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Jens L. Hougaard (University of Copenhagen)

Juan D. Moreno-Ternero (U. Pablo de Olavide and CORE)

Lars P. Osterdal (University of Southern Denmark)

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Keywords: rationing, operators, baselines, composition, claims

Department of Economics

A unifying framework for the problem of adjudicating conflicting claims*

Jens L. Hougaard[†] **Juan D. Moreno-Ternero**[‡] **Lars P. Østerdal**[§]

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Abstract

In a recent paper, Thomson and Yeh [Operators for the adjudication of conflicting claims, Journal of Economic Theory 143 (2008) 177-198] introduced the concept of operators on the space of rules for the problem of adjudicating conflicting claims. They focussed on three operators in order to uncover the structure of such a space. In this paper, we generalize their analysis upon presenting and studying a general family of operators inspired by three apparently unrelated approaches to the problem of adjudicating conflicting claims. We study the structural properties of this family and show, in particular, that most of Thomson and Yeh's results are specific cases of our study.

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[†]Department of Food and Resource Economics, University of Copenhagen.

[‡]Department of Economics, Universidad Pablo de Olavide, and CORE, Université catholique de Louvain.

[§]Department of Business and Economics, University of Southern Denmark.

1 Introduction

The problem of adjudicating conflicting claims describes a situation in which an arbitrator has to allocate a given amount of a perfectly divisible (and homogeneous) resource among a group of agents when the available amount is not enough to fully honor their claims. Most rationing problems can be given this form (e.g., the division of an estate that is insufficient to cover all the debts incurred by the deceased, the collection of a given tax from taxpayers, sharing the cost of a public facility). The reader is referred to Moulin (2002) or Thomson (2003) for reviews of the sizable related literature initiated by O'Neill (1982). In most of this literature the aim is to single out rules that assign for each problem an allocation indicating how much each agent obtains. A recent study by Thomson and Yeh (2008) uncovers the structure of the space of rules upon studying *operators* on such a space. An operator is a mapping on the space of rules that associates with each rule another one. Thomson and Yeh (2008) consider three operators. First, the *duality operator*, which assigns to each rule R, its dual (R^d), that allocates awards in the same way as R allocates losses. Second, the so-called *claims truncation operator*, which assigns to each rule R the rule R^t defined, for each problem, by applying R after each claim has been truncated at the available amount. Third, the *attribution of minimal rights operator*, which associates with each rule R the rule R^m defined by the following two-step procedure. For each problem, first each agent receives her minimal right (to be understood as the part of the available amount that remains, if anything, when all other agents have been fully honored); second, each agent gets an award according to R applied to the revised problem obtained by reducing agents' claims by their minimal burdens, and the available amount by the sum of the minimal rights. Thomson and Yeh (2008) establish a number of results linking them and determine which properties of rules are preserved under each of these operators, and which are not.

In this paper, we generalize the analysis of Thomson and Yeh (2008) upon providing a systematic analysis of a family of operators generalizing the last two mentioned above. Our family is inspired by three apparently unrelated approaches to the problem of adjudicating conflicting claims that, as we shall see later, are intimately connected. First, the so-called baseline rationing, a generalization of the benchmark model upon adding a vector in the awards space representing some reference point, or *baseline*, judged relevant to the division (see Hougaard et al., (2010) for further details). Second, two of the principles most frequently employed in the literature on the problem of adjudicating conflicting claims, pertaining to the way rules react to tentative allocations of the available amount, and known as *composition* properties (e.g., Young, 1988; Moulin, 2000). Third, the analysis of *lower bounds*, an important aspect within the theory of fair allocation, (e.g., Thomson, 2011) that has only been recently explored for the problem of adjudicating conflicting claims (e.g., Moreno-Ternero and Villar, 2004; Dominguez and Thomson, 2006).

More precisely, think of the following situation: after having divided the allocation of the available amount among its creditors, it turns out that the actual value of the amount is larger than was initially assumed. Then, two options are open: either the tentative division is canceled altogether and the actual problem is solved, or we add to the initial distribution the result of applying the rule to the remaining amount. The requirement of *composition up* is that both ways of proceeding should result in the same awards vectors. Think now of the dual case. Namely, after having divided the available amount among its creditors one finds that the actual value of the amount to divide falls short of what was assumed. Here again we can ignore the initial division and apply the rule to the revised problem, or we can apply the rule to the problem in which the initial claims are substituted by the (unfeasible) allocation initially proposed. The requirement of *composition down* is that both ways of proceeding should result in the same awards vectors.

A simple examination of the two composition principles described above will suffice to detect a close relationship between the attribution of minimal rights operator and the principle of composition up, and the claims truncation operator and the principle of composition down, respectively. In the former case, one just needs to interpret minimal rights as a baseline reflecting the tentative allocation of the available amount, whereas in the latter case, truncated claims would refer to a baseline reflecting the unfeasible allocation initially proposed. We propose in this paper a family of operators to generalize those two upon associating to each baseline an operator reflecting the composition principles with respect to it.

In other words, all operators within our family will share a common feature inspired by the composition properties. Namely, if all (individual) baselines cannot be granted, then no agent will receive an amount above her baseline. Similarly, if all baselines could be granted, then each agent will receive at least her baseline. More precisely, each rule will be mapped into a new rule that will solve problems according to a two-stage process, depending on whether the available amount in each problem is above or falls short the aggregate baseline for that problem. In the former case, and inspired by the axiom of composition up, the resulting rule assigns first to each agent her baseline and then adds to that distribution the result of applying the original rule to the remaining amount, once claims have been adjusted. In the latter case, and inspired by the axiom of composition down, the resulting rule solves each problem upon applying the original rule to the problem in which the initial claims are substituted by the (unfeasible) allocation proposed by the baselines profile.

As one can easily infer, both the claims truncation operator, and the attribution of minimal rights operator are specific members of the family of (baseline) composition operators described above. We shall study some logical relations among the operators within this family, as well as a number of results linking them, and determine which properties of rules are preserved under each of these operators, and which are not. In doing so, we shall generalize and scrutinize the results by Thomson and Yeh (2008). We shall also provide a thorough study of some other focal members of the family.

The rest of the paper is organized as follows. In Section 2, we describe the model and basic definitions. In Section 3, we introduce the family of composition operators. In Section 4, we relate the operators by means of several relationships among them. In Section 5, we explore the preservation of axioms under these operators. Finally, we conclude in Section 6.

