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Department of Economics

A new analysis of a simple model of fair allocation

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Abstract

In a recent article, Fragnelli and Gagliardo [Cooperative models for allocating an object, Economics Letters 117 (2012) 227-229] propose several procedures to solve a basic problem of fair allocation. We scrutinize their proposal and contextualize it into recent developments of the literature on bankruptcy problems. Our analysis supports two of the procedures they propose; namely, the Shapley and Talmud rules.

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Keywords*: Fair allocation, Bankruptcy problems, Lower bounds, Upper bounds, Shapley, Talmud, Concede-and-Divide.*

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1 Preliminaries

In a recent paper, Fragnelli and Gagliardo (2012) (in what follows, "FG") analyze a simple distribution problem in which one object (divisible or indivisible) has to be allocated among a group of agents, each having a (strictly) positive valuation of the object. FG propose several methods to solve this problem. One traces back to a classical procedure due to Knaster (1946). Another uses classical values at a coalitional game associated to the problem. Yet a third one is based on the literature on bankruptcy problems, initiated by O'Neill, (1982). Our aim in this note is to extend FG's analysis, as well as to contextualize it into recent developments of the fast-expanding literature on bankruptcy problems (e.g., Thomson, 2003; 2012).

Formally, suppose one object (divisible or indivisible) has to be allocated among a group of agents $N = \{1, \ldots, n\}$. For each agent $i \in N$, her valuation of the object is denoted by $v_i > 0$. Without loss of generality, let us assume that agents are ranked according to their valuations, i.e., $v_1 \leq v_2 \leq \cdots \leq v_n$.

It is straightforward to see that, by Pareto efficiency, the object should be assigned to agent n, whereas the others should be compensated. Thus, an admissible (efficient) allocation for the problem just described is simply a vector $x = (x_1, x_2, \ldots, x_n)$, such that $\sum_{i=1}^{n} x_i = v_n$ (balance condition), and where each $x_i \geq 0$ is interpreted as the compensation for agent $i \in N \setminus \{n\}$.

Besides Pareto efficiency, FG propose several properties for allocation rules: **Individual Equal Sharing** requires that agents obtain at least one n-th of their valuation, i.e., $x_i \geq v_i/n$; **Selfish monotonicity**, on the other hand, states that agents cannot obtain more than a portion of the highest valuation, where this portion is inversely correlated to the number of agents with a higher valuation, i.e., for each $i \in N$, $x_i \leq v_n/|K_i|$, where $K_i = \{j \in N : v_j \geq v_i\}$.

FG consider several allocation rules and study their performance with respect to the above properties. First, they consider the so-called **Knaster** rule, which awards each agent one n-th of her valuation and then divides equally the remainder. Formally, for each $i \in N$,

$$K_i(v) = \frac{1}{n} v_i + \lambda,$$

where $\lambda \geq 0$ is chosen so that the balance condition is guaranteed. Second, they take a game-theoretical approach and consider the rules emerging from using two well-known values (**Shapley** and **Tau**) to solve a coalitional game associated with each problem, whose characteristic function yields as the worth of a coalition the maximum valuation of the members of that coalition, i.e., $u : 2^N \to \mathbb{R}$ such that $u(S) = \max_{i \in S} v_i$, for each $S \subset N$. Formally,

$$S_i(v) = \frac{v_1}{n} + \frac{v_2 - v_1}{n-1} + \frac{v_3 - v_2}{n-2} + \cdots + \frac{v_i - v_{i-1}}{n-i+1},$$

for each $i \in N$, and

$$\tau(v) = \left(\frac{v_{n-1}}{n}, \frac{v_{n-1}}{n}, \ldots, \frac{v_{n-1}}{n}, v_n - v_{n-1} + \frac{v_{n-1}}{n} \right).$$

