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On Rationing with Baselines: The Composition Extension Operator

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JEL Classification numbers: D63

Keywords: rationing, baselines, claims, operators, composition

Department of Economics

Rationing with Baselines: The Composition Extension Operator*

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Abstract

We introduce a new operator for general rationing problems in which, besides conflicting claims, individual baselines play an important role in the rationing process. The operator builds onto ideas of composition, which are not only frequent in rationing, but also in related problems such as bargaining, choice, and queuing. We characterize the operator and show how it preserves some standard axioms in the literature on rationing. We also relate it to recent contributions in such literature.

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1 Introduction

In a seminal contribution, O'Neill (1982) introduced a simple model to analyze the problem in which a group of individuals have conflicting claims over an insufficient amount of a (perfectly divisible) good. Ever since, a sizable literature has emerged dealing with standard *rationing* problems fitting such a model.[1] Recently, there has been a growing interest in analyzing more complex rationing situations in which not only claims, but also rights, references, or other objective entitlements, play an important role in the rationing process (e.g., Young, 1987, 1994; Pulido et al., 2002, 2008; Kaminski, 2004, 2006; Ju et al., 2007; Hougaard et al., 2012, 2013).

The aim of this paper is to explore an operator that allows us to move from the domain of standard rationing rules to the domain of rules in which a *baselines* profile complements the claims profile of a standard rationing problem.[2] Our operator is inspired by the properties of *composition*, which pertain to the way rules react to tentative allocations of the available amount to allocate (e.g., Young, 1988). More precisely, think of the following situation: after having divided the allocation of the available amount among its creditors, it turns out that the actual value of the amount is larger than was initially assumed. Then, two options are open: either the tentative division is canceled altogether and the actual problem is solved, or we add to the initial distribution the result of applying the rule to the remaining amount. The requirement of *composition up* is that both ways of proceeding should result in the same awards vectors. Think now of the dual case. Namely, after having divided the available amount among its creditors one finds that the actual value of the amount to divide falls short of what was assumed. Here again we can ignore the initial division and apply the rule to the revised problem, or we can apply the rule to the problem in which the initial claims are substituted by the (unfeasible) allocation initially proposed. The requirement of *composition down* is that both ways of proceeding should result in the same awards vectors.[3]

The operator we present here behaves in a similar way. More precisely, it associates with

[1] The reader is referred to Thomson (2003) for a survey.

[2] The notion of operators for the domain of rationing rules was first introduced by Thomson and Yeh (2008).

[3] These properties are reminiscent of the so-called "path independence" axiom for choice functions (e.g., Plott, 1973). They also have a relative in the theory of axiomatic bargaining: the so-called "step-by-step negotiations" axiom introduced by Kalai (1977), which is the basis for the characterization of the egalitarian solution in such context. The same principle has also been frequently used in other related contexts like taxation, queuing, or resource allocation (e.g., Moulin, 2000; Moulin and Stong, 2002; Moreno-Ternero and Roemer, 2012).

each rule defined for the standard rationing model a rule for the general model with baselines, which solves problems following a two-stage process. In the first stage, baselines (truncated by claims) either replace claims (if they are not collectively feasible) or are awarded to agents (if they are collectively feasible). In the second stage, the primitive rule is used to solve the standard problem that results after adjusting claims and endowment according to the process of the first stage. We characterize such a *composition extension operator* and show how it preserves some standard axioms in the literature on rationing.

A main focus of this paper is to compare the composition extension operator directly to similar types of operators that have appeared in the literature. In particular, the so-called *baselines first operator* introduced in Hougaard et al., (2013). Both operators agree on how to extend a standard rationing rule to the setting of rationing in the presence of baselines, when truncated baselines are