2 Model and basic concepts

2.1 The benchmark framework

We study claims problems in a variable-population model. The set of potential claimants, or *agents*, is identified with the set of natural numbers $\mathbb{N}$. Let $\mathcal{N}$ be the set of finite subsets of $\mathbb{N}$, with generic element N. We denote by $\mathbb{R}^N_+$ the cross-product of $|N|$ copies of $\mathbb{R}_+$ indexed by the members of N.[1] For each $i \in N$, let $c_i \in \mathbb{R}_+$ be i's *claim* and $c \equiv (c_i)_{i \in N}$ the claims profile.[2] A *problem* is a triple consisting of a population $N \in \mathcal{N}$, a claims profile $c \in \mathbb{R}^N_+$, and an *endowment* $E \in \mathbb{R}_+$ such that $\sum_{i \in N} c_i \geq E$. Let $C \equiv \sum_{i \in N} c_i$. To avoid unnecessary complication, we assume $C > 0$. Let $\mathcal{D}^N$ be the set of problems with population N and $\mathcal{D} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{D}^N$.

[1] Alternatively, the superscript N may refer to a set pertaining to the agents in N. Whichever interpretation is intended should be unambiguous from the context.

[2] For each $N \in \mathcal{N}$, each $M \subseteq N$, and each $z \in \mathbb{R}^N$, let $z_M \equiv (z_i)_{i \in M}$.

2.2 Rules

An *allocation* for a problem $(N, c, E) \in \mathcal{D}$ is a vector $x \in \mathbb{R}^N$ such that, for each $i \in N$, $0 \leq x_i \leq c_i$ (*boundedness*) and $\sum_{i \in N} x_i = E$ (*balance*). A *rule* on $\mathcal{D}$, $R\colon \mathcal{D} \to \bigcup_{N \in \mathcal{N}} \mathbb{R}^N$, associates with each problem $(N, c, E) \in \mathcal{D}$ an allocation $R(N, c, E)$ for the problem. Some classical rules are the **constrained equal-awards** rule, A, which makes awards as equal as possible, subject to the condition that no agent gets more than her claim, and the **constrained equal-losses** rule, L, which makes losses as equal as possible, subject to the condition that no agent gets a negative amount. Formally, $A(N, c, E) = (\min\{c_i, \lambda\})_{i \in N}$, and $L(N, c, E) = (\max\{0, c_i - \lambda\})_{i \in N}$, where $\lambda > 0$ is chosen, in each case, so that the balance is guaranteed. Two other prominent rules are the **proportional** rule, P, which makes awards proportional to claims, and the **Talmud** rule (e.g., Aumann and Maschler, 1985), which is a compromise between the constrained equal-awards and the constrained equal-losses rules. Formally, $T(N, c, E) = (\min\{\frac{1}{2}c_i, \lambda\})_{i \in N}$ if $E \leq \frac{1}{2}C$ and $(\max\{\frac{1}{2}c_i, c_i - \mu\})_{i \in N}$ if $E \geq \frac{1}{2}C$, where λ and μ are chosen so that the balance condition is guaranteed.

2.3 Operators

An *operator* is a mapping on the space of rules that associates with each rule another one. Three operators have been proposed in the literature (e.g., Thomson and Yeh, 2008). The first one expresses the idea of duality. For any given rule R, the *dual rule of* R, denoted as R^d, associates with each $(N, c, E) \in \mathcal{D}$, $R^d(N, c, E) \equiv c - R(N, c, C - E)$. A rule is *self-dual* if it coincides with its dual. The constrained equal awards and the constrained equal losses are dual rules and the proportional and the Talmud are self-dual rules, i.e., $A^d = L$, $P^d = P$, $T^d = T$. The *duality operator* O^d is the operator assigning to each rule R its dual, i.e., $O^d(R) = R^d$. The duality operator is an involution, i.e., $O^d(O^d(R)) = R$.

The second one refers to the concept of *truncated claims*. In some claims problems, it makes sense to postulate that the part of a claim that is above the endowment should be ignored. For a given problem (N, c, E), we denote by $t(N, E, c)$ its corresponding vector of truncated claims, i.e., $t(N, E, c) = (t_i(N, c, E))_{i \in N}$, where $t_i(N, c, E) = \min\{E, c_i\}$ for all $i \in N$. The *claims truncation operator* O^t is the operator assigning to each rule R the rule arising from applying R to each problem once claims have been truncated. Formally, $O^t(R) = R^t$, where $R^t(N, c, E) = R(N, t(N, E, c), E)$, for each problem (N, c, E).

The third one refers to the concept of *minimal rights*. The dual to the previous idea is the principle that a minimal amount should be ensured for each agent. The most natural amount, for each agent i, would be $m_i(N,c,E) = \max\{0, E - \sum_{j\in N\setminus\{i\}} c_j\}$, which is the portion of the endowment that is left to her when the claims of all other agents are fully honored, provided this amount is nonnegative. We interpret this amount as the minimal right of agent i. Let $m(N,c,E) = (m_i(N,c,E))_{i\in N}$ and $M(N,c,E) = \sum_{i\in N} m_i(N,c,E)$. The *attribution of minimal rights operator* O^m is the operator assigning to each rule R the rule arising from allocating minimal rights first and applying then R to the resulting problem once claims (and the endowment) have been adjusted. Formally, $O^m(R) = R^m$, where $R^m(N,c,E) = m(N,c,E) + R(N, c - m(N,E,c), E - M(N,c,E))$, for each problem (N,c,E).