Third, they propose four rules coming from the literature on bankruptcy problems. On the one hand, the so-called **constrained equal-awards** rule, which makes awards as equal as possible, subject to the condition that no agent gets more than her valuation, and the so-called **constrained equal-losses** rule, which makes losses as equal as possible, subject to the condition that no agent gets a negative amount. Formally, for each $i \in N$,

$$A_i(v) = \min\{v_i, \lambda\},$$

and

$$L_i(v) = \max\{0, v_i - \lambda\},$$

where $\lambda > 0$ is chosen, in each case, so that the balance condition is guaranteed. On the other hand, the so-called **proportional** rule, which makes awards proportional to valuations, i.e.,

$$P(v) = \frac{v_n}{V} \cdot v,$$

where $V = \sum_{i \in N} v_i$, and the **Talmud** rule (e.g., Aumann and Maschler, 1985), which is a compromise between the constrained equal-awards and the constrained equal-losses rules. Formally, for each $i \in N$,

$$T_i(v) = \begin{cases} \min\{\frac{1}{2}v_i, \lambda\} & \text{if } v_n \leq \frac{1}{2}V \\ \max\{\frac{1}{2}v_i, v_i - \mu\} & \text{if } v_n \geq \frac{1}{2}V \end{cases}$$

where λ and μ are chosen so that the balance condition is guaranteed.

2 The analysis

The property of individual equal sharing is a specific lower bound condition. Lower bound conditions have a long tradition in models of distributive justice (e.g., Moulin, 1991; Maniquet, 1996). Even though some specific lower bounds have been consider in the literature on bankruptcy problems from its beginning (for instance, the very definition of a rule includes the requirement that awards be non-negative), Moreno-Ternero and Villar (2004) introduced a focal one, under the name of **securement**. Formally, a (bankruptcy) rule satisfies securement if it guarantees to each agent one n-th of her truncated valuation (claim). Formally, $x_i \geq \frac{1}{n}\min\{v_i, E\}$, for each $i \in N$, where E denotes the endowment to be allocated in the bankruptcy problem. In FG's setting, $E = v_n \geq v_i$, for each $i \in N$ and, thus, securement becomes individual equal sharing. The performance of the bankruptcy-like rules mentioned above, with respect to this property, is summarized in Proposition 3 in FG, which is a straightforward consequence of Theorem 1, and Propositions 2 and 3 in Moreno-Ternero and Villar (2004). The fact that the remaining rules (namely; Knaster, Shapley and Tau) satisfy individual equal sharing is straightforward, as noted by FG.

As for selfish monotonicity, one simply has to note that it is a weakening of the standard notion of order preservation (in awards), i.e., agents with higher valuations receive higher awards. As a consequence, selfish monotonicity is satisfied by each of the rules considered above, as stated in Proposition 5 of FG.[1] Now, selfish monotonicity can be seen as a sort of counterpart of individual equal sharing, as it imposes an upper bound on individuals' awards. Nevertheless, and as shown above, such upper bound is rather weak. An upper bound with a deeper bite could be obtained by imposing the **dual** notion of securement (also introduced by Moreno-Ternero and Villar, 2004), which formally says that $v_i - x_i \geq \frac{1}{n}\min\{v_i, \sum_{j\in N} v_j - E\}$, for each $i \in N$. In FG's setting, this property translates into the requirements $x_i \leq \frac{n-1}{n}v_i$, for each $i \in N \setminus \{n\}$, and $x_n \leq \max\{\frac{n+1}{n}v_n - \frac{V}{n}, \frac{n-1}{n}v_n\}$, to which we shall refer as the **dual of individual equal sharing**. The following proposition summarizes the performance of the above rules with respect to this property.

Proposition 1 *The constrained equal losses, Talmud and Shapley rules satisfy the dual notion of individual equal sharing, whereas the constrained equal awards, proportional, Tau, and Knaster rules violate it.*

Proof. The statements regarding the constrained equal losses and Talmud rules are a consequence of Proposition 3 and Theorem 2 in Moreno-Ternero and Villar (2004). The statements regarding the constrained equal awards, proportional, Tau, and Knaster rules are shown, for

[1] It is worth noting that, in the proof of Proposition 5, FG wrongly consider "claims monotonicity", the property saying that if an agent's claim (valuation) increases then she should receive at least as much as she did initially (e.g., Thomson, 2003), instead of order preservation of awards.

