse multi-utilitarian bargaining solutions constitute two classes of solutions whose intersection is the class of weighted utilitarian solutions. They both intersect the family of Choquet bargaining solutions. The class of inequality averse multi-utilitarian solutions contains that of inequality averse Choquet bargaining solutions and therefore also the class of generalized Gini solutions. Figure 10 represents the inclusions.

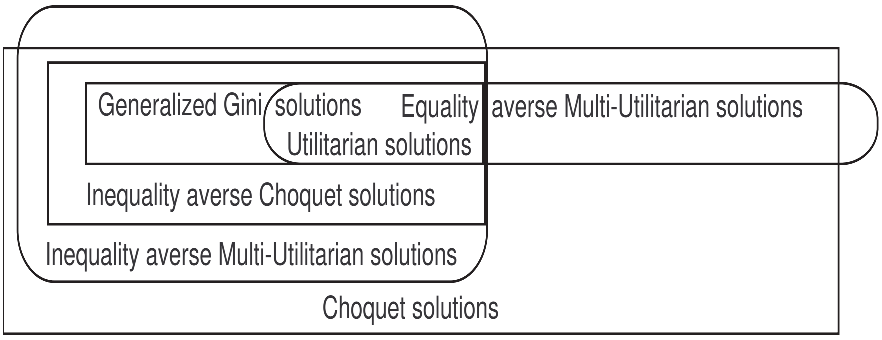

Figure 10: Inclusions.

Note that both inequality and equality averse multi-utilitarian solutions, together with Choquet bargaining solutions are all obtained from generalized Gini solutions when the Anonymity axiom is erased. Nevertheless, the class of solutions that arises from the elimination of Anonymity is wider if the invariance axiom is adapted to accommodate more general polyhedrons. If polyhedrons included in $\Delta_p^n = \{\lambda \in \mathbb{R}_+^n \,|\, \lambda \cdot p = 1\}$ with $p \gg 0^n$ are considered instead of being included in the simplex Δ^n, then two more general classes of solutions verifying SPO, ACA, CON and Co-INV emerge, that could be called p-inequality averse multi-utilitarian and p-equality averse multi-utilitarian solutions.

6 Appendix

Proof of Proposition 3.3: Since the proof of 3 is straightforward, we prove 1,2 and 4:

1. $C_\Lambda(\hat{\lambda}) = \{x \in \mathbb{R}_+^n \,|\, \hat{\lambda} \cdot x \leq \lambda \cdot x,\ \forall \lambda \in\ ext(\Lambda),\ \lambda \neq \hat{\lambda}\} = \{x \in \mathbb{R}_+^n \,|\, (\hat{\lambda} - \lambda) \cdot x \leq 0,\ \forall \lambda \in\ ext(\Lambda),\ \lambda \neq \hat{\lambda}\}$ and the set $\{t1^n,\ t \in \mathbb{R}_+\}$ is included in every hyperplane $(\hat{\lambda} - \lambda) \cdot x = 0$ since $\lambda \cdot 1^n = 1$ for all $\lambda \in ext(\Lambda)$.

2. If $\hat{\lambda}$ is a extreme point of the polyhedron Λ, it is possible to find a supporting hyperplane of Λ at $\hat{\lambda}$, $c \cdot x = a$, with $a \in \mathbb{R}$, $c \in \mathbb{R}^n$, such that $c \cdot \hat{\lambda} = a$ and $c \cdot \lambda < a$ for all $\lambda \in ext(\Lambda)$, $\lambda \neq \hat{\lambda}$. Let $b = \alpha c + (1-\alpha)1^n$, with a sufficiently small α such that $b \geq 0$. It is easy to check that $b \in int\big(C_\Lambda(\hat{\lambda})\big)$.

4. Consider $x \in int\Big(C_\Lambda(\hat{\lambda}^1) \cap C_\Lambda(\hat{\lambda}^2)\Big)$. Then $\hat{\lambda}^1 \cdot x = \hat{\lambda}^2 \cdot x$. However, since $x \in int\Big(C_\Lambda(\hat{\lambda}^1)\Big)$ and $x \in int\Big(C_\Lambda(\hat{\lambda}^2)\Big)$, $\hat{\lambda}^1 \cdot x < \hat{\lambda}^2 \cdot x$ and $\hat{\lambda}^1 \cdot x > \hat{\lambda}^2 \cdot x$, respectively. This is a contradiction. □

