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Welfare-poverty measurement

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Keywords: poverty, welfare loss, individual poverty measure, distributive impact of poverty.

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Abstract

This paper provides an approach to poverty measurement that relies on the interpretation of poverty as a *welfare loss*. Our contribution is twofold. On the one hand, we analyse the relationship between individual and aggregate indicators, by introducing the notion of "distributive impact of poverty" (a measure of the poverty loss due to the inequality among the poor). We show that a welfare inequality measure can be expressed as the sum of the average individual welfare poverty plus the distributive impact of poverty. On the other hand, we extend the analysis to the case of a society made of several population subgroups, by using a decomposability principle consistent with this approach. An empirical application, regarding educational poverty in the OECD, out of the data in PISA 2009, illustrates the extent of our method.

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1. Introduction

The persistence of inequalities in the distribution of income and wealth (e.g. Shorrocks & Davis (2011)) is one of the main economic and social problems and the subject of many policy measures. The assessment on the extent of inequality and poverty, and on the efficacy of the policies designed to improve equality is a relevant part of the on-going research agenda. Theoretical and applied contributions abound and the concern for inequality and poverty seems to expand in these times of economic turbulences.

Since the pioneering work of Amartya Sen (1976) the measurement of poverty is assumed to involve three different aspects: *incidence*, *intensity* and *inequality*. That is, how many poor people are in society, how poor they are, and how unequal is the distribution of the achievements among the poor. Poverty measurement typically starts by defining a poverty line, as the minimum achievement deemed acceptable, and then proceeds to construct an evaluation function that applies to those agents whose incomes are below that threshold.

A *poverty index* is a function that associates real numbers to income distribution vectors, which incorporates some basic value judgements. There are many properties that we can ask for a function to qualify as a poverty index (see Chakravarty (2009, Ch. 2) for a discussion). The most basic ones are: *Symmetry* (permuting people's incomes does not change the value of the index), *Continuity* (the index is continuous in the incomes of the poor), *Focus* (the index is independent on the incomes of the non-poor), *Monotonicity* (reducing the income a poor, other things equal, increases the value of the index), *Scale Independence* (multiplying incomes and the poverty threshold by a positive constant does not change the value of the index), and *Normalization* (the index takes on the value zero when there are no poor people and is positive otherwise).

The measurement of poverty may refer to the distribution of a single variable or a combination of them, adopt a subjective or an objective approach, refer to some absolute or relative threshold to define who are the poor, and consider quantitative and/or categorical variables (see Wagle (2008), Chakravarty (2009), or Haughton & Khandker (2009) for comprehensive discussions of those topics). We focus here on objective poverty measurement for quantitative variables with respect to a given poverty line. We consider the case of single reference variable (that we call income) as well as the problem of measuring multidimensional poverty.

Our approach to the construction of poverty indices relies on the interpretation of poverty as a *welfare loss*, following the standard Atkinson-Kolm-Sen approach used in the normative theory of income inequality. This idea already appears in the works of Blackorby & Donaldson (1980) and was later developed in a series of papers by Clark, Hemming & Ulph (1981), Vaughan (1987), Pyatt (1987), and Lewis & Ulph (1988) among others (see also the discussion in Kakwani (1997) and Chakravarty (2009, 2.3.3)). Our contribution here is neither to define new poverty measures nor to provide alternative axiomatic characterizations of those already existing. We rather aim at: (i) Reviewing welfare poverty measures in order to establish clear links between individual and aggregate indicators, both in the uni-dimensional and the multi-dimensional case; and (ii) Extending the analysis to the case of a society made of several population subgroups, by using a decomposability principle consistent with this approach.

Concerning point (i) we introduce in Section 2 the notion of "distributive impact of poverty", a term that measures the poverty loss due to the inequality among the poor. Then we show that a welfare inequality measure can be expressed as the sum of the average individual welfare poverty plus the distributive impact of poverty. This allows computing the impact of the dispersion of achievements among the poor on the poverty index by means of standard inequality measures. We also show that this approach can be easily extended to a multidimensional context, provided we assume factor decomposability. Interestingly enough, in this case the set of the poor is endogenously determined out of the values of the different thresholds.

Concerning point (ii) we introduce in Section 3 a notion of decomposability between population subgroups that differs from the standard one in this literature (e.g. Foster & Shorrock (1991)). It consists of the weighted sum of the poverty measures of the population subgroups, with weights given by their population shares, plus the distributive impact of poverty between those groups. We show that the decomposability of a welfare poverty measure resolves in the decomposability of the inequality index involved in the distributive impact of poverty component. This has an obvious implication on the class of inequality measures that are admissible and hence helps closing the evaluation formula in a natural way.

We illustrate our approach in Section 4 by means of an empirical application concerning the measurement of educational poverty, out of the data of the 2009 wave of the Program for International Students Assessment (PISA). We identify educational poverty with the insufficient educational achievements, according to the PISA, regarding three different dimensions: mathematics, reading competence and science. The report identifies six different levels of competence, parameterized by certain ranges of the test scores, and declares level 2 as the minimum admissible level of competence. We analyse the educational poverty in terms of the individual scores for the three educational dimensions. The results show that educational poverty in the OECD countries gives some relevant information about the distribution of the students' performance, information that is not reflected in the average scores of the countries.

2. The reference model

2.1. The single-dimensional case

Let $\mathbf{y} \in \Re_{+}^{n}$ stand for an income distribution vector in a society *N* consisting of *n* agents and let $z > 0$ be a positive scalar that defines the *poverty line*, to be understood as the minimum income admissible for an agent in this society. An agent will be considered poor whenever her income is smaller than *z*. We shall not discuss here how this poverty line is determined, even though this is obviously one of the key elements in poverty measurement. We denote by $Q(\mathbf{y}, z)$ the set of the poor, that is, $Q(\mathbf{y},z)=\{i \in N \ / \ y_i < z\}$. Let Ω the set of all possible income distributions for any (finite) population size. A **poverty index** is a function $P:\Omega \times \Re_{+} \to \Re$ that associates to each pair $(\mathbf{y}, z)$ a real number $P(\mathbf{y}, z)$ to be interpreted as a synthetic measure of poverty.