instance, by considering the problem $v = (1, 3, 10)$, as

$$\min\{A_1(v), P_1(v), \tau_1(v), K_1(v)\} > \frac{2}{3} = \frac{n-1}{n} v_1.$$

To conclude, let us show the statement regarding the Shapley rule. Note first that $S_1(v) = \frac{v_1}{n} \leq \frac{n-1}{n} v_1$. Reasoning by induction, let us assume that $S_i(v) \leq \frac{n-1}{n} v_i$, for each $i \in \{1, 2, \dots, m\}$, with $m \leq n-2$. Then,

$$S_{m+1}(v) = S_m(v) + \frac{v_{m+1} - v_m}{n-m} \leq \frac{n-1}{n} v_m + \frac{v_{m+1} - v_m}{n-m} \leq \frac{n-1}{n} v_{m+1},$$

where the last inequality follows from the fact that $n \leq (n-1)(n-m)$. Thus, it only remains to show that $S_n(v) \leq \max\{\frac{n+1}{n} v_n - \frac{V}{n}, \frac{n-1}{n} v_n\}$. Now,

$$S_n(v) = v_n - \sum_{i=1}^{n-1} x_i \leq v_n - \sum_{i=1}^{n-1} \frac{v_i}{n} = \frac{n+1}{n} v_n - \frac{V}{n},$$

where the last inequality follows from the fact that the Shapley rule obeys securement, as mentioned above. ■

It follows from the above proposition, and the discussion preceding it, that the Talmud and Shapley rules emerge as superior procedures in FG's setting, at least regarding their performance with respect to lower and upper bounds. It turns out that these two rules coincide in the two-agent case, but diverge in the case of more than two agents. Actually, the two-agent version of these two rules is the so-called **concede-and-divide** procedure (e.g., Thomson, 2003), denoted by CD, which rationalizes the classical example of the contested garment that appears in the Talmud, which states that the contested portion of the garment is divided equally, and the non-contested parts are conceded to the other agent. Formally, if $v = (v_1, v_2)$ is a two-agent problem, then $CD(v) = (v_1/2, v_2 - v_1/2)$. As the next proposition shows, the two bounds described above actually characterize the concede-and-divide procedure (as shown by Moreno-Ternero and Villar (2004) in the more general setting of bankruptcy problems).

Proposition 2 *In the two-agent case, a rule satisfies individual equal sharing and its dual, if and only if it agrees with concede-and-divide.*

Proof. It is straightforward to see that concede-and-divide satisfies individual equal sharing and its dual, for each two-agent problem. Conversely, let $v = (v_1, v_2)$ be a two-agent problem and R be a rule satisfying individual equal sharing and its dual. Then, by individual equal sharing, $R_1(v) \geq v_1/2$ and, by its dual, $R_1(v) \leq v_1/2$. Consequently, $R_1(v) = v_1/2$ and, by the balance condition, $R(v) = (v_1/2, v_2 - v_1/2) = CD(v)$. ■

For problems involving more than two agents, the Shapley and Talmud rules diverge, although they still satisfy the general version of the two bounds. This is in contrast with the Tau rule, which also coincides with concede-and-divide in the two-agent case (and, hence, satisfies both bounds in such case) but, as shown above, it does not obey the dual of individual equal sharing in the case of more than two agents.

3 Further insights

The previous analysis extends (as well as contextualizes) that of FG by scrutinizing how the rules they considered performed with respect to focal lower and upper bound conditions, which have played an important role in the theory of fair allocation. We have singled out a unique procedure to solve two-agent problems in FG's setting by means of a lower and an upper bound condition. For problems involving more than two agents, no unique procedure exists satisfying those two conditions, and we have actually presented two among those considered by FG. It remains as an open question to characterize the set of procedures satisfying the two bounds in the general case of more than two agents and, ideally, to add some other ethically appealing axioms that, when combined with the two bounds, single out a unique procedure.

Apart from the previous notions, the literature on bankruptcy problems has provided a long list of properties reflecting ethical or operational properties. Most of them are relational properties which do not preserve the constraint $E = v_n$, and hence cannot be brought to FG's context. That is the case of the standard notions modeling the principle of solidarity in the model of bankruptcy problems (e.g., resource monotonicity, population monotonicity, consistency). Nevertheless, there are some punctual properties that could still be considered in this context. Instances are properties modeling the principle of impartiality (such as equal treatment of equals, or anonymity), or the principle of priority (such as the notions of order preservation, in awards and losses). Each of these properties are satisfied by all the rules described above, which enhances their fairness nature (which was the initial motivation of FG's analysis).[2]

To conclude, it is worth mentioning that, as indicated by FG, their model is similar to the model of the so-called **airport problem**.[3] Counterparts of the rules presented in the previous

[2]The principles of impartiality, priority and solidarity have a long tradition in the theory of justice (e.g., Moreno-Ternero and Roemer, 2006).