Proof of Theorem 3.7. First, we will prove that $\sum_{i=1}^n a_i x_{(i)} \leq \sum_{i=1}^n a_{\pi(i)} x_{(i)}$ for any $\pi \in \Pi$. Consider an elemental permutation, π_k, $1 \leq k < n$, that consists of $\pi_k(k) = k+1$, $\pi_k(k+1) = k$ and $\pi_k(i) = i$ for all $i \neq k,\ k+1$. In this case $\sum_{i=1}^n a_{\pi_k(i)} x_{(i)} =$

$\sum_{i\neq k,k+1}^{n} a_i x_{(i)} + a_{k+1}x_{(k)} + a_k x_{(k+1)} = \sum_{i=1}^{n} a_i x_{(i)} + (a_k - a_{k+1})(x_{(k+1)} - x_{(k)}) \geq \sum_{i=1}^{n} a_i x_{(i)}$, since $a_k \geq a_{k+1}$ and $x_{(k)} \leq x_{(k+1)}$.
Now, for any $\pi \in \Pi$, let π^{n+1} be the identity permutation and recursively define π^k, $k = n, n-1, \ldots, 3, 2$, as follows: π^k is the identity permutation if $\pi(k) = \pi^{k+1}\pi^{k+2}\ldots\pi^{n+1}(k)$, and otherwise, (that is, $\pi(k) = \pi^{k+1}\pi^{k+2}\ldots\pi^{n+1}(h)$) for some $h \in \{1, 2, \ldots, k-1\}$, then π^k is the composition of permutations, $\pi^k = \pi_{k-1}\pi_{k-2}\ldots\pi_{h+1}\pi_h$. It is easy to see that $\pi = \pi^2\pi^3\ldots\pi^n$, and the result follows. As an immediate consequence, the inequality $\sum_{i=1}^{n} a_i x_{(i)} \leq \sum_{i=1}^{n} a_{\pi(i)} x_i$ also holds for any $\pi \in \Pi$. Hence $g_a(x) = \sum_{i=1}^{n} a_i x_{(i)} = \min_{\pi\in\Pi}\{a_\pi \cdot x\}$. Moreover, in terms of the extreme points of the polyhedron generated by the permutations of a, $g_a(x)$ can be written as $g_a(x) = \min\{\alpha^1\cdot x, \alpha^2\cdot x, \ldots, \alpha^k\cdot x\} = m_{\Lambda_a}(x)$. □

Proof of Theorem 3.9: The result follows if we prove that $\min_{\pi\in\Pi}\sum_{i=1}^{n} c_i^{\pi}(v)x_i$ is attained when $\pi = \pi_x$, that is, $W_v(x) = \sum_{i=1}^{n} c_i^{\pi_x}(v)x_i = \min_{\pi\in\Pi}\sum_{i=1}^{n} c_i^{\pi}(v)x_i$.
Let π be any permutation function in Π and let $x \in S$ be any feasible outcome.
If $P(\pi_x, i) \subseteq P(\pi, i)$ holds for all $i \in N$ corresponding to non-zero components of vector x, then it follows from the convexity of v that $\sum_{i=1}^{n} c_i^{\pi_x}(v)x_i \leq \sum_{i=1}^{n} c_i^{\pi}(v)x_i$.
Otherwise, consider the smallest non-zero component of outcome x, which we denote by x_{i^1} $(1 \leq i^1 \leq n)$. If we divide the inequality $\sum_{i=1}^{n} c_i^{\pi_x}(v)x_i \leq \sum_{i=1}^{n} c_i^{\pi}(v)x_i$ by x_{i^1}, we obtain an equivalent inequality:

$$\sum_{i\in P(\pi_x,i^1)\cup\{i^1\}} c_i^{\pi_x}(v) + \sum_{i\in P(\pi_x,i^1)} c_i^{\pi_x}(v)x_i' \leq$$

$$\leq \sum_{i\in P(\pi_x,i^1)\cup\{i^1\}} c_i^{\pi}(v) + \sum_{i\in P(\pi_x,i^1)} c_i^{\pi}(v)x_i',$$

where $x_i' = \frac{x_i}{x_{i^1}} - 1$. Notice that $P(\pi_x, i^1) \cup \{i^1\}$ constitutes the whole set of agents that obtain non-null outcomes with the allocation x.
It is easy to check that $\sum_{i\in P(\pi_x,i^1)\cup\{i^1\}} c_i^{\pi_x}(v) = v(P(\pi_x, i^1) \cup \{i^1\})$. It follows from Lemma 3.8 that $\sum_{i\in P(\pi_x,i^1)\cup\{i^1\}} c_i^{\pi_x}(v) \leq \sum_{i\in P(\pi_x,i^1)\cup\{i^1\}} c_i^{\pi}(v)$ for all $\pi \in \Pi$, and therefore, the proof is reduced to show that:

$$\sum_{i\in P(\pi_x,i^1)} c_i^{\pi_x}(v)x_i' \leq \sum_{i\in P(\pi_x,i^1)} c_i^{\pi}(v)x_i'.$$