$W(\mathbf{y})$ stands for the welfare evaluation of an income distribution vector, $\mathbf{y} \in \Re_{+}^{n}$, where *W* is some social welfare function with the conventional properties. When inequality matters in the evaluation of income distribution, the society's aggregate welfare associated with an income vector **y** can be expressed in terms of the *egalitarian equivalent income*, $y^{e}(\mathbf{y})$. That is, the value of the income that equally distributed would yield the same welfare level as the actual distribution. This value is implicitly defined by the equation: $W\left[\mathbf{1}_n y^{e}(\mathbf{y})\right] = W(\mathbf{y})$, where $\mathbf{1}_n$ is the unit vector of dimension *n*. Following the standard approach in the normative theory of income distribution, total welfare can then be expressed as *n* times the egalitarian equivalent income. That is,

$$W(\mathbf{y}) = ny^e(\mathbf{y}) = n\mu(\mathbf{y})\left[1 - I(\mathbf{y})\right] \qquad [1]$$

This expression tells us that we can approximate the social welfare associated with an income distribution **y** as the aggregate income deflated by inequality, measured by some inequality index $I(\mathbf{y}) \geq 0$. Needless to say, setting the inequality index defines the welfare function and viceversa. Here the term $n\mu(\mathbf{y})I(\mathbf{y})$ provides a measure of the total *welfare loss* due to the unequal distribution of income.

Let us apply that approach to the measurement of poverty, assuming that the poverty threshold is given. Let *z* stand for the *poverty line* and let *p* be the number of poor people in society, that is, the number of agents whose income is below the threshold *z*. We concentrate our attention on the set of poor agents. The very notion of the threshold *z* implies that the expression:

$$W(\mathbf{1}_p z) = pz\left[1 - I(\mathbf{1}_p z)\right] = pz$$

corresponds to the minimum welfare that is admissible for the set of the poor, where $\mathbf{1}_p z$ is a vector with *p* components, all equal to *z*. That is, we can interpret *pz* as the minimum welfare for a society made of *p* agents. The actual welfare level of the poor, according to [1], is given by:

$$W(\mathbf{y}^p) = p\mu(\mathbf{y}^p)\left[1 - I(\mathbf{y}^p)\right]$$

where $\mathbf{y}^p$ is the income vector of the poor and $p\mu(\mathbf{y}^p)I(\mathbf{y}^p)$ is the welfare loss due to the inequality among the poor.

We can define the **welfare poverty gap**, $D(\mathbf{y}, z)$, as follows:

$$D(\mathbf{y}, z) = W(\mathbf{1}_p z) - W(\mathbf{y}^p) = p\left(z - \mu(\mathbf{y}^p)\left[1 - I(\mathbf{y}^p)\right]\right) \qquad [2]$$

The welfare poverty gap tells us how far away is society from ensuring the minimum admissible welfare to all its members. This function has some of the attributes of a poverty index: it increases with the number of the poor, with the distance of the incomes of the poor to the poverty line, and with the inequality within the poor. Yet it fails to satisfy scale independence and does not take well into account the incidence (as the size of the population is not computed).

From the argument above it follows that we can build a welfare-poverty index as a **relative welfare poverty gap**, which is simply the ratio between the welfare poverty gap and the minimum welfare admissible for the whole society, $nz = W(\mathbf{1}_n z)$. Formally:

$$P(\mathbf{y}, z) = \frac{D(\mathbf{y}, z)}{nz} = \frac{p}{n}\left(1 - \frac{y^e(\mathbf{y}^p)}{z}\right) \qquad [3]$$

The family of poverty indices so constructed appears as the product of the incidence and the intensity of poverty, once intensity has been adjusted by inequality. That is, *the share of poor people times how poor and unequal they are*. This is an intuitive formula, derived from a standard normative approach, easy to interpret, which integrates nicely and rather explicitly the three key aspects of poverty measurement.

We shall refer to the family of poverty indices that derive from equation [3] as **welfare poverty measures**. Those indices generalize the well-known Sen's poverty measure in which $y^e(\mathbf{y}^p) = \mu(\mathbf{y}^p)\left[1 - G(\mathbf{y}^p)\right]$, where $G(\mathbf{y}^p)$ is the Gini index of the poor (see also the extension in Blackorby and Donaldson (1980)).

A welfare poverty measure can be regarded as the sum of the (relative) poverty gap plus a term that measures the distributive impact of poverty. That is,

$$\begin{aligned} P(\mathbf{y},z) &= \frac{p}{n}\left(1 - \frac{\mu(\mathbf{y}^p)}{z}\right) + \frac{p\mu(\mathbf{y}^p)I(\mathbf{y}^p)}{nz} \\ &= HI(\mathbf{y},z) + f(\mathbf{y},z) \end{aligned} \qquad [4]$$

where $HI(\mathbf{y},z)$ is the standard (relative) poverty gap.

This leads to the following:

Definition 1.- *The* ***distributive impact of poverty****, $f(\mathbf{y}, z)$, is the ratio between the welfare loss due to the inequality among the poor and the society's minimum welfare. That is,*

$$f(\mathbf{y},z) := \frac{p\mu(\mathbf{y}^p)I(\mathbf{y}^p)}{nz} \geq 0 \qquad [5]$$

A welfare poverty index satisfies, by construction, the following properties:

- *Normalization.-* Poverty is zero if and only if the set of the poor is empty: $P(\mathbf{y},z) = 0 \Leftrightarrow Q(\mathbf{y},z) = \emptyset$.
- *Focus.-* Changes in the income of the non-poor do not affect the poverty measure. To be precise: Let $\mathbf{y}, \hat{\mathbf{y}}$ be two income distribution vectors relative to a society N with n members, such that $\mathbf{y}^p = \hat{\mathbf{y}}^p$. Then, $P(\mathbf{y},z) = P(\hat{\mathbf{y}},z)$.

Other properties of the welfare poverty measure derive directly from the properties of the associated inequality index that appears in the distributive impact of poverty term (see [4]). This is true, in particular, with respect to the most standard properties, that is:

- *Symmetry.-* Permuting the incomes does not change the value of the index.