2.4 Baselines

A *baseline* is a mapping associating to each problem a vector satisfying the boundedness condition, but not necessarily the balance condition. Formally, $b\colon \mathcal{D} \to \bigcup_{N\in\mathcal{N}} \mathbb{R}^N$, associates with each problem $(N,c,E) \in \mathcal{D}$ a vector $b(N,c,E)$ such that $0 \le b_i(N,c,E) \le c_i$ for each $i \in N$.[3] We single out two important types of baselines. A *lower bound* on $\mathcal{D}$, $\underline{b}\colon \mathcal{D} \to \bigcup_{N\in\mathcal{N}} \mathbb{R}^N$, is a baseline that associates with each problem $(N,c,E) \in \mathcal{D}$ a feasible vector. Formally, $\sum_i \underline{b}_i(N,c,E) \le E$, for each $(N,c,E) \in \mathcal{D}$. On the other hand, an *upper bound* on $\mathcal{D}$, $\bar{b}\colon \mathcal{D} \to \bigcup_{N\in\mathcal{N}} \mathbb{R}^N$, is a baseline that associates with each problem $(N,c,E) \in \mathcal{D}$ an unfeasible vector. Formally, $\sum_i \bar{b}_i(N,c,E) \ge E$, for each $(N,c,E) \in \mathcal{D}$. An instance of a lower bound is the application that assigns the minimal right to each agent, whereas an instance of an upper bound is the application that assigns the truncated claim for each agent. Another interesting instance of a lower bound is the one that assigns to each agent one n-th of her truncated claim (e.g., Moreno-Ternero and Villar, 2004). Formally, $\mu(N,c,E) = (\mu_i(N,c,E))_{i\in N}$, where $\mu_i(N,c,E) = \frac{1}{n} t_i(N,c,E) = \frac{1}{n}\min\{c_i, E\}$.[4] The corresponding upper bound is obtained when requiring that if agent i's claim is at most as large as the deficit $C - E$, she should receive at most $c_i - \frac{1}{n}c_i$, and otherwise, she should receive at most $c_i - \frac{1}{n}(C-E)$. Formally, $\mu^d(N,c,E) = \left(\mu_i^d(N,c,E)\right)_{i\in N}$, where $\mu_i^d(N,c,E) = c_i - \frac{1}{n} t_i(N,c,C-E) = c_i - \frac{1}{n}\min\{c_i, C-E\}$.

Other lower bounds that will play a role in the analysis of this paper are the so-called *proportional lower bounds*, which assign to each agent a fixed portion of the proportional al-

[3] See Hougaard et al., (2010) for an alternative modeling of baseline rationing in which baselines are considered as arbitrary and exogenously given.

[4] Note that using claims instead of truncated claims while defining the bound is not a meaningful option, as the bounds so obtained would not necessarily be feasible for all problems.

location. Formally, for each $\alpha \in (0,1)$, $\rho_\alpha(N,c,E) = \alpha P(N,c,E) = \alpha \frac{E}{C} c$. The corresponding (proportional) upper bounds are easily obtained: $\rho_\alpha^d(N,c,E) = \frac{(1-\alpha)C+\alpha E}{C} c$.

It might well be the case that a baseline is neither a lower bound nor an upper bound. Instances are those baselines that do not depend on the available amount. A focal example will be the case in which baselines are a given percentage of claims. Formally, for each $\theta \in (0,1)$, $\theta(N,c,E) = \theta c$, for all $(N,c,E) \in \mathcal{D}$.

3 Composition Operators

We now introduce the concept of a composition operator. More precisely, for a given baseline b, the *composition operator* O^b is the operator assigning to each rule R the rule R^b arising from composing the tentative allocation of b with the allocation that R proposes for the revised problem. Formally,

$$R^b(N,c,E) = \begin{cases} R(N, b(N,c,E), E) & \text{if } E \leq \sum_i b_i(N,c,E) \\ b(N,c,E) + R(N, c - b(N,c,E), E - \sum_i b_i(N,c,E)) & \text{if } E \geq \sum_i b_i(N,c,E) \end{cases} \tag{1}$$

A consequence of (1) is that R^b yields allocations satisfying $x_i \leq b_i(N,c,E)$ for each $i \in N$ if and only if $E \leq \sum_i b_i(N,c,E)$, and $x_i \geq b_i(N,c,E)$ for each $i \in N$ if and only if $E \geq \sum b_i(N,c,E)$. In words, R^b imposes a rationing of the same sort for each individual and the whole society according to the profile of baselines.

It can be argued that R^b rations agents relative to the feasible or unfeasible baselines in the spirit of composition up and down respectively. More precisely, if b is a feasible baseline (i.e., a lower bound), then

$$R^b(N,c,E) = b(N,c,E) + R(N, c - b(N,c,E), E - \sum_{i \in N} b_i(N,c,E)),$$

as in the spirit of *composition up*, where R^b first allocates b and then allocates the residual endowment using the rule R with respect to residual claims $c - b$.

If b is an unfeasible baseline (i.e., an upper bound), then

$$R^b(N,c,E) = R(N, b(N,c,E), E),$$

as in the spirit of *composition down*, where R^b allocates E by using the rule R with respect to the baseline itself.

It follows from the above that the minimal rights operator and the claims truncation operator are specific instances of composition operators. More precisely, if $b(N,c,E) = m(N,c,E)$ for each $(N,c,E) \in \mathcal{D}$, then the corresponding composition operator O^b is precisely the operator O^m. Similarly, if $b(N,c,E) = t(N,c,E)$ for each $(N,c,E) \in \mathcal{D}$, then the corresponding composition operator O^b is precisely the operator O^t.

As mentioned above, a natural class of baselines arises when baselines profiles are assumed to be a given percentage of claims, i.e., for some $\theta \in (0,1)$, $b(N,E,c) = \theta c$, for each $(N,c,E) \in \mathcal{D}$. It turns out that the proportional rule is a fixed point for each of the corresponding composition operators $\{O^\theta\}_{\theta\in(0,1)}$. Formally, $O^\theta(P) = P$, for each $\theta \in (0,1)$.[5] As for the specific case in which $\theta = 1/2$, which could be interpreted as a psychological threshold, the other two classical rules (namely, the constrained equal awards and the constrained equal losses rules) also provide interesting outcomes. More precisely, it is straightforward to show that $O^{1/2}(A)$ would precisely be the so-called Piniles rule (e.g., Thomson, 2003), whereas $O^{1/2}(L)$ would be its dual rule.

4 Relating operators

4.1 Duality relationships

In this section, we relate some of the composition operators described above. Note first that, as the duality operator is an involution, then R is the dual of S if and only if S is the dual of R. Hence, if R is the dual of S we can refer to R and S as dual rules.