[3]See Thomson (2007) for a survey of the literature on these problems.

sections and characterizations of them, based on axioms related to the cost-sharing nature of the airport problem, exist in the literature. For instance, the sequential equal contributions rule (which corresponds to the Shapley rule) is the only rule satisfying equal treatment of equals and a property indicating that an agent's contribution should not depend on the costs of the segments he does not use (e.g., Moulin and Shenker, 1992). Similarly, the constrained equal benefits rule (which corresponds to the constrained equal losses rule) is the only rule satisfying equal treatment of equals, a limited version of the independence property mentioned above, and a specific version of the consistency principle (e.g., Hu et al., 2012).

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Appendix that is not part of the submission for possible publication

To save space, we have included in this appendix, which is not for publication, formal statements and proofs of some aspects, as well as some further insights, that have been dismissed from the body of the paper.

- Two other properties (neither connected to lower bounds, nor to upper bounds) were also considered by FG. **Equitability** requires that agents are awarded equally, i.e., $x_i = x_j$, for each $i, j \in N$; **No-Envy** requires that no agent perceives another agent obtains a higher value than herself after the allocation, i.e., $v_{n-1}/n \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n$.[4] The two properties are very restrictive in this context and that is why we have discarded them from our analysis in this note. As a matter of fact, no rule satisfies the two properties and, as noted by FG, only some do so for specific restricted domains.

- A bankruptcy problem is a triplet consisting of a population N, a claims profile $c \in \mathbb{R}^N_+$, and an endowment $E \in \mathbb{R}_+$ such that $\sum_{i \in N} c_i \geq E$. The problem proposed by FG can therefore be seen as a bankruptcy problem in which each claim is interpreted as the valuation of the corresponding agent, and the endowment corresponds with the highest valuation, i.e., $E = c_n$. Four focal rules are considered in this setting. On the one hand, the so-called **constrained equal-awards** rule, A, which makes awards as equal as possible, subject to the condition that no agent gets more than her claim, and the so-called **constrained equal-losses** rule, L, which makes losses as equal as possible, subject to the condition that no agent gets a negative amount. Formally, $A(N, c, E) = (\min\{c_i, \lambda\})_{i \in N}$, and $L(N, c, E) = (\max\{0, c_i - \lambda\})_{i \in N}$, where $\lambda > 0$ is chosen, in each case, so that the balance is guaranteed, i.e., $\sum_{i \in N} x_i = E$. Two other prominent rules are the so-called **proportional** rule, P, which makes awards proportional to claims, i.e., $P(N, c, E) = \frac{E}{C} \cdot c$, and the **Talmud** rule (e.g., Aumann and Maschler, 1985), which is a compromise between the constrained equal-awards and the constrained equal-losses rules.

[4] It is assumed by FG that the value obtained by any agent $i \in N \setminus \{n\}$ is perceived by any other agent $j \in N$ equal to her compensation, whereas the value obtained by agent n is perceived by any other agent $j \in N \setminus \{n\}$ as her evaluation of the object minus the compensations, i.e., $v_j - \sum_{i=1}^{n-1} x_i = v_j - v_n + x_n$. Consequently, the no-envy axiom translates into $v_{n-1}/n \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n$. The first inequality follows from the fact that agent $n-1$ cannot envy agent n. Given our assumption that agents are ranked according to their valuations, such inequality also guarantees that no other agent envies agent n.

Formally, $T(N, c, E) = (\min\{\frac{1}{2}c_i, \lambda\})_{i\in N}$ if $E \leq \frac{1}{2}C$ and $(\max\{\frac{1}{2}c_i, c_i - \mu\})_{i\in N}$ if $E \geq \frac{1}{2}C$, where λ and μ are chosen so that the balance condition is guaranteed.