- *Population Principle.*- Replicating the population does not change the value of the index.
- *Dalton Principle of Transfers.*- A transfer from a poor to a poorer, without changing their ranking, reduces de value of the index.
- *Scale Independence.*- The index does not depend on the units in which incomes are measured.
- *Restricted continuity* (resp. *differentiability*).- The index is continuous (resp. differentiable) in the incomes of the poor.

That motivates the following:

Definition 2: *We say that a welfare poverty index P(.) is* ***regular*** *(resp.* ***regular****)if it satisfies the following properties: Symmetry, Population Principle, Dalton Principle of Transfers, Scale Independence, and Restricted Continuity (resp. Restricted Differentiability).*

Let P be a welfare poverty measure that satisfies the Population Principle and let $i \in N = \{1, 2, \ldots, n\}$. We define agent *i*'s **individual welfare poverty index**, $P^i(\mathbf{y}, z)$, as:

$$P^i(\mathbf{y}, z) = P(\mathbf{1}_n y_i, z) = \max\left\{0, \ \left(1 - \frac{y_i}{z}\right)\right\} \qquad [6]$$

That is, the individual welfare poverty index of a single agent *i* in a society *N*, corresponds to the welfare poverty index of a society with *n* agents identical to agent *i* (here $\mathbf{1}_n$ is a vector with *n* components all equal to 1). This index will be equal to zero whenever $y_i \geq z$, is such that *p/n* is either 1 or 0, and has, trivially, zero inequality. Note that this notion of individual poverty is a direct implication of the Population Principle.

It is immediate to check that *a welfare poverty index* $P(\mathbf{y}, z)$ *that satisfies the Population Principle corresponds to the sum of the mean individual welfare poverty plus the distributive impact of poverty*. That is,

$$P(\mathbf{y}, z) = \frac{1}{n}\sum_{i=1}^{n} P^i(\mathbf{y}, z) + f(\mathbf{y}, z) \qquad [7]$$

To see this simply observe that:

$$\frac{1}{n}\sum_{i=1}^{n}P^{i}(\mathbf{y},z)+f(\mathbf{y},z)=\frac{1}{n}\sum_{i=1}^{n}\max\left\{0,\left(1-\frac{y_i}{z}\right)\right\}+f(\mathbf{y},z)$$
$$=\frac{1}{n}\sum_{i=1}^{p}\left(1-\frac{y_i}{z}\right)+f(\mathbf{y},z)=$$
$$=\frac{p}{n}\left(1-\frac{\mu(\mathbf{y}^{p})}{z}\right)+f(\mathbf{y},z)=HI(\mathbf{y},z)+f(\mathbf{y},z)$$

Note that the mean individual welfare poverty coincides with the (relative) poverty gap. Therefore, the decomposition provided in [4] or [7] permits one to disentangle neatly the three different components of poverty.

2.2. Multi-dimensional welfare poverty: factor decomposability and the set of the poor

The precedent analysis can be readily extended to the case of multidimensional poverty measurement (see Dardadoni (1995), Tsui (2002), or Bourguignon & Chakravarty (2003) for a discussion), provided we assume factor decomposability.

Suppose now that we have to assess the poverty of a society consisting of *n* agents with respect to *K* different dimensions, all of which can be measured in terms of quantitative variables. Let *Y* denote the *n* x *K* matrix that describes the achievements of the *n* agents with respect to the *K* dimensions, let **z** stand for the *K*-vector of poverty thresholds (one threshold for every dimension), and let **b** the vector of relative weights that we attach to the different dimensions, which are taken as externally given. All entries in *Y*, **z** and **b** are assumed to be strictly positive. A multidimensional poverty index is now a function that applies the space of tuples (*Y*, **z**, **b**) into the real numbers. We, therefore, write *P*(*Y*, **z**, **b**) for our multidimensional poverty index.

When there is a single dimension involved we can define who are the poor independently on the way of measuring the incidence of poverty (see however Subramanian (2012)). The literature on multidimensional poverty has kept the tradition of determining who are the poor as a separate issue of the choice of the poverty index. Yet counting the poor in a multidimensional context is not that simple, because the poverty threshold is now a vector with n >1 components and we may find that some agents' achievements exceed the threshold levels in some dimensions and fall short in some others. Who are the poor in that case? There are two extreme positions in the literature concerning this aspect: the *union approach*, that declares poor anyone who is below the reference value in some dimension, and the *intersection approach*, according to which one person is poor only if all her achievements are simultaneously below the reference values. There is consensus on the fact that the union approach may overestimate our assessment of poverty whereas the intersection approach may underestimate it. That is why we also find some intermediate proposals, as those in Bourguignon and Chakravarty (2003), Alkire and Foster (2007), Lugo and Maasoumi (2008) or Alkire and Santos (2010), even though it does not seem to be a clear-cut principle.

Yet the freedom to decide on the way of counting the poor evaporates when we require the poverty index to satisfy some elementary properties. In particular,

Claim 1: *Let P be a poverty index that satisfies Normalization and the Population Principle. Then, the set of the poor consists exactly of those agents whose individual poverty measure is positive.*

To see this simply notice that, when the poverty index satisfies the Population Principle, we can always define *agent i's individual welfare poverty index*, $P^i(Y,\mathbf{b},\mathbf{z})$, as follows:

$$P^i(Y,\mathbf{b},\mathbf{z}) = P\left(Y\left(\mathbf{y}_i\right),\mathbf{b},\mathbf{z}\right)$$

(where $Y\left(\mathbf{y}_i\right)$ is a matrix all whose rows are equal to the *i*th agent's vector of achievements). Then, Normalization implies that i is poor if and only if $P^i(Y,\mathbf{b},\mathbf{z})>0$, and the result follows. That is, the set of the poor is given by:

$$Q(Y,\mathbf{b},\mathbf{z}) = \left\{ i\in N \;\;/\;\; P^i(Y,\mathbf{b},\mathbf{z})>0 \right\}$$

Consider now the following:

Definition 3: *A multidimensional poverty index P(Y,* ***z****,* ***b****) satisfies* ***Factor Decomposability*** *when it satisfies the following:*

$$P(Y,\mathbf{b},\mathbf{z}) = \sum_{j=1}^{K} b_j P\left(\mathbf{y}(j),z_j\right) \qquad [8]$$

where $P\left(\mathbf{y}(j),z_j\right)$ *is the poverty index relative to the jth dimension considered in isolation.*

Assuming factor decomposability we can define the family of multidimensional welfare poverty measures as those corresponding to the following equation:

$$P(Y,\mathbf{b},\mathbf{z}) = \frac{p}{n}\sum_{j=1}^{K} b_j\left(1-\frac{y^e(\mathbf{y}(j))}{z_j}\right) \qquad [9]$$

In this context, when the welfare poverty index satisfies Factor Decomposability ant the Population Principle, *agent i's individual welfare poverty index*, $P^i(Y,\mathbf{b},\mathbf{z})$, is given by:

$$P^i(Y,\mathbf{b},\mathbf{z}) = P(Y(\mathbf{y}_i),\mathbf{b},\mathbf{z}) = \max\left\{0, \quad \sum_{j=1}^{K} b_j\left(1-\frac{y_{ij}}{z_j}\right)\right\} \qquad [10]$$

That is we also take here the individual poverty index as the poverty index of a society with *n* agents identical to agent *i*, as an immediate derivation from the Population Principle and the assumption of Factor Decomposability.

Note that factor decomposability and the very definition of the individual poverty index imply the existence of substitutability between dimensions for each individual, at a constant rate given by the corresponding relative weights attached to the different dimensions. Consequently, the assumption of factor decomposability determines, together with the Population Principle, the specific way of counting the poor out of equation [10]. That is,

$$Q(Y,\mathbf{b},\mathbf{z}) = \left\{i \in N \ / \ \sum_{j=1}^{K} b_j\left(1-\frac{y_{ij}}{z_j}\right) > 0\right\}$$

The distributive impact of poverty of a welfare poverty measure that satisfies Factor Decomposability will be given by:

$$f(Y,\mathbf{z},\mathbf{b}) = \sum_{j=1}^{K} b_j \frac{p\mu^p(j)}{nz_j} I(\mathbf{y}^p(j)) \qquad [11]$$

where $\mu^p(j)$ is the mean value of the poor with respect to dimension *j* and $I(\mathbf{y}^p(j))$ is the corresponding inequality index concerning the distribution of the *j*th variable among the poor.

The relationship between individual welfare poverty measures and the society's poverty index in equation [7] can be easily extended to this more general context:

Claim 2: *A multidimensional welfare poverty index* $P(Y,\mathbf{b},\mathbf{z})$ *that satisfies Factor Decomposability is the sum of the mean individual welfare poverty plus the distributive impact of poverty. That is,*

$$P(Y,\mathbf{b},\mathbf{z}) = \frac{1}{n}\sum_{i=1}^{p} P^i(Y,\mathbf{b},\mathbf{z}) + f(Y,\mathbf{b},\mathbf{z}) \qquad [12]$$

To see this notice that:

$$\frac{1}{n}\sum_{i=1}^{n} P^i(Y,\mathbf{b},\mathbf{z}) + f(Y,\mathbf{b},\mathbf{z}) =$$

$$\frac{1}{n}\sum_{i=1}^{n} \max\left\{0,\ \sum_{j=1}^{K} b_j\left(1-\frac{y_{ij}}{z_j}\right)\right\} + \sum_{j=1}^{K} b_j \frac{p\mu^p(j)}{nz_j} I\left(\mathbf{y}^p(j)\right)$$

$$= \frac{1}{n}\sum_{i=1}^{p}\sum_{j=1}^{K} b_j\left(1-\frac{y_{ij}}{z_j}\right) + \sum_{j=1}^{K} b_j \frac{p\mu^p(j)}{nz_j} I\left(\mathbf{y}^p(j)\right)$$

$$= \frac{p}{n}\sum_{j=1}^{K} b_j\left(1-\frac{\mu^p(j)}{z_j}\right) + \sum_{j=1}^{K} b_j \frac{p\mu^p(j)}{nz_j} I\left(\mathbf{y}^p(j)\right)$$

$$= \frac{p}{n}\sum_{j=1}^{K} b_j\left(1-\frac{\mu^p(j)\left[1-I\left(\mathbf{y}^p(j)\right)\right]}{z_j}\right) = P(Y,\mathbf{b},\mathbf{z})$$

Now the expression,

$$\frac{1}{n}\sum_{i=1}^{n} P^i(Y,\mathbf{b},\mathbf{z}) = \frac{p}{n}\sum_{j=1}^{K} b_j\left(1-\frac{\mu^p(j)}{z_j}\right) = HI(Y,\mathbf{b},\mathbf{z})$$

gives us the (relative) *poverty gap in a multidimensional and factor decomposable context.*

3. Decomposability by population subgroups

3.1. The single-dimensional case

Consider now that the society under study can be classified into G different population subgroups, g = 1, 2, …, G, according to some socio-demographic criterion. It is interesting in this context to know how much poverty can be attributed to each subgroup, in order to understand the differences between the poor due to those socio-demographic features. The notion of decomposability comes immediately to mind to address this question.

The literature on poverty measurement defines decomposability as follows (see Foster & Shorrocks (1991), Chakravarty (2009, Chapter 2)):

Definition 4: *A poverty index is **decomposable** with respect to G population subgroups if we can write:*

$$P(\mathbf{y},z)=\sum_{g=1}^{G}\frac{n_g}{n}P(\mathbf{y}_g,z) \qquad [13]$$

where $P(\mathbf{y}_g,z)$ *is the poverty index of subgroup g considered as a society on its own and* n_g/n *is the population share of subgroup g.*

By applying repeatedly the property, we conclude that:

$$P(\mathbf{y},z)=\frac{1}{n}\sum_{i=1}^{p}p(y_i,z) \qquad [14]$$

where $p(y_i, z)$ is the corresponding individual poverty measure and the sum runs over the set of the poor.