We now define the concept of a *dual baseline*. Formally, for a given baseline b, we define its dual b^d by

$$b^d(N,c,E) = c - b(N,c,C-E). \tag{2}$$

In words, as it happens with rules, baseline b^d allocates awards in the same way as baseline b allocates losses. The dual of a lower bound is an upper bound and vice versa. Note also that $E \geq \sum b_i(N,c,E)$ is equivalent to saying that $C - E \leq \sum b_i^d(N,c,C-E)$. Then, we have the following result:

Theorem 1 *Let R and S be dual rules. Then R^b and S^{b^d} are dual rules.*

Proof. We need to show that for each (N,c,E), $R^b(N,c,E) = c - S^{b^d}(N,c,C-E)$. We distinguish two cases.

[5] That is also the case for the *proportional bounds operators* $\{O^{\rho_\alpha}\}_{\alpha\in(0,1)}$ and $\{O^{\rho_\alpha^d}\}_{\alpha\in(0,1)}$.

Case 1: $E \leq \sum_{i \in N} b_i\,(N, c, E)$.

In this case, $R^b\,(N, c, E) = R\,(N, b(N, c, E), E)$ and

$$\begin{aligned} c - S^{b^d}\,(N, c, C - E) &= c - b^d\,(N, c, C - E) - S\left(N, c - b^d\,(N, c, C - E)\,, C - E - \sum_{i \in N} b_i^d\,(N, c, C - E)\right) \\ &= b(N, c, E) - S(N, b(N, c, E), \sum_{i \in N} b_i(N, c, E) - E). \end{aligned}$$

Now, as R and S are dual rules, it follows that

$$R\,(N, b(N, c, E), E) = b(N, c, E) - S\left(N, b(N, c, E), \sum_{i \in N} b_i\,(N, c, E) - E\right).$$

Thus, $R^b\,(N, c, E) = c - S^{b^d}\,(N, c, C - E)$, as desired.

Case 2: $E \geq \sum_{i \in N} b_i\,(N, c, E)$.

In this case,

$$R^b\,(N, c, E) = b\,(N, c, E) + R(N, c - b\,(N, c, E)\,, E - \sum_{i \in N} b_i\,(N, c, E)),$$

and

$$c - S^{b^d}\,(N, c, C - E) = c - S\left(N, b^d\,(N, c, C - E)\,, C - E\right) = c - S\,(N, c - b(N, c, E), C - E)\,.$$

Now, as R and S are dual rules, it follows that

$$R(N, c - b\,(N, c, E)\,, E - \sum_{i \in N} b_i\,(N, c, E)) = c - b\,(N, c, E) - S\,(N, c - b\,(N, c, E)\,, C - E)\,,$$

from where the desired equality follows. ■

Theorem 1 says, in particular, that if R is a self-dual rule, then R^b and R^{b^d} are dual rules. Somewhat related, we also have that if b is a baseline, S is a rule such that S^b is self-dual, and R is the dual rule of S, then $R^{b^d} \equiv S^b$.

The truncated claims and the minimal rights are dual baselines, i.e., $t^d = m$ and $m^d = t$. We then have the following corollary, which corresponds to Theorems 1 and 5 in Thomson and Yeh (2008).

Corollary 1 *Let R and S be two dual rules. Then, R^m and S^t are dual rules.*

Other similar results can also be obtained as straightforward consequences of the above theorem.

Corollary 2 *Let R and S be two dual rules. Then,*

- R^{μ} *and* S^{μ^d} *are dual rules.*
- R^{ρ_α} *and* $S^{\rho_\alpha^d}$ *are dual rules.*
- R^{θ} *and* $S^{1-\theta}$ *are dual rules.*[6]

4.2 Commutative relationships

We now turn to explore how composition operators behave when they are applied sequentially. More precisely, our next result shows that the composition operators corresponding to two dual baselines commute, provided they satisfy an additional condition. Formally, let b be a lower bound. We say that b is *duality stable* if

$$b(N, b^d(N, c, E), E) = b(N, c, E),$$

for each $(N, c, E) \in \mathcal{D}$. Then, we have the following:

Theorem 2 *If a lower bound is duality stable then the composition operator O^b commutes with the composition operator O^{b^d}, i.e., for each rule R,*

$$O^{b^d}(O^b(R(N, c, E))) = O^b(O^{b^d}(R(N, c, E))).$$

Proof. Let $(N, c, E) \in \mathcal{D}$ be given.

On the one hand,

$$\begin{aligned} O^{b^d}(O^b(R(N,c,E))) &= R^b(N, b^d(N,c,E), E) \\ &= b(N, b^d(N,c,E), E) \\ &+ R(N, b^d(N,c,E) - b(N, b^d(N,c,E), E), E - \sum_{i \in N} b_i(N, b^d(N,c,E), E)). \end{aligned} \tag{3}$$

On the other hand,

$$\begin{aligned} O^b(O^{b^d}(R(N,c,E))) &= b(N,c,E) + R^{b^d}(N, c - b(N,c,E), E - \sum_{i \in N} b_i(N,c,E)) \\ &= b(N,c,E) \\ &+ R(N, b^d\left(N, c - b(N,c,E), E - \sum_{i \in N} b_i(N,c,E)\right), E - \sum_{i \in N} b_i(N,c,E)). \end{aligned} \tag{4}$$

[6]In particular, if R is a self-dual rule then $R^{1/2}$ is self-dual too.

It follows from (2) that

$$\begin{aligned} b^d\left(N, c-b(N,c,E), E-\sum_{i\in N} b_i(N,c,E)\right) &= c-b(N,c,E)-b(N,c-b(N,c,E),C-E) \\ &= c-b(N,c,E)-b(N,b^d(N,c,C-E),C-E) \\ &= c-b(N,c,E)-b(N,c,C-E) \\ &= b^d(N,c,E)-b(N,c,E). \end{aligned}$$

Thus, by dual stability, both (3) and (4) are equal to

$$b(N,c,E)+R(N,b^d(N,c,E)-b(N,c,E),E-\sum_{i\in N} b_i(N,c,E)). \blacksquare$$

It is not difficult to show that the minimal rights lower bound is duality stable. Consequently, we have the following corollary, which corresponds to Theorem 3 in Thomson and Yeh (2008).

Corollary 3 *O^m and O^t commute.*

The proportional lower bounds are duality stable too. Consequently, O^{ρ_α} and $O^{\rho_\alpha^d}$ commute, for any $\alpha \in (0,1)$. The lower bound μ, however, is not duality stable and, actually, it turns out that O^μ and O^{μ^d} do not commute. The operators O^θ and $O^{1-\theta}$ do not commute either, for any $\theta \in (0,1)$.