- Moreno-Ternero and Villar (2004) introduced the property of securement in the context of bankruptcy problems. Formally, a bankruptcy rule R satisfies securement if $R_i(N, c, E) \geq \frac{1}{n}\min\{c_i, E\}$, for each $i \in N$.[5] Moreno-Ternero and Villar (2006) define **lower securement** as the half of the securement property guaranteeing the lower bound for claims below the endowment, i.e., $R_i(N, c, E) \geq \frac{1}{n}c_i$, for each $i \in N$, such that $c_i \leq E$. This notion is, therefore, also identical to individual equal sharing in FG's setting.

- The notion of duality is proposed by Aumann and Maschler (1985) in the context of bankruptcy problems. It refers to the possibility of associating each rule (or property) with a *mirror* procedure that allocates awards in the same manner as the initial one allocates losses. The notion of duality can also be applied to the properties a rule satisfies. That is, $\mathcal{P}^*$ is the *dual property of* $\mathcal{P}$ if for every rule R it is true that R satisfies $\mathcal{P}$ if and only if its dual rule R^* satisfies $\mathcal{P}^*$. The dual of securement in the bankruptcy literature (also introduced by Moreno-Ternero and Villar, 2004), is an upper bound axiom formally defined as $R_i(N, c, E) \leq c_i - \frac{1}{n}\min\{c_i, \sum_{j\in N} c_j - E\}$, for each $i \in N$, which, in FG's setting, translates into the requirements $x_i \leq \frac{n-1}{n}v_i$, for each $i \in N \setminus \{n\}$, and $x_n \leq \max\{\frac{n+1}{n}v_n - \frac{V}{n}, \frac{n-1}{n}v_n\}$.[6]

- Theorem 1 in Moreno-Ternero and Villar (2004) shows that the Talmud rule satisfies both securement and its dual. Proposition 2 (3) shows that the constrained equal awards (losses) rule satisfies securement (its dual) but not its dual (securement). Proposition 1 shows that the proportional rule fails to satisfy securement and its dual in the general domain of bankruptcy problems. However, it obeys securement (albeit not its dual) in the restricted framework considered by FG, as $\frac{v_i}{n} \leq \frac{v_n}{V}v_i$, for each $i \in N$.

- The Shapley rule satisfies individual equal sharing:

$$S_i(v) = \frac{v_1}{n} + \frac{v_2 - v_1}{n-1} + \frac{v_3 - v_2}{n-2} + \cdots + \frac{v_i - v_{i-1}}{n-i+1},$$

[5] Note that using claims, instead of truncated claims, while defining the bound is not a meaningful option, as the bounds so obtained would not necessarily be feasible for all problems.

[6] Note that an upper bound can also be interpreted as a lower bound on losses (the difference between the amount awarded and the initial claim).

for each $i \in N$. Then, for each $i \in N$, $S_i(v) \geq \frac{v_i}{n} \iff$

$$\frac{v_i - v_1}{n} \leq \frac{v_2 - v_1}{n-1} + \frac{v_3 - v_2}{n-2} + \cdots + \frac{v_i - v_{i-1}}{n-i+1} \iff$$

$$\frac{v_2 - v_1}{n} + \frac{v_3 - v_2}{n} + \cdots + \frac{v_i - v_{i-1}}{n} \leq \frac{v_2 - v_1}{n-1} + \frac{v_3 - v_2}{n-2} + \cdots + \frac{v_i - v_{i-1}}{n-i+1} \quad \square$$

- In the bankruptcy literature, there exists a focal principle in the two-agent case, known as *concede-and-divide*, whose motivation can be traced back to the Talmud.[7] It amounts to solve two-agent problems upon conceding each agent the portion of the endowment that is not claimed by the other agent, and dividing the remainder equally. Formally, for each two-agent problem $P = (\{i,j\}, (c_i, c_j), E)$,

$$CD(P) = \left(\min\{0, E - c_j\} + \frac{E - M(P)}{2}, \min\{0, E - c_i\} + \frac{E - M(P)}{2}\right),$$

  where $M(P) = \min\{0, E - c_j\} + \min\{0, E - c_i\}$. It turns out that the two bounds (securement and its dual) actually characterize the concede-and-divide procedure in bankruptcy problems (e.g., Moreno-Ternero and Villar, 2004; Theorem 2).