Equations [7] and [14] show the difference between the family of welfare poverty indices and the family of subgroup decomposable (or, more generally, *consistent*) poverty measures.[2] In particular, welfare poverty measures are not decomposable in the sense of [14] and decomposable poverty measures do not take explicitly into account the inequality between the poor.[3]

In view of equation [7] we define decomposability in our framework as follows:

Definition 5.- *A welfare poverty index P is **decomposable by population subgroups** if it can be written as:*

$$P(\mathbf{y},z)=\sum_{g=1}^{G}\frac{n_g}{n}P(\mathbf{y}_g,z)+f\left[\left(\mu_1^p,\mu_2^p,\dots,\mu_G^p\right),z\right] \qquad [15]$$

where $P(\mathbf{y}_g,z)$ *is the poverty index of subgroup g considered as a society on its own,* n_g/n *is the population share of subgroup g, and* $f\left[\left(\mu_1^p,\mu_2^p,\dots,\mu_G^p\right),z\right]$ *is the distributive impact of poverty in a fictitious population whose members are the representative agents of the different groups.*

Clearly, a repeated iteration of this principle leads to equation [7]. Since the distributional impact of poverty is basically a transformation of the inequality

[2] Note that *consistent* poverty indices resolve in a monotone transformation of decomposable indices under rather general conditions (see Foster & Shorrocks (1991)).

[3] Inequality between the poor does not appear explicitly in the family of decomposable or consistent poverty indices (e.g. the well known family of poverty measures proposed in Foster, Greer & Thorbecke (1984)). The reason is that inequality among the poor is incorporated in an indirect way through the functional form of *P*, by computing the distance of each individual income to the poverty threshold.

among the poor, the decomposability of a welfare poverty measure is closely related to the additive decomposability of the associated inequality index. Recall that an inequality index *I*(.) is additively decomposable by population subgroups if it can be expressed as the weighted sum of the inequality within population subgroups plus the inequality between those subgroups. That is, $I(\mathbf{y})=\sum_{g=1}^{G}\omega_g I(\mathbf{y}_g)+I(\mu_1,\mu_2,\ldots,\mu_G)$, where $\mathbf{y}_g$ is the income vector of group g, ω_g is the weight attached to group g, and μ_g is the mean income of that group.

That connection between inequality and welfare poverty measures yields the following result:

Proposition 1.- *A welfare poverty index is decomposable by population subgroups if and only if the associated inequality measure is additively decomposable with the coefficients of the within groups inequality equal to the subgroups' income shares.*

Proof.-

We know from [7] that,

$$P(\mathbf{y},z)=\frac{1}{n}\sum_{i=1}^{n}P^i(\mathbf{y},z)+f(\mathbf{y},z)$$
$$=\frac{1}{n}\sum_{g=1}^{g}\sum_{i=1}^{p_g}\left(1-\frac{y_{ig}}{z}\right)+\frac{p\mu^p}{nz}I(\mathbf{y}^p)$$

Note that

$$P(\mathbf{y}_g,z)=\frac{1}{n_g}\sum_{i=1}^{p_g}\left(1-\frac{y_{ig}}{z}\right)+f(\mathbf{y}_g^p,z)$$
$$\Rightarrow n_g P(\mathbf{y}_g,z)-n_g f(\mathbf{y}_g^p,z)=\sum_{i=1}^{p_g}\left(1-\frac{y_{ig}}{z}\right)$$

so that, we can write:

$$P(\mathbf{y},z)=\sum_{g=1}^{G}\frac{n_g}{n}P(\mathbf{y}_g,z)-\sum_{g=1}^{G}\frac{n_g}{n}f(\mathbf{y}_g^p,z)+\frac{p\mu^p}{nz}I(\mathbf{y}^p)$$
$$=\sum_{g=1}^{G}\frac{n_g}{n}P(\mathbf{y}_g,z)-\sum_{g=1}^{G}\frac{p_g\mu_g^p}{nz}I(\mathbf{y}_g^p)+\frac{p\mu^p}{nz}I(\mathbf{y}^p)$$

Moreover, decomposability by population subgroups implies that:

$$\frac{p\mu^p}{nz}I(\mathbf{y}^p)-\sum_{g=1}^{G}\frac{p_g\mu_g^p}{nz}I(\mathbf{y}_g^p)=f\left[\left(\mu_1^p,\mu_2^p,\ldots,\mu_G^p\right),z\right]$$

Applying the definition of distributive impact of poverty we get:

$$f\left[\left(\mu_1^p,\mu_2^p,\ldots,\mu_G^p\right),z\right]=\frac{p\mu(\mathbf{y}^p)}{nz}I\left(\mu_1^p,\mu_2^p,\ldots,\mu_G^p\right)$$

From that it follows:

$$\frac{p\mu(\mathbf{y}^p)}{nz} I\left(\mu_1^p, \mu_2^p, \ldots, \mu_G^p\right) = \frac{p\mu^p}{nz} I\left(\mathbf{y}^p\right) - \sum_{g=1}^{G} \frac{p_g \mu_g^p}{nz} I\left(\mathbf{y}_g^p\right)$$

$$\Rightarrow I\left(\mathbf{y}^p\right) = \sum_{g=1}^{G} \frac{p_g \mu_g^p}{p\mu^p} I\left(\mathbf{y}_g^p\right) + I\left(\mu_1^p, \mu_2^p, \ldots, \mu_G^p\right)$$

Q.e.d.

The next result follows:

Corollary: *A regular* welfare poverty index is decomposable by population subgroups if and only if it adopts the following form:*

$$P(\mathbf{y}, z) = \frac{1}{n} \sum_{i=1}^{p} \left(1 - \frac{y_i}{z}\right) + \frac{p\mu^p}{nz} T\left(\mathbf{y}^p\right) \qquad [16]$$

where T(.) is the first index of Theil.

<u>Proof.-</u>

We know from Shorrocks (1984) that an inequality index that satisfies the properties of scale, symmetry, population principle, Dalton principle of transfers, differentiability and scale independence has to be a member of the entropy family that extends Theil's inequality measures. Moreover, among the members of this family, only the first and second indices of Theil satisfy the property that the within groups weights that appear in the decomposition add up to one. And the first index of Theil is the one that uses income shares as weights. Then, in view of Proposition 1, the result follows.

Q.e.d.