Theorem 2 can be further extended in the two-agent case for rules satisfying a mild notion of impartiality stating that agents with equal claims are treated equally. In order to present this result, let us refer first to a focal principle in the two-agent case, known as *concede-and-divide*, whose motivation can be traced back to the Talmud.[7] It amounts to solve two-agent problems upon conceding each agent a portion of the endowment and dividing the remainder equally. A point in case for such concessions are the minimal rights described above, but any other lower bound could be considered too. Formally, for any lower bound b, the so-called *generalized concede-and-divide* rule, CD_b, selects, for each two-agent problem $(\{i,j\},(c_i,c_j),E)$, the allocation

$$\left(b_i(\{i,j\},(c_i,c_j),E)+\frac{E-B(\{i,j\},(c_i,c_j),E)}{2}, b_j(\{i,j\},(c_i,c_j),E)+\frac{E-B(\{i,j\},(c_i,c_j),E)}{2}\right),$$

where $B(\{i,j\},(c_i,c_j),E) = b_i(\{i,j\},(c_i,c_j),E)+b_j(\{i,j\},(c_i,c_j),E)$. The so-called concede-and-divide rule, CD, which corresponds to the two-agent version of the Talmud rule, is obtained when $b=m$.

[7]The principle was first modeled by Aumann and Maschler (1985) although the term *concede-and-divide* was later coined by Thomson (2003).

We now introduce an additional condition for lower bounds, which says that all agents should face a constant gap between a lower bound and its dual upper bound. Formally, let b be a lower bound. We say that b satisfies *constant duality gap* if, for each $(N,c,E) \in \mathcal{D}$,

$$b_i^d(N,c,E) - b_i(N,c,E) = b_j^d(N,c,E) - b_j(N,c,E),$$

for each $i,j \in N$.

The following result strengthens Theorem 2 in the two-agent case.

Theorem 3 *If R is a two-agent rule satisfying equal treatment of equals, and b is a two-agent lower bound satisfying dual stability and constant duality gap, then $O^b(O^{b^d}(R)) = O^{b^d}(O^b(R)) = CD_b$.*

Proof. Let R be a two-agent rule satisfying *equal treatment of equals*, and let b be a two-agent lower bound satisfying *dual stability* and *constant duality gap*. Let $(\{i,j\},(c_i,c_j),E) \in \mathcal{D}$ be a two-agent problem. Then, by the proof of Theorem 2,

$$\begin{aligned}&O^{b^d}(O^b(R(\{i,j\},(c_i,c_j),E))) = O^b(O^{b^d}(R(\{i,j\},(c_i,c_j),E))) = \\ &b(\{i,j\},(c_i,c_j),E) + R(\{i,j\}, b^d(\{i,j\},(c_i,c_j),E) - b(\{i,j\},(c_i,c_j),E), E - B(\{i,j\},(c_i,c_j),E)),\end{aligned}$$

where $B(\{i,j\},(c_i,c_j),E) = b_i(\{i,j\},(c_i,c_j),E) + b_j(\{i,j\},(c_i,c_j),E)$. By constant duality gap (of b) and equal treatment of equals (of R) the result follows.[8] ■

The minimal rights lower bound satisfies constant duality gap in the two-agent case. Consequently, we have the following corollary, which corresponds to Theorem 2 in Thomson and Yeh (2008).

Corollary 4 *If R is a two-agent rule satisfying equal treatment of equals, then $O^m(O^t(R)) = O^t(O^m(R)) = CD$.*

4.3 Fixed points and recursive iterations

Another interesting issue related to operators is to explore the rules that remain unchanged when applied to them. A similar question has been recently addressed by Dominguez (2010) in the specific case of lower bounds. More precisely, Dominguez (2010) shows that if a lower

[8]Note that the result could be extended to the general case of n claimants upon imposing constant duality gap for that general case, and obtaining the rule that allocates the lower bound first and then divides the remainder equally among all agents.

bound b is continuous and *positive* (i.e., for each $(N,c,E) \in \mathcal{D}$, $b_i(N,c,E) \geq 0$ for each $i \in N$, with some strict inequality), then there is a unique rule satisfying invariance under the assignment of such lower bound (in the parlance of our paper, the composition operator O^b has a unique fixed point, i.e., there is a unique S such that $S^b = S$). Furthermore, such a rule is obtained by applying the lower bound recursively.[9] Thanks to Theorem 1, we can extend this result to general upper bounds and baselines. We say that an upper bound is *strict* if for each $(N,c,E) \in \mathcal{D}$, $b_i(N,c,E) \leq c_i$ for all $i \in N$, with at least one strict inequality. It is straightforward to show that a lower bound is positive if and only if its dual upper bound is strict. The next result summarizes our findings in this section.[10]

Theorem 4 *The following statements hold:*

- *If a continuous lower bound b is positive then there is a unique fixed point of O^b. This fixed point is the rule obtained by applying the lower bound b recursively.*

- *If a continuous upper bound b is strict then there is a unique fixed point of O^b. This fixed point is the rule obtained by applying the upper bound b recursively, and it corresponds to the dual rule of the unique fixed point of O^{b^d}.*

- *If a continuous baseline b is strict and positive, then there is a unique fixed point of O^b. This fixed point is the rule obtained by applying the baseline b recursively.*

Proof. As mentioned above, the first statement is shown by Dominguez (2010). As for the second statement, let b be a strict continuous upper bound. Then, b^d is a positive continuous lower bound. By the first statement, there is a unique rule S, such that $S^{b^d} = S$ (and, moreover, S is the rule that is obtained by applying the bound iteratively). Then, $O^d(S^{b^d}) = O^d(S)$. Let $R = O^d(S)$. As R and S are dual rules, then, by Theorem 1, so are S^{b^d} and R^b. Thus, $R^b = O^d(S^{b^d}) = O^d(S) = R$, which shows that R is indeed the unique fixed point of O^b. It also follows, by duality, that R is the rule obtained upon applying recursively the upper bound b. This proves the second statement. The third one follows from the first two statements. More precisely, positivity guarantees that the residual endowment to be allocated decreases for those

[9] It is worth mentioning that if positivity is dropped then the result does not hold, and the corresponding composition operator may well have many fixed points. For instance, all rules within the TAL-family (e.g., Moreno-Ternero and Villar, 2006) from the constrained equal losses rule to the Talmud rule are fixed points of O^m.