- FG take a game-theoretical approach and associate with each problem the coalitional game defined by the characteristic function selecting as the worth of a coalition the maximum valuation of the members of that coalition. Formally, $u : 2^N \to \mathbb{R}$ is such that $u(S) = \max_{i \in S} v_i$, for each $S \subset N$. It turns out that such is the same formula proposed by Aumann (2010) to associate bankruptcy problems with coalitional games. Nevertheless, the formula is overly optimistic for small coalitions. A different (and perhaps more realistic) proposal was provided by O'Neill (1982), who considered the worth of each coalition to be the difference between the amount available and the sum of the valuations of the members of the complementary coalition, if this difference is non-negative, and 0 otherwise. Formally, $u : 2^N \to \mathbb{R}$ such that $u(S) = \max\{v_n - \sum_{i \in N \setminus S} v_i, 0\}$, for each $S \subset N$. It is well-known that the nucleolus of that coalitional game yields the same solutions as the Talmud rule for the associated bankruptcy problems (e.g., Aumann and Maschler, 1985). If, instead of the nucleolus, we consider the Shapley and τ values, we end up with the two following well-known rules in the bankruptcy literature (e.g., Thomson, 2003).

[7] The principle was first modeled by Aumann and Maschler (1985) although the term *concede-and-divide* was later coined by Thomson (2003).

First, imagine a bankruptcy situation in which agents arriving one at a time to get compensated, and suppose that each claim is fully honored until money runs out. To remove the unfairness of the first-come first-serve scheme associated with any particular order of arrival, let us take the average of the awards vectors calculated in this way when all orders are equally probable. The so-called **Random Arrival** rule results (e.g., O'Neill, 1982). Formally, $RA(v) = (\frac{1}{n!}\sum_{\succ \in \mathcal{R}^N} \min\{v_i, \max\{v_n - \sum_{j\in N, j\succ i} v_j, 0\}\})_{i\in N}$, where $\mathcal{R}^N$ denotes the class of strict orders in N. It turns out that this rule satisfies securement and its dual upper bound described above.

Now, consider the adjustment of the proportional rule resulting from assigning first to each agent his minimal right (to be understood as the remaining amount, if positive, after honoring all other agents' valuations; and zero otherwise), revise subsequently his valuation down, truncate revised valuations at the amount that remains to divide, and apply the proportional rule. Formally, the **Adjusted Proportional** rule (e.g., Curiel et al., 1987) is defined, in this context, as $AP(v) = P(v)$ if $v_n < V/2$ and $AP(v) = (v_1/2, \ldots, v_{n-1}/2, v_n + (v_n - V)/2)$ otherwise. It turns out that this rule also satisfies securement and its dual upper bound described above (that is actually a consequence of Proposition 5 in Moreno-Ternero and Villar, 2004).

- The Knaster procedure not only violates the dual of individual equal sharing, but even the weakest possible notion of upper bound by which awards cannot be above individual valuations. A natural modification of this procedure to account for such natural upper bound, while keeping the spirit of the rule, would amount to consider the constrained equal-awards procedure in the second stage, rather than just imposing equal shares. Formally, $\widehat{K}(v) = (\frac{1}{n}v + \min\{\frac{n-1}{n}v_i, \lambda\})_{i\in N}$, where $\lambda > 0$ is chosen, so that the balance condition is guaranteed. It is straightforward to see that $\widehat{K}$ guarantees resulting allocations satisfy valuations upper boundedness.[8] It does not obey the dual of securement though (e.g., some agents with small valuations could be fully honored with this procedure). For that to happen we would need to modify the second stage of the procedure in a way that would alter the spirit of the original procedure proposed by Knaster.

- The Shapley rule considered in FG has also been studied in bankruptcy problems under the name of Ibn-Ezra's rule. The focus therein has been to extend such rule to the

[8] In the more general setting of bankruptcy, $\widehat{K}$ is an instance of the rules obtained by the application of the so-called **baseline composition operators** on the space of bankruptcy rules (e.g., Hougaard et al., 2012).

universal domain of bankruptcy problems (e.g., Bergantiños and Mendez-Naya, 2001; Alcalde et al., 2005).

- Solidarity properties with respect to population changes were famously introduced by Thomson (1983a,b). A related notion with respect to changes in the endowment was first considered by Roemer (1986). In the bankruptcy literature, similar notions have been considered by several authors (e.g., Young, 1988; Chun, 1999).

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