3.2. The multi-dimensional case

Take now the case in which society can be partitioned into G different population subgroups in a multidimensional context, assuming both factor decomposability and decomposability between population subgroups (see Chakravarty, Mukerjee & Ranade (1998)). Let $\mathbf{y}_g^p(j)$, p_g denote the vector of achievements of subgroup g with respect to dimension *j* within the set of the poor and let p_g stand for its population size. In view of the decomposability properties assumed and the results in Proposition 1 and its Corollary (for regular welfare poverty measures), we can write:

$$f_T(Y,\mathbf{b},\mathbf{z}) = \sum_{j=1}^{K} b_j \frac{p\mu^p(j)}{nz_j}\left[\sum_{g=1}^{G} \frac{p_g\mu_g^p(j)}{p\mu^p(j)} T\left(\mathbf{y}_g^p(j)\right) + T\left(\mu_1^p(j),\, \mu_2^p(j),\, \ldots,\, \mu_G^p(j)\right)\right]$$

$$= \sum_{j=1}^{K} b_j \frac{p\mu^p(j)}{nz_j} \sum_{g=1}^{G} \frac{p_g\mu_g^p(j)}{p\mu^p(j)} T\left(\mathbf{y}_g^p(j)\right) + \sum_{j=1}^{K} b_j \frac{p\mu^p(j)}{nz_j} f_T\left[\left(\mu_1^p(j),\, \mu_2^p(j),\, \ldots,\, \mu_G^p(j)\right),\, z_j\right]$$

where the last term represents the *between groups* distributive impact of poverty, that we may simply call $f_T^B(Y,\mathbf{b},\mathbf{z})$, in order to save notation. That is,

$$f_T^B(Y,\mathbf{b},\mathbf{z}) := \sum_{j=1}^{K} b_j \frac{p\mu^p(j)}{nz_j} f_T\left[\left(\mu_1^p(j),\, \mu_2^p(j),\, \ldots,\, \mu_G^p(j)\right),\, z_j\right] \qquad [17]$$

We can also define *subgroup g's distributive impact of poverty relative to dimension j* as:

$$f_T\left(\mathbf{y}_g^p(j),\, z_j\right) = \frac{p_g\mu_g^p(j)}{n_g z_j} T\left(\mathbf{y}_g^p(j)\right)$$

Therefore,

$$f_T(Y,\mathbf{b},\mathbf{z}) = \sum_{j=1}^{K} \sum_{g=1}^{G} b_j \frac{n_g}{n} f_T\left(\mathbf{y}_g^p(j),\, z_j\right) + f_T^B(Y,\mathbf{b},\mathbf{z})$$

an expression that says that the distributive impact of poverty in society consists of the sum of the distributive impact of poverty of the population subgroups, relative to the K dimensions, weighted by their corresponding coefficients, plus de between subgroups component.

Now observe that the poverty index of subgroup g, considered as a society on its own, is given by:

$$P\left(Y_g,\mathbf{b},\mathbf{z}\right) = \sum_{j=1}^{K} \sum_{i=1}^{p_g} \frac{1}{n_g}\left(1 - \frac{y_{ig}(j)}{z_j}\right) + f_T\left(Y_g,\mathbf{b},\mathbf{z}\right)$$

where,

$$f_T\left(Y_g,\mathbf{b},\mathbf{z}\right) = \sum_{j=1}^{K} b_j \frac{p_g\mu_g^p(j)}{n_g z_j} T\left(\mathbf{y}_g^p(j)\right)$$

is the distributive impact of poverty in group g.

Bearing this in mind and plugging equation [17] into the multidimensional welfare poverty measure, we conclude:

Claim 3.- *The multidimensional poverty index of a population consisting of several subgroups can be expressed as the weighted sum of the poverty indices of the population subgroups, with weights equal to their population shares, plus the between subgroups component of the distributive impact of poverty. That is,*

$$P(Y,\mathbf{b},\mathbf{z}) = \sum_{g=1}^{G} \frac{n_g}{n} P(Y_g,\mathbf{b},\mathbf{z}) + f^B(Y,\mathbf{b},\mathbf{z}) \qquad [18]$$

Moreover, this corresponds to:

$$P(Y,\mathbf{b},\mathbf{z}) = \frac{1}{n}\sum_{g=1}^{G}\sum_{i=1}^{p_g}\sum_{j=1}^{K} b_j\left(1-\frac{y_{ig}(j)}{z_j}\right) + \sum_{g=1}^{G}\sum_{j=1}^{K} b_j \frac{p_g \mu_g^p(j)}{nz_j} T\left(\mathbf{y}_g^p, j\right) \qquad [19]$$

4. An application: measuring educational poverty

We apply now the poverty analysis presented above to the measurement of educational poverty out of the results in the 2009 wave of the OECD's *Programme for International Student Assessment* (PISA). This Programme provides the richest and most comprehensive database for the evaluation of the educational achievements of 15 year-old students in three different subjects: reading, mathematics and science. Note that the age of the students corresponds to the end of compulsory education for most of the participating countries. Those results, therefore, inform about the basic knowledge provided by the different countries to their citizens. Sixty-five countries and large economies participated in the 2009 wave of PISA; moreover, some countries have also enlarged their samples in order to get relevant data at regional level. Half a million students participated in the study, representing 26 million 15 year-old students of those 65 countries and large economies involved. See OECD (2010).

The PISA establishes six different levels of educational achievements, parameterized in terms of the scores of the tests that students perform for each subject. It is understood that Level 2 is the minimum admissible level of competence to be able to develop a reasonable integration in the labour market and, more generally, in society. Students who do not reach that level are considered to have an *insufficient knowledge*. It is only natural to interpret insufficient knowledge as educational poverty and use the thresholds that define those minimum levels to set the corresponding poverty lines in reading competence, mathematics and science, respectively.

According to the PISA report, the thresholds that define insufficient competence in those aspects are: 407 test score points for mathematics (m), 409 points for reading competence (r), and 420 points for science (s). Therefore, our vector $\mathbf{z}$ of poverty lines is given by: $\mathbf{z} = (407,\ 409,\ 420)$. We consider that those three dimensions are equally important so that the vector of weights is $\mathbf{b}=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$.