[10] Statement 1 is due to Dominguez (2010).

steps in which the recursive iteration yields a feasible vector, as in the case of lower bounds; strictness guarantees that claims are reduced for those steps in which the recursive iteration yields an unfeasible vector, as in the case of upper bounds. Continuity concludes. ■

It is worth mentioning that, as with the case of positivity, strictness is necessary to guarantee the uniqueness stated in the previous statements.[11] Note that the upper bound μ^d satisfies strictness. The unique fixed point of its corresponding composition operator is hence the dual rule of the so-called recursive rule, introduced and analyzed by Dominguez and Thomson (2006). Proportional lower (upper) bounds satisfy positivity (strictness). As mentioned above, the proportional rule is a fixed point of the corresponding operators. Theorem 4 shows uniqueness.[12] As for the operators $\{O^{\theta}\}_{\theta\in(0,1)}$, Theorem 4 also shows that they each have a unique fixed point.

5 Preservation of axioms

In this final section we undertake a systematic investigation of which properties are preserved under the composition operators. An axiom is said to be *preserved* under an operator if any rule that satisfies the axiom is mapped by the operator into a rule that also satisfies the axiom. The literature has provided a wide variety of axioms for rules reflecting ethical or operational principles. Here we shall concentrate on those formalizing the principles of impartiality, priority, and solidarity, which have a long tradition in the theory of justice (e.g., Moreno-Ternero and Roemer, 2006), but also some operational (independence, strategic, and procedural) properties widely used in the literature on the problem of adjudicating conflicting claims (e.g., Thomson, 2003). In what follows, and in order to save space, we simply enumerate the properties and refer readers to earlier literature for motivation and formal definitions. We also dismiss some proofs, which are available upon request.

Equal Treatment of Equals: agents with equal claims should receive equal amounts; *Anonymity*: any "renaming" of claimants should be accompanied by a parallel reassignment of awards; *Or-*

[11] If strictness is dropped then the corresponding composition operator may well have many fixed points. For instance, all rules within the TAL-family (e.g., Moreno-Ternero and Villar, 2006) from the Talmud rule to the constrained equal awards rule are fixed points of O^t.

[12] One could extend the idea of proportional lower bounds by assigning to each agent a fixed portion of any (not necessarily proportional) allocation. Formally, for each $\alpha \in (0,1)$, and rule R, $b_{(R,\alpha)}(N,c,E) = R(N,c,\alpha E)$. It turns out that if the rule R satisfies the property of composition up, then it is a fixed point of the operator $O^{b_{(R,\alpha)}}$. A counterpart dual result can also be obtained for the corresponding upper bounds, when composition down is considered.

der Preservation: agents with larger claims receive larger awards but face larger losses too; *Resource Monotonicity*: if there is more to be divided, nobody should lose; *Claims Monotonicity*: if an agent's claim increases, she should receive at least as much as she did initially; *Linked Monotonicity*: if an agent's claim and the endowment increase by the same amount, the agent's award should increase by at most that amount; *Population monotonicity*: if new claimants arrive, each claimant initially present should receive at most as much as he did initially; *Resource-and-Population Monotonicity*: if new claimants arrive and the endowment increases by the sum of their claims, then each claimant initially present should receive at least as much as he did initially; *Resource-and-Population Uniformity*: the arrival of new agents should affect all the incumbent agents in the same direction; *Consistency*: if some claimants leave with their awards and the problem of dividing among the remaining claimants what is left is considered, these claimants should receive the same awards as initially;[13] *Scale Invariance*: if claims and endowment are multiplied by the same positive number, then so should all awards.

To conclude with the inventory of properties, we state two new independence properties that generalize two existing properties in the literature, known as *Minimal Rights First* and *Claims Truncation Invariance*. Formally, for any lower bound b, we say that a rule satisfies *b-first* if $R(N,c,E) = b(N,c,E) + R(N, c - b(N,c,E), E - \sum_i b_i(N,c,E)) = R^b(N,c,E)$, for each $(N,c,E) \in \mathcal{D}$. Similarly, for any upper bound b, we say that a rule satisfies *b-invariance*, if $R(N,c,E) = R(N, b(N,c,E), E) = R^b(N,c,E)$, for each $(N,c,E) \in \mathcal{D}$.[14]

For any given property $\mathcal{P}$, $\mathcal{P}^d$ is the *dual property of* $\mathcal{P}$ if for each rule R, R satisfies $\mathcal{P}$ if and only if its dual rule R^d satisfies $\mathcal{P}^d$. Claims monotonicity and linked monotonicity are dual properties. The same occurs for population monotonicity and resource-and-population monotonicity. A property is said to be self-dual if it coincides with its dual. Equal treatment of equals, order preservation, and resource monotonicity are instances of self-dual properties. Similarly, if b^d is the dual baseline of a given lower bound b then *b-first* and *b^d-invariance* are dual properties.

Our next result helps to identify the preservation of some axioms.

Theorem 5 *A property is preserved under the composition operator O^b if and only if its dual*

[13] See Young (1987, 1988) or Moulin (2000) for important implications of the idea of consistency in this context and Thomson (1996) for a survey of the many applications that have been made on this principle in this and related contexts.

[14] Using a different parlance, a rule satisfies *b-first* if b is a lower bound and the rule is a fixed point of O^b, and *b-invariance* if b is an upper bound and the rule is a fixed point of O^b.

property is preserved under the composition operator O^{b^d}.

Proof. Let $\mathcal{P}$ be a property that is preserved under O^b, $\mathcal{P}^d$ its dual, and let R be a rule satisfying $\mathcal{P}^d$. We need to show that R^{b^d} satisfies $\mathcal{P}^d$. As $\mathcal{P}$ is dual to $\mathcal{P}^d$, $S = R^d$, the dual rule of R, satisfies $\mathcal{P}$. As $\mathcal{P}$ is preserved under O^b, it follows that S^b satisfies $\mathcal{P}$. By Theorem 1, $S^b = R^{b^d}$, which concludes the proof of the "if" statement. The proof of the "only if" part goes along the same lines. ■

The next corollary summarizes some straightforward consequences of the above theorem. Its first statement corresponds to Theorem 4 in Thomson and Yeh (2008).