If $P(\pi_x, i) \subseteq P(\pi, i)$ for all $i \in P(\pi_x, i^1)$, then the result follows from convexity.
Otherwise, by dividing the inequality by x_{i^2}' (the smallest non-null component of vector x'), the proof reduces to:

$$\sum_{i\in P(\pi_x,i^2)} c_i^{\pi_x}(v)x_i'' \leq \sum_{i\in P(\pi_x,i^2)} c_i^{\pi}(v)x_i'',$$

where $x_i'' = \frac{x_i'}{x_{i^2}'} - 1$.
This procedure of reduction obviously ends in a finite number of steps and this concludes the proof. □

Proof of Theorem 3.10: The necessity part of the proof is straightforward because F_Λ verifies:

- SPO, since m_Λ is a increasing function.
- ACA: Consider $x \in F_\Lambda(S) \cap T$. $y \in F_\Lambda(T)$, $y \neq x$, implies $m_\Lambda(y) = m_\Lambda(x)$ and therefore $y \in F(S)$. Conversely, $y \in F_\Lambda(S) \cap T$, $y \neq x$, implies also $m_\Lambda(y) = m_\Lambda(x)$ and therefore $y \in F(T)$.
- Co-INV, since m_Λ is linear in $C_\Lambda(\hat{\lambda})$, for each $\hat{\lambda} \in ext(\Lambda)$.
- CON, since m_Λ is piecewise linear.
- COM, since m_Λ is concave and S is convex.

To prove the sufficiency part, consider the class of equivalence Ω, for which Co-INV is satisfied. We will show that there exists $\Lambda \in \Omega$ for which the solution F that satisfies SPO, ACA, Co-INV, CON and COM is F_Λ. To do that we consider, as in Ok and Zhou (2000), a real function on $\mathbb{R}^n_+$ defined by:

$$m(x) = inf\left\{t \in \mathbb{R} \,|\, t1^n \in F(cch\{x, t1^n\})\right\}.$$

The following lemmas state that $m(x)$ is linear, increasing, homogeneous of degree one, additive in each cone generated by any $\Lambda \in \Omega$ and concave function and it result to be that $F(S) = \arg\max_{x \in S} m(x)$. These results and their proofs are adaptation of that in Ok and Zhou (2000) and are included for completeness of the paper.

Lemma 6.1. *Let $t_2 > t_1$ and $x \in \mathbb{R}^n_+$. If $t_1 1^n \in F(cch\{x, t_1 1^n\})$, then $F(cch\{x, t_2 1^n\}) = \{t_2 1^n\}$.*

Proof. By applying Co-INV we obtain $t_2 1^n \in F(cch\{x, t_1 1^n\} + \{(t_2 - t_1)1^n\}) = F(cch\{x + (t_2 - t_1)1^n, t_2 1^n\})$. Since $cch\{x, t_2 1^n\} \subset cch\{x + (t_2 - t_1)1^n, t_2 1^n\}$ and $PO(cch\{x + (t_2 - t_1)1^n, t_2 1^n\}) \cap cch\{x, t_2 1^n\} = \{t_2 1^n\}$, by applying SPO and ACA, we have $F(cch\{x, t_2 1^n\}) = \{t_2 1^n\}$. □

Lemma 6.2. *$m(x)1^n \in F(cch\{x, m(x)1^n\})$.*

Proof. From the definition of $m(x)$ and Lemma 6.1, it follows that $\{(m(x) + 1/l)1^n\} = F(cch\{x, (m(x) + 1/l)1^n\})$ for all $l \geq 1$.
Hence, by applying CON, $m(x)1^n \in F(cch\{x, m(x)1^n\})$. □

Lemma 6.3. *Let $t \in \mathbb{R}_+$. If $t < m(x)$, then $F(cch\{x, t1^n\}) = \{x\}$.*