The micro-data of the PISA report provide information about the test scores of individuals that conform the representative sample for each country. We take those individual micro-data as our starting point, focusing on the test scores on mathematics, reading and science. Out of these data we are able to compute, for each student *i* in the sample of every OECD country, the corresponding individual poverty index. That is, the number:

$$\frac{1}{3}\left[\left(1-\frac{y_i(m)}{407}\right)+\left(1-\frac{y_i(r)}{409}\right)+\left(1-\frac{y_i(s)}{420}\right)\right]$$

A student is considered as educationally poor whenever this number is positive and non-poor otherwise. Once the set of the poor has been determined, we have calculated all the elements required to compute the poverty index, using the first index of Theil as the right inequality measure. That is, for each OECD country we compute:

$$P(.)=\frac{p}{3n}\left[\left(1-\frac{\mu^p(m)}{409}T\left(\mathbf{y}^p(m)\right)\right)+\left(1-\frac{\mu^p(r)}{407}T\left(\mathbf{y}^p(r)\right)\right)+\left(1-\frac{\mu^p(s)}{420}T\left(\mathbf{y}^p(s)\right)\right)\right]$$

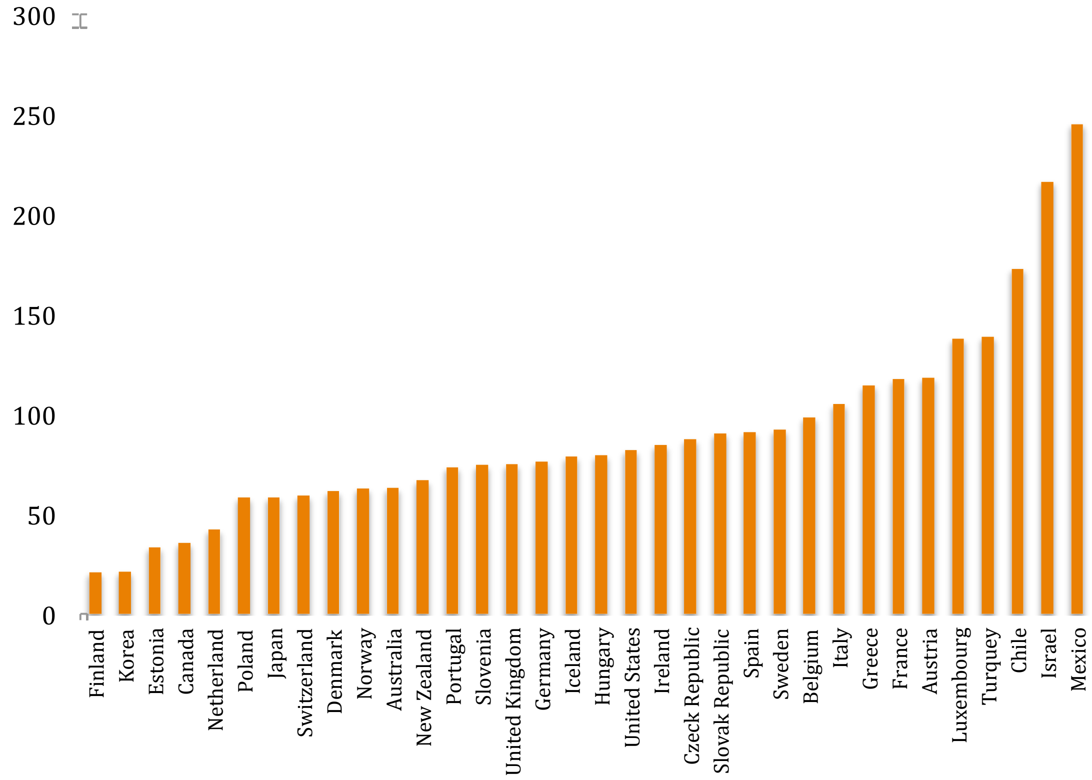
Table 1 below gives us the results we have obtained, both in absolute terms and as percentages of the OECD mean. Figure 1 illustrates those values ordering the countries from best to worse. We also provide the welfare loss due to the distributional impact of poverty (in %), which yields an average value of some 11,6 % loss with a relatively small dispersion. The last two columns of Table 1 tell us the rank of the different countries regarding poverty (from less to more) and average mean scores of the tests (from more to less). The comparison of those ranks shows that poverty analysis provides some information about the performance of the schooling systems that is not reflected in the average scores.[4] Even though the rank correlation is high (a Spearman coefficient of 0.874 for the whole OECD), there are substantial differences in particular cases (e.g. Belgium, France, and Portugal whose rankings differ in 10 or more positions).

[4] The distribution of the students' performance (densities) among the different countries varies substantially, which makes of the mean score an indicator that is not very informative. See the discussion in Villar (2012).

Table 1: OECD Educational Poverty Index and distributional impact of poverty (PISA 2009)

	Poverty index	% of the OECD	Distributional Impact of poverty (%)	Poverty rank	Mean Score rank
Australia	0,0177	62,95	11,28	10	6
Austria	0,0333	118,22	12,53	29	26
Belgium	0,0277	98,41	12,37	25	11
Canada	0,0100	35,65	11,11	4	4
Chile	0,0486	172,72	10,56	32	33
Czeck Republic	0,0246	87,35	12,37	21	23
Denmark	0,0173	61,35	11,34	9	16
Estonia	0,0094	33,35	11,26	3	9
Finland	0,0059	20,90	10,65	1	1
France	0,0331	117,37	11,29	28	18
Germany	0,0215	76,27	12,46	16	10
Greece	0,0322	114,41	11,64	27	30
Hungary	0,0224	79,38	10,89	18	21
Iceland	0,0222	78,68	11,41	17	12
Ireland	0,0238	84,54	10,60	20	19
Israel	0,0609	216,21	12,28	33	31
Italy	0,0296	104,95	12,03	26	27
Japan	0,0164	58,27	10,24	6	3
Korea	0,0060	21,17	9,64	2	2
Luxembourg	0,0388	137,62	11,74	30	29
Mexico	0,0690	245,09	10,15	34	34
Netherland	0,0119	42,08	15,38	5	7
New Zealand	0,0188	66,74	12,15	12	5
Norway	0,0177	62,79	11,06	11	14
Poland	0,0164	58,20	12,04	7	13
Portugal	0,0207	73,44	11,62	13	24
Slovak Republic	0,0255	90,40	12,62	22	25
Slovenia	0,0210	74,44	12,49	14	17
Spain	0,0256	90,86	10,65	23	28
Sweden	0,0259	92,11	11,30	24	22
Switzerland	0,0167	59,14	11,94	8	8
Turquey	0,0391	138,67	11,85	31	32
United Kingdom	0,0211	75,00	12,04	15	15
United States	0,0231	81,87	12,41	19	20