Corollary 5 *The following statements hold:*

- *A property is preserved under O^m if and only if its dual property is preserved under O^t.*
- *A property is preserved under O^μ if and only if its dual property is preserved under O^{μ^d}.*
- *A property is preserved under O^{ρ_α} if and only if its dual property is preserved under $O^{\rho_\alpha^d}$.*
- *A property is preserved under O^θ if and only if its dual property is preserved under $O^{1-\theta}$.*

For arbitrary baseline profiles it is not difficult to show that almost none of the above axioms are preserved by the composition operators. A notable exception are the so-called independence properties described above, which generalize the properties of minimal rights first and claims truncation invariance.

Proposition 1 *The following statements hold:*

- *If b is a lower bound then O^b preserves b-first.*
- *If b is an upper bound then O^b preserves b-invariance.*

Proof. By Theorem 5, we only need to prove one statement. Let b be an upper bound, R a rule satisfying b-invariance, and $S = O^b(R) = R^b$. For any $(N, c, E) \in \mathcal{D}$, $S^b(N, c, E) = S(N, b(N, c, E), E) = R^b(N, b(N, c, E), E)$. As R satisfies b-invariance, it follows that

$$R^b(N, b(N, c, E), E) = R(N, b(N, c, E), E) = R^b(N, c, E) = S(N, c, E).$$

Altogether, we have that $S^b(N, c, E) = S(N, c, E)$, as desired. ■

As mentioned above, most of the usual axioms are not preserved by the composition operators. It turns out, however, that many axioms are *consequently preserved*. By consequent preservation we mean that if a rule R satisfies a property $\mathcal{P}$, and the baseline does that too, then R^b also satisfies this property.

Proposition 2 *Equal treatment of equals, anonymity, order preservation and scale invariance are consequently preserved.*

Proof.

Let us start with equal treatment of equals. A baseline b satisfies equal treatment of equals if for each $(N, c, E) \in \mathcal{D}$, and $i, j \in N$ such that $c_i = c_j$ then $b_i(N, c, E) = b_j(N, c, E)$. Let R and b be a rule and a baseline, respectively, satisfying equal treatment of equals. Let $(N, c, E) \in \mathcal{D}$ be given and let $i, j \in N$ be such that $c_i = c_j$. Then, $b_i(N, c, E) = b_j(N, c, E)$ and $c_i - b_i(N, c, E) = c_j - b_j(N, c, E)$ and therefore, $R_i^b(N, c, E) = R_j^b(N, c, E)$.

A baseline b is anonymous if for each $(N, c, E) \in \mathcal{D}$, $\pi \in \Pi^N$, and $i \in N$, $b_{\pi(i)}\left(N, (c_{\pi(i)})_{i \in N}, E\right) = b_i(N, c, E)$. Let R and b be an anonymous rule and baseline, respectively. Let $(N, c, E) \in \mathcal{D}$ be given and let also $\pi \in \Pi^N$ and $i \in N$ be given. Then, if $E \leq \sum_i b_i(N, c, E)$, $R^b_{\pi(i)}(N, (c_{\pi(i)})_{i \in N}, E) = R_{\pi(i)}(N, b(N, (c_{\pi(i)})_{i \in N}, E), E) = R_{\pi(i)}(N, (b_{\pi(i)}(N, (c_{\pi(i)})_{i \in N}, E))_{i \in N}, E) = R_i(N, b(N, c, E), E) = R_i^b(N, c, E)$. Similarly, if $E \geq \sum_i b_i(N, c, E)$, then $R^b_{\pi(i)}(N, (c_{\pi(i)})_{i \in N}, E) = b_{\pi(i)}(N, (c_{\pi(i)})_{i \in N}, E) + R_{\pi(i)}(N, (c_{\pi(i)} - (b_i(N, (c_{\pi(i)})_{i \in N}, E))_{i \in N}, E - \sum_i b_i(N, (c_{\pi(i)})_{i \in N}, E)) = b_i(N, c, E)) + R_i(N, (c - b(N, c, E), E - \sum_i b_i(N, c, E)) = R_i^b(N, c, E))$, as desired.

We now move to order preservation. A baseline b is order preserving if for each $(N, c, E) \in \mathcal{D}$, and $i, j \in N$ such that $c_i < c_j$ then $b_i(N, c, E) \leq b_j(N, c, E)$ and $c_i - b_i(N, c, E) \leq c_j - b_j(N, c, E)$. Let R and b be an order-preserving rule and baseline, respectively. Let $(N, c, E) \in \mathcal{D}$ be given and let $i, j \in N$ be such that $c_i < c_j$. Then, $b_i(N, c, E) \leq b_j(N, c, E)$ and $c_i - b_i(N, c, E) \leq c_j - b_j(N, c, E)$. As R is order preserving, it follows that $R_i(N, b, E) \leq R_j(N, b, E)$ if $E \leq \sum_i b_i(N, c, E)$ and that $b_i(N, c, E) + R_i(N, c - b, E - \sum_i b_i(N, c, E)) \leq b_j(N, c, E) + R_j(N, c - b, E - \sum_i b_i(N, c, E))$ if $E \geq \sum_i b_i(N, c, E)$. Then, $R_i^b(N, c, E) \leq R_j^b(N, c, E)$. Similarly, as R is order preserving, it follows that $b_i(N, c, E) - R_i(N, b, E) \leq b_j(N, c, E) - R_j(N, b, E)$ if $E \leq \sum_i b_i(N, c, E)$. As b is order preserving, this implies that $c_i - R_i(N, b, E) \leq c_j - R_j(N, b, E)$ if $E \leq \sum_i b_i(N, c, E)$. Finally, as R and b are order preserving, it follows that $c_i - b_i(N, c, E) - R_i(N, c - b, E - \sum_i b_i(N, c, E)) \leq c_j - b_j(N, c, E) - R_j(N, c - b, E - \sum_i b_i(N, c, E))$ if $E \geq \sum_i b_i(N, c, E)$. Thus, $c_i - R_i^b(N, c, E) \leq c_j - R_j^b(N, c, E)$, which concludes the proof.