Proof. First, notice that as a consequence of SPO, $F(cch\{x, t1^n\}) \subseteq co\{x, t1^n\}$ holds, and since $t < m(x)$, $t1^n \notin F(cch\{x, t1^n\})$. Now let $y = \mu x + (1 - \mu)t1^n$ for some $\mu \in (0, 1)$, and suppose on the contrary that $y \in F(cch\{x, t1^n\})$.
Suppose first that $\mu \in (0, 1/2]$ and $y = \mu x + (1 - \mu)t1^n \in F(cch\{x, t1^n\})$. By ACA $y \in F(cch\{x, y\})$ and by Co-INV, $t1^n \in F(cch\{x + (t1^n - y), t1^n\})$. ACA can be applied now because $F(cch\{x + (t1^n - y), t1^n\}) \subseteq cch\{x, t1^n\}$ and $y \in cch\{x + (t1^n - y), t1^n\} \cap F(cch\{x, t1^n\})$ to obtain $t1^n \in F(cch\{x + (t1^n - y), t1^n\}) = cch\{x + (t1^n - y), t1^n\} \cap F(cch\{x, t1^n\})$. This contradicts that $t < m(x)$.

Next suppose that $\mu \in (1/2, 1)$. Define:

$$\begin{aligned}
y^0 =\ & y \\
y^1 =\ & y^0 + (y^0 - x) = y + (y - x) = 2y - x = \\
& (2\mu - 1)x + 2(1-\mu)t1^n = \mu^1 x + (1-\mu^1)t1^n \\
y^2 =\ & y^1 + (y^1 - x) = 2y^1 - x = \\
& (4\mu - 3)x + 4(1-\mu)t1^n = \mu^2 x + (1-\mu^2)t1^n \\
& \vdots \\
y^k =\ & y^{k-1} + (y^{k-1} - x) = 2y^{k-1} - x = \\
& (2^k\mu - (2^k - 1))x + 2^k(1-\mu)t1^n = \mu^k x + (1-\mu^k)t1^n
\end{aligned}$$

Again by ACA $y \in F(cch\{x, y\})$. Hence, by recursively applying Co-INV, we obtain:

$$y^k \in F(cch\{x, y^{k-1}\} + (y^k - y^{k-1})) = F(cch\{y^{k-1}, y^k\}), \text{ for } k = 1, 2, \ldots \tag{6.1}$$

We will prove by induction that $y^k \in F(cch\{x, t1^n\})$ for every $k = 1, 2, \ldots$. By hypothesis $y^0 \in F(cch\{x, t1^n\})$ and assume that $y^{k-1} \in F(cch\{x, t1^n\})$. Hence, $cch\{y^{k-1}, y^k\} \cap F(cch\{x, t1^n\}) \neq \emptyset$ and $y^k \in F(cch\{y^{k-1}, y^k\})$ by (6.1). Hence, by ACA, it follows that $y^k \in F(cch\{x, t1^n\})$.
Now notice that there exists $\bar{k} \in \mathbb{N}$ such that $\mu^k \in (0, 1/2]$. We have $y^{\bar{k}} \in F(cch\{x, t1^n\})$, and we can apply the first part of the proof to conclude that $t1^n \in F(cch\{x, t1^n\})$. This again contradicts $t < m(x)$. □

Lemma 6.4. $\{x, m(x)1^n\} \subseteq F(cch\{x, m(x)1^n\})$.

Proof. It follows from Lemma 6.2 that $m(x)1^n \in F(cch\{x, m(x)1^n\})$ and from Lemma 6.3 that $F(cch\{x, (m(x) - 1/l)1^n\}) = \{x\}$ for all $l \geq 1$. Therefore, by applying CON $x \in F(cch\{x, m(x)1^n\})$. □

Lemma 6.5. $F(cch\{x, m(x)1^n\}) = co\{x, m(x)1^n\}$.

Proof. Take $y = \mu x + (1-\mu)m(x)1^n$ and $\mu \in (0, 1)$.
First, suppose that $\mu \in [1/2, 1)$ and let $z = (1-\mu)x + \mu m(x)1^n$. Since $x \in cch\{x, z\} \cap F(cch\{x, m(x)1^n\})$, by ACA, $x \in F(cch\{x, z\})$, and thus by Co-INV,

$$y = x + (y - x) \in F(cch\{x + (y - x), z + (y - x)\}) = F(cch\{y, m(x)1^n\}). \tag{6.2}$$

Notice also that $m(x)1^n \in F(cch\{x, m(x)1^n\}) \cap cch\{y, m(x)1^n\}$. Hence, by ACA, $F(cch\{y, m(x)1^n\}) = F(cch\{x, m(x)1^n\}) \cap cch\{y, m(x)1^n\}$. Now, by using (6.2), we obtain $y \in F(cch\{x, m(x)1^n\})$.
For the case $\mu \in (0, 1/2)$ an analogous reasoning can be used. Let $z = (1-\mu)x + \mu m(x)1^n$. Since $m(x)1^n \in cch\{z, m(x)1^n\} \cap F(cch\{x, m(x)1^n\})$, by ACA, $m(x)1^n \in F(cch\{z, m(x)1^n\})$, and thus by Co-INV,

$$\begin{aligned}
& y = m(x)1^n + (y - m(x)1^n) \in \\
& \in F(cch\{z + (y - m(x)1^n), m(x)1^n + (y - m(x)1^n)\}) = F(cch\{x, y\})
\end{aligned} \tag{6.3}$$