Figure 1: Educational Poverty in the OECD according to PISA 2009 (OECD = 100)



A striking feature of educational poverty is the immense variability that exhibits among OECD countries (and it is even much larger in those countries outside the OECD!). While the coefficient of variation of the test scores is low (around 0,053), the coefficient of variation of educational poverty is more than ten times larger (about 0,548). There seems to be a very different concern for those who do not reach a minimum level of competence among developed countries, differences which are substantial even *within* countries. In the case of Italy, with a poverty index slightly above the OECD mean, we find that the coefficient of variation for its regions is of 0,45, below that of the OECD. In the case of Spain, a country with a poverty index below the OECD mean, we find a regional coefficient of variation of 0,78, much larger than that of the OECD.

It is worth pointing out (see Table 2 in the Appendix) that the variability of the educational poverty index among countries contrasts with the homogeneity of the corresponding distributional impact of poverty. That is, the performance of educationally poor students is relatively homogenous within each country and also between countries (values of Theil index around 0,018 with a small dispersion), whereas they differ substantially with respect to the percentage of poor students (incidence of poverty), whose coefficient of variation is of 0,45.

5. Final remarks

We have presented in this paper a welfare-based approach to poverty measurement, much in line with the standard normative inequality analysis. There are two key concepts derived from this approach that lead the discussion. First, the notion of "individual poverty index", a simple application of the population principle. Second, the notion of "distributive impact of poverty", a way of measuring the poverty loss due to the inequality among the poor. A welfare poverty measure can then be expressed as the sum of the average individual welfare poverty plus the distributive impact of poverty.

The approach can be readily extended to the multidimensional case, assuming factor decomposability. The treatment of decomposability by population subgroups differs from the usual one and is much closer to the standard in inequality measurement. The notions of individual poverty indices and the distributive impact of poverty are also helpful in the context of decomposable poverty indices. Those notions also make it clear that, under very general conditions, the way of counting the poor is essentially endogenous.

To illustrate the working of this approach we have analysed the educational poverty in the OECD countries, out of the 2009 PISA report on the performance of 15-year old students regarding reading, mathematics and science. The analysis of educational poverty provides interesting insights on the working of the compulsory education in developed countries and shows the existence of very large differences.

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APPENDIX

Table 2: Average scores and Theil indices of the poor

	Average score of poor students			% of poor students	Theil indices		
	READ	MATHS	SCIE		READ	MATH	SCIE
AUS	351,20	368,49	362,26	0,13	0,01935	0,01615	0,01877
AUT	339,70	377,49	361,05	0,23	0,02342	0,01803	0,02156
BEL	354,21	363,54	347,60	0,18	0,02172	0,02114	0,02481
CAN	363,56	376,93	369,92	0,09	0,015	0,01322	0,01416
CHE	352,46	380,06	364,32	0,13	0,01774	0,01682	0,01711
CHL	369,18	346,77	370,40	0,36	0,01654	0,01665	0,01536
CZE	353,78	373,94	368,05	0,19	0,01858	0,01684	0,01877
DEU	355,15	369,35	367,97	0,16	0,01874	0,0183	0,01904
DNK	366,44	379,59	359,76	0,15	0,01428	0,01399	0,01682
ESP	355,33	359,40	366,61	0,18	0,01741	0,01746	0,01623
EST	358,17	379,79	387,79	0,09	0,01367	0,01168	0,01199
FIN	360,50	385,23	376,53	0,06	0,01312	0,01067	0,01248
FRA	341,96	357,26	349,32	0,19	0,02507	0,0208	0,02237
GBR	355,23	370,61	369,54	0,16	0,01906	0,01512	0,01852
GRC	362,53	362,94	358,61	0,23	0,01981	0,01711	0,01834
HUN	356,93	358,26	372,22	0,17	0,01726	0,0177	0,01504
IRL	350,28	358,29	361,78	0,16	0,02009	0,01585	0,01905
ISL	354,64	377,54	353,64	0,16	0,01922	0,01558	0,01859
ISR	350,20	335,64	340,44	0,32	0,03058	0,027	0,02792

ITA	356,07	364,60	358,85	0,21	0,02061	0,01867	0,02009
JPN	336,99	372,09	358,21	0,11	0,02036	0,01584	0,01808
KOR	370,31	373,55	368,15	0,05	0,01096	0,01288	0,01184
LUX	335,62	368,47	348,96	0,23	0,02601	0,01928	0,0241
MEX	355,99	356,10	354,56	0,45	0,02041	0,01711	0,01618
NLD	376,93	390,43	372,44	0,13	0,01495	0,01396	0,01699
NOR	363,11	373,25	363,90	0,14	0,01665	0,01375	0,01556
NZL	354,13	370,01	357,23	0,13	0,02064	0,01698	0,02157
POL	363,89	368,13	381,60	0,15	0,01614	0,01503	0,01387
PRT	365,14	363,03	377,95	0,17	0,01587	0,01679	0,01353
SVK	356,25	372,83	362,52	0,19	0,01798	0,01846	0,0208
SVN	350,20	373,48	378,73	0,17	0,01812	0,01733	0,0165
SWE	353,39	364,94	352,23	0,17	0,02093	0,01747	0,02057
TUR	376,44	352,11	369,84	0,31	0,0154	0,02026	0,01465
USA	365,05	365,21	367,00	0,18	0,01863	0,01647	0,01849
OCDE	354,48	361,83	359,44	0,20	0,02036	0,01938	0,02023

Source: OCDE (2010)