Finally, a baseline b is scale invariant if for each $(N, c, E) \in \mathcal{D}$, and $\lambda > 0$, $b(N, \lambda c, \lambda E) = \lambda b(N, c, E)$. Let R and b be a scale-invariant rule and baseline, respectively. Let $(N, c, E) \in \mathcal{D}$

be given and let $\lambda > 0$ be given. Then, if $E \leq \sum_i b_i(N,c,E)$, $R^b(N,\lambda c,\lambda E) = R(N, b(N,\lambda c,\lambda E),\lambda E) = R(N,\lambda b(N,c,E),\lambda E) = \lambda R(N,b(N,c,E),E) = \lambda R^b(N,c,E)$. Similarly, if $E \geq \sum_i b_i(N,c,E)$, then $R^b(N,\lambda c,\lambda E) = b(N,\lambda c,\lambda E) + R(N,\lambda c - b(N,\lambda c,\lambda E),\lambda E) = \lambda b(N,c,E) + \lambda R(N,b(N,c,E),E) = \lambda R^b(N,c,E)$, as desired. ■

We now move to a set of properties that require additional conditions on baselines, to the consequent counterpart properties of the baselines, to be preserved. Borrowing the term from Hokari and Thomson (2008), we refer to this aspect as *assisted preservation*.

Let us start with claims monotonicity. We say that an upper bound b satisfies *strong claims monotonicity* if, for all $(N,c,E) \in \mathcal{D}$ and $i \in N$, $c_i < c'_i$ implies $b_i(N,(c_i,c_{N\setminus\{i\}}),E) < b_i(N,(c'_i,c_{N\setminus\{i\}}),E)$ and $b_j(N,(c_i,c_{N\setminus\{i\}}),E) = b_j(N,(c'_i,c_{N\setminus\{i\}}),E)$ for each $j \in N \setminus \{i\}$.[15] It is not difficult to show that claims monotonicity is preserved if assisted by the previous condition. More precisely, if R is claims monotonic and b is an upper bound that satisfies *strong claims monotonicity*, then R^b satisfies claims monotonicity too.

As for population monotonicity, we say that an upper bound satisfies *strong population monotonicity* if, for each $(N,c,E) \in \mathcal{D}$ and $(N',c',E) \in \mathcal{D}$ such that $N \subseteq N'$ and $c'_N = c$, then $b_i(N',c',E) = b_i(N,c,E)$, for each $i \in N$.[16] It is not difficult to show that population monotonicity is preserved if assisted by the previous condition.

Thanks to Theorem 5, we also have that linked monotonicity and resource-and-population monotonicity are preserved if assisted by the dual properties of strong claims monotonicity and strong population monotonicity, respectively. To summarize:

Proposition 3 *The following statements hold:*

- *If R satisfies claims monotonicity and b is an upper bound that satisfies strong claims monotonicity, then R^b satisfies claims monotonicity.*

- *If R satisfies linked monotonicity and b is a lower bound that satisfies strong linked monotonicity, then R^b satisfies linked monotonicity.*

- *If R satisfies population monotonicity and b is an upper bound that satisfies strong linked monotonicity, then R^b satisfies population monotonicity.*

- *If R satisfies resource-and-population monotonicity and b is a lower bound that satisfies strong resource-and-population monotonicity, then R^b satisfies resource-and-population*

[15] Note that this condition implies claims monotonicity.

[16] Note that this condition implies population monotonicity.

monotonicity.

The remaining properties described above are highly disruptive. This is certainly the case of the two composition properties, which might be somewhat surprising given the underlying connection between these properties and the composition operators. Something similar happens with resource monotonicity, consistency, resource-and-population uniformity, and self-duality. The reason for such a disruptive behavior in all these cases is that the effect of each of these properties on the baseline and the primitive rule cannot be disentangled, in contrast with the previous properties.

We conclude referring to the properties dealing with the immunity of rules to coalitional manipulation. It is well known that the proportional rule is essentially the only non-manipulable rule (e.g., Ju et al., 2007).[17] It follows from here that if the proportional rule is not a fixed point of a given composition operator, then such operator does not preserve any of the properties of non manipulability.

The following corollary, whose content is shown directly by Thomson and Yeh (2008), is a straightforward consequence of the results in this section.

Corollary 6 *The following statements hold:*

- O^t *preserves invariance under claims truncation, equal treatment of equals, anonymity, order preservation, scale invariance, claims monotonicity and population monotonicity.*
- O^m *preserves minimal rights first, equal treatment of equals, anonymity, order preservation, scale invariance, linked monotonicity and resource-and-population monotonicity*

6 Final remarks

We have presented in this paper a unifying framework for the problem of adjudicating conflicting claims. Thomson and Yeh (2008) uncovered the structure of the space of rules for that problem upon studying operators on such a space. In this paper, we have generalized their contribution with a systematic analysis of a family of operators generalizing two of theirs. This has allowed us to scrutinize further the structure of the space of rules, deriving new lessons and testing the robustness of other known aspects.

[17]This is not only the case for manipulations via merging or splitting claims, but also with respect to any other similar form of manipulation such as reallocation or transfers of claims (e.g., Ju et al, 2007).

We believe that the main important lesson of our analysis is that three apparently unrelated approaches to the problem of adjudicating conflicting claims (namely; baseline rationing, composition properties, and lower bounds) can be unified. It is left for further research to explore whether such feature extends to other related models.

Our analysis and results are also relevant for the so-called Talmud rule, whose initial analysis originated a relevant portion of the literature dealing with the problem of adjudicating conflicting claims. As mentioned above, the Talmud rule is a compromise between two other focal rules (the constrained equal-awards and the constrained equal-losses rules). More precisely, the Talmud rule behaves as follows. First, apply equal division until the claimant with the smallest claim has obtained one half of her claim. Then, that agent stops receiving additional units and the remaining amount is divided equally among the other agents until the claimant with the second smallest claim gets one half of her claim. The process continues until every agent has received one half of her claim, or the available amount is distributed. If there is still something left after this process, agents are invited back to receive additional shares. Now agents receive additional amounts sequentially starting with those with larger claims and applying equal division of their losses. As such, the rule can be seen as imposing a rationing of the same sort for each individual and the whole society, according to the profile of half claims. If the endowment falls short of half of the aggregate claim, then no agent gets more than half of her claim. Similarly, if the endowment is above one half of the aggregate claim, no agent gets less than half of her claim. It turns out, as mentioned above, that our family of operators introduced here are also somehow reflecting this same Talmudic dictum, but in a more general way, as half claims are replaced by arbitrary baselines.

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