Notice now that $x \in F(cch\{x, m(x)1^n\}) \cap cch\{x, y\}$.
Hence $F(cch\{x, y\}) = F(cch\{x, m(x)1^n\}) \cap cch\{x, y\}$, by ACA, and it follows from (6.3) that $y \in F(cch\{x, m(x)1^n\})$. □

As a consequence of the above Lemmas we obtain:

$$F(cch\{x, t1^n\}) = \begin{cases} \{t1^n\}, & \text{if } t > m(x) \\ co\{x, m(x)1^n\}, & \text{if } t = m(x) \\ \{x\} & \text{if } t < m(x) \end{cases} \tag{6.4}$$

Lemma 6.6. *If $x \in F(cch\{x, y\})$, then $m(x) \geq m(y)$.*

Proof. Suppose that $m(x) < m(y)$, and take t such that $m(x) < t < m(y)$. Consider $z \in F(cch\{x, y, t1^n\})$. By SPO, there exists $w \in co\{x, y\}$ such that $z \in co\{w, t1^n\}$. It follows from (6.4) that , either $w \in F(cch\{w, t1^n\})$ or $t1^n \in F(cch\{w, t1^n\})$ (or both). Assume first that $w \in F(cch\{w, t1^n\})$. In this case $z \in F(cch\{x, y, t1^n\}) \cap cch\{w, t1^n\}$, and it follows from ACA that $w \in F(cch\{x, y, t1^n\})$. Hence, $w \in F(cch\{x, y, t1^n\}) \cap cch\{x, y\}$, and by ACA $F(cch\{x, y\}) = F(cch\{x, y, t1^n\}) \cap cch\{x, y\}$. Therefore, $x \in F(cch\{x, y, t1^n\})$. Now $x \in F(cch\{x, y, t1^n\}) \cap cch\{x, t1^n\}$. By applying ACA again $x \in F(cch\{x, t1^n\})$, but by (6.4), this contradicts $t > m(x)$. Now assume $t1^n \in F(cch\{w, t1^n\})$. Then $F(cch\{x, y, t1^n\}) \cap cch\{w, t1^n\} = F(cch\{w, t1^n\})$, by ACA, since $z \in F(cch\{x, y, t1^n\}) \cap cch\{w, t1^n\}$. This implies that $t1^n \in F(cch\{x, y, t1^n\})$. Now $t1^n \in F(cch\{x, y, t1^n\}) \cap cch\{y, t1^n\}$. Hence, by applying ACA again, we obtain $t1^n \in F(cch\{y, t1^n\})$. This contradicts $t < m(y)$. □

Lemma 6.7. *If $x \in F(cch\{x, y\})$ and $m(x) \leq m(y)$, then $y \in F(cch\{x, y\})$.*

Proof. Take $z \in F(cch\{x, y, m(x)1^n\})$. By SPO, there exists $w \in co\{x, y\}$ such that $z \in co\{w, m(x)1^n\}$. First we shall prove that $m(y)1^n \in F(cch\{x, y, m(x)1^n\})$.
Either $w \in F(cch\{w, m(y)1^n\})$, or $m(y)1^n \in F(cch\{w, m(y)1^n\})$ is followed from (6.4). In addition, $z \in F(cch\{x, y, t1^n\}) \cap cch\{w, m(y)1^n\}$, hence it follows from ACA that either $w \in F(cch\{x, y, m(y)1^n\})$ or $m(y)1^n \in F(cch\{x, y, m(y)1^n\})$.

Let us see that $w \in F(cch\{x, y, m(y)1^n\})$ implies $m(y)1^n \in F(cch\{x, y, m(y)1^n\})$. Notice that $w \in F(cch\{x, y, m(y)1^n\}) \cap cch\{x, y\}$, and from ACA it follows that $F(cch\{x, y, m(y)1^n\}) \cap cch\{x, y\} = F(cch\{x, y\})$. Thus $x \in F(cch\{x, y, m(y)1^n\})$. We can apply again ACA and $F(cch\{x, y, m(y)1^n\}) \cap cch\{x, m(y)1^n\} = F(cch\{x, m(y)1^n\})$. Hence, $m(y)1^n \in F(cch\{x, y, m(y)1^n\})$ because $m(y)1^n \in F(cch\{x, m(y)1^n\})$, as a consequence of $m(x) \leq m(y)$ and (6.4).
Now we are going to show that $y \in F(cch\{x, y\})$.
Since $m(y)1^n \in F(cch\{x, y, m(x)1^n\}) \cap cch\{y, m(y)1^n\}$, it follows from ACA that $co\{y, m(y)1^n\} = F(cch\{y, m(y)1^n\}) = F(cch\{x, y, m(y)1^n\}) \cap cch\{y, m(y)1^n\}$, and therefore $y \in F(cch\{x, y, m(y)1^n\})$.
Besides, since $y \in F(cch\{x, y, m(y)1^n\}) \cap cch\{x, y\}$, by ACA $F(cch\{x, y, m(y)1^n\}) \cap cch\{x, y\} = F(cch\{x, y\})$. Therefore, $y \in F(cch\{x, y\})$. □

Lemma 6.8. *m is an increasing function.*

Proof. If $x \geq y$, it follows from SPO that $x \in F(cch\{x,y\})$, and hence, by Lemma 6.6, $m(x) \geq m(y)$ holds. Moreover, if $x > y$, let us see that $m(x) > m(y)$. If $m(x) \leq m(y)$, then by (6.4) $m(y)1^n \in F(cch\{x, m(y)1^n\})$ holds. Since $cch\{y, m(y)1^n\} \subset cch\{x, m(y)1^n\}$ and $F(cch\{x, m(y)1^n\}) \cap cch\{y, m(y)1^n\} \neq \emptyset$ ($m(y)1^n \in F(cch\{x, m(y)1^n\})$), it follows from ACA that

$$F(cch\{x, m(y)1^n\}) \cap cch\{y, m(y)1^n\} = F(cch\{y, m(y)1^n\}). \tag{6.5}$$

As a consequence of (6.4), we have that $y \in F(cch\{y, m(y)1^n\})$, and this together with (6.5) implies $y \in F(cch\{x, m(y)1^n\})$. This contradicts SPO. □

Lemma 6.9. *a) m is Ω-coregionally additive, that is, if $x, y \in \mathbb{R}^n_+$ are Λ-coregional then $m(x+y) = m(x) + m(y)$.*

b) m is homogeneous of degree one.

Proof. Let us first prove Λ-coregional additivity. Let $x, y \in \mathbb{R}^n_+$ be Λ-coregional, and $\varepsilon > 0$. Since $F(cch\{x, (m(x)+\varepsilon)1^n\}) = \{(m(x)+\varepsilon)1^n\}$, and $PO(cch\{x, (m(x)+\varepsilon)1^n\})$ and $PO(cch\{x, (m(x)+\varepsilon)1^n\} + y)$ are Λ-coregional, by Co-INV we have:

$$F(cch\{x+y, (m(x)+\varepsilon)1^n + y\}) = (m(x)+\varepsilon)1^n + y,$$

Analogously, since $F(cch\{y, m(y)+\varepsilon)1^n\}) = \{(m(y)+\varepsilon)1^n\}$, and $PO(cch\{y, m(y) + \varepsilon)1^n\})$ and $PO(cch\{y, m(y)+\varepsilon)1^n\} + (m(x)+\varepsilon)1^n)$ are Λ-coregional, by Co-INV,

$$F(cch\{(m(x)+\varepsilon)1^n + y, (m(x)+\varepsilon)1^n + (m(y)+\varepsilon)1^n\}) = (m(x)+\varepsilon)1^n + (m(y)+\varepsilon)1^n,$$

Hence by Lemma 6.6, we obtain:

$$m(x) + m(y) + 2\varepsilon \geq m((m(x)+\varepsilon)1^n + y) \geq m(x+y).$$

By letting $\varepsilon \downarrow 0$, we obtain $m(x) + m(y) \geq m(x+y)$.
Reciprocally, since $F(cch\{x, m(x)-\varepsilon)1^n\}) = \{x\}$, and $PO(cch\{x, m(x)+\varepsilon)1^n\})$ and $PO(cch\{x, m(x)+\varepsilon)1^n\} + y)$ are Λ-coregional, as a consequence of Co-INV,

$$F(cch\{x+y, (m(x)-\varepsilon)1^n + y\}) = x + y$$

Analogously, since $F(cch\{y, m(y)-\varepsilon)1^n\}) = \{y\}$, and $PO(cch\{y, m(y)-\varepsilon)1^n\})$ and $PO(cch\{y, m(y)-\varepsilon)1^n\} + (m(x)-\varepsilon)1^n)$ are Λ-coregional, by Co-INV,

$$F(cch\{(m(x)-\varepsilon)1^n + (m(y)-\varepsilon)1^n, (m(x)-\varepsilon)1^n + y\}) = (m(x)-\varepsilon)1^n + y,$$

Hence, it follows from Lemma 6.6 that:

$$m(x+y) \geq m((m(x)-\varepsilon)1^n + y) \geq m(x) + m(y) - 2\varepsilon.$$

By letting $\varepsilon \downarrow 0$, we obtain $m(x)+m(y) \leq m(x+y)$. Therefore $m(x+y) = m(x)+m(y)$.

Now let us prove that m is linear homogenous. Co-regional additivity implies that $m(kx) = km(x)$ for all $k \in \mathbb{N}$. Thus $\frac{1}{k}m(x) = m\left(\frac{1}{k}x\right)$ for all $k \in \mathbb{N}$ and therefore, $m(\mu x) = \mu m(x)$ for all $\mu \in \mathbb{Q}_{++}$. Then, by choosing rational sequences $\{a_l\}$, $\{b_l\} \in \mathbb{Q}_{++}$ such that $a_l \uparrow \mu$ and $b_l \downarrow \mu$, as $l \to \infty$, as a consequence of the monotonicity of m we obtain that $a_l m(x) = m(a_l x) \leq m(\mu x) \leq m(b_l x) = b_l m(x)$ for all $l > 1$. And by letting $l \to \infty$, we find $m(\mu x) = \mu m(x)$. □

Lemma 6.10. $F(S) = arg\max_{x \in S} m(x)$.

Proof. Let $x \in F(S)$ and $y \in S$. Since S is convex, it follows from ACA that $x \in F(cch\{x, y\})$. Therefore, $m(x) \geq m(y)$ as a consequence of 6.6.
Reciprocally consider $x \in \arg\max_{x \in S} m(x)$ and by contradiction suppose $x \notin F(S)$. Take $y \in F(S)$. Since $x \in \arg\max_{x \in S} m(x)$, $m(x) \geq m(y)$. Since S is convex, $y \in F(cch\{x, y\})$ by ACA. Therefore, by Lemma 6.6, $m(y) \geq m(x)$ and $m(y) = m(x)$. Hence, by Lemma 6.7, $x \in F(cch\{x, y\})$. Now, since $cch\{x, y\} \subseteq S$ and $F(S) \cap cch\{x, y\} \neq \emptyset$ ($y \in F(S)$), it follows from ACA that $F(S) \cap cch\{x, y\} = F(cch\{x, y\})$. Therefore $x \in F(S)$ and this contradicts the assumption. □

Lemma 6.11. *m is concave.*

Proof. In Lemma 3 in Ok and Zhou (1999), they show that if a choice correspondence, F, defined by $F(S) = \arg\max_{x \in S} m(x)$ for all $S \in \Sigma$ (where m is continuous and strictly increasing), verifies COM, then $m(x)$ must be quasi-concave on $\mathbb{R}^n_+$. As a consequence of Lemma 6.9, m is also linear homogeneous. Hence m is concave because any quasi-concave, linearly homogeneous, and non-negative-valued function on $\mathbb{R}^n_+$ must be concave. □

In what follows we show that for all $x \in \mathbb{R}^n_+$, $m(x) = m_\Lambda(x)$ for some $\Lambda \subseteq \Delta^n$. Consider the family of cones $\{C_j\}_{j=1,2,\ldots,k}$, in the orthant $\mathbb{R}^n_+$ associated to Ω. Since m is a piecewise linear increasing function, as stated in Lemmas 6.7 and 6.9, for each $x \in C_j$, $m(x) = \lambda^j \cdot x$ for some $\lambda^j \in \mathbb{R}^n_{++}$. We are going to prove that $m(x) = \min\{\lambda^j \cdot x,\ j = 1, 2, \ldots, k\}$. Suppose that there exists $i, j \in \{1, 2, \ldots, k\}$ and $x \in C_i$ such that $\lambda^i \cdot x > \lambda^j \cdot x$. Since $int(C_j) \neq \emptyset$, consider $y \in int(C_j)$ and $z \in co\{x, y\}$ such that $z = \beta x + (1 - \beta)y \in int(C_j)$. Now, it follows from the concavity of m that:

$$\lambda^j \cdot z = m(z) \geq \beta m(x) + (1 - \beta)m(y) = \beta\lambda^i \cdot x + (1 - \beta)\lambda^j \cdot y >$$

$$> \beta\lambda^j \cdot x + (1 - \beta)\lambda^j \cdot y = \lambda^j \cdot (\beta x + (1 - \beta)y) = \lambda^j \cdot z$$

and this is a contradiction.
Since $m(1^n) = min\{t \in \mathbb{R} \,|\, t1^n \in F(cch\{1^n, t1^n\})\} = 1$, then $\lambda^j \cdot 1^n = 1$, for all $j = 1, 2, \ldots, k$ and $\lambda^j \in int(\Delta^n)$.
Consider now the polyhedron $\Lambda \subseteq \Delta^n$ generated by $\{\lambda^j,\ j = 1, 2, \ldots, k\}$. All these generators are extreme points of Λ. Suppose, on the contrary, that λ^l, $l \in \{1, 2, \ldots, k\}$ is in the convex hull of the remaining vectors. This means that λ^l can be written as $\lambda^l = \sum_{j \neq l} \beta_j \lambda^j$, $\sum_{j \neq l} \beta_j = 1$. Then, if $x \in int(B_l)$, $\lambda^l \cdot x < \lambda^j \cdot x$ for all $j \neq l$ by Lemma

6.11. It follows that $\lambda^l \cdot x = \sum_{j\neq l} \beta_j \lambda^l \cdot x < \sum_{j \neq l} \beta_j \lambda^j \cdot x = \lambda^l \cdot x$, which is a contradiction. □

Proof of Theorem 4.2 : Since, for two-person bargaining problems, as a consequence of Lemma 6.8 and of Lemma 6.9, m is an increasing linear function on $C_1 = \{x \in \mathbb{R}^2_+ \,|\, x_1 - x_2 \geq 0\}$ and $C_2 = \{x \in \mathbb{R}^2_+ \,|\, x_1 - x_2 \leq 0\}$, m is either concave or convex. Therefore, we only need to show that for all $x \in \mathbb{R}^n_+$, $m(x) = M_\Lambda(x)$ for some $\Lambda \subseteq \Delta^n$ if m is convex. Consider the family of cones $\{C_j\}_{j=1,2,\ldots,k}$, in the orthant $\mathbb{R}^n_+$ associated to Ω_l. Since m is a piecewise linear increasing function, as stated in Lemmas 6.7 and 6.9, for each $x \in C_j$, $m(x) = \lambda^j \cdot x$ for some $\lambda^j \in \mathbb{R}^n_{++}$. We are going to prove that $m(x) = \max\{\lambda^j \cdot x,\ j = 1, 2, \ldots, k\}$. Suppose that there exists $i, j \in \{1, 2, \ldots, k\}$ and $x \in C_i$ such that $\lambda^i \cdot x < \lambda^j \cdot x$. Since $int(C_j) \neq \emptyset$, consider $y \in int(C_j)$ and $z \in co\{x, y\}$ such that $z = \beta x + (1 - \beta)y \in int(C_j)$. Since m is convex it follows:

$$\lambda^j \cdot z = m(z) \leq \beta m(x) + (1-\beta)m(y) = \beta\lambda^i \cdot x + (1-\beta)\lambda^j \cdot y <$$

$$< \beta\lambda^j \cdot x + (1-\beta)\lambda^j \cdot y = \lambda^j \cdot (\beta x + (1-\beta)y) = \lambda^j \cdot z$$

and this is a contradiction.

Since $m(1^n) = min\{t \in \mathbb{R} \,|\, t1^n \in F(cch\{1^n, t1^n\})\} = 1$, then $\lambda^j \cdot 1^n = 1$, for all $j = 1, 2, \ldots, k$ and $\lambda^j \in int(\Delta^n)$.

Consider now the polyhedron $\Lambda \subseteq \Delta^n$ generated by $\{\lambda^j,\ j = 1, 2, \ldots, k\}$. All these generators are extreme points of Λ. Suppose, on the contrary, that λ^l, $l \in \{1, 2, \ldots, k\}$ is in the convex hull of the remaining vectors. This means that λ^l can be written by $\lambda^l = \sum_{j\neq l} \beta_j \lambda^j$, $\sum_{j \neq l} \beta_j = 1$. Then, if $x \in int(B_l)$, $\lambda^l \cdot x > \lambda^j \cdot x$, $j \neq l$ since m is convex. Therefore $\lambda^l \cdot x = \sum_{j\neq l} \beta_j \lambda^l \cdot x > \sum_{j\neq l} \beta_j \lambda^j \cdot x = \lambda^l \cdot x$, and this is a contradiction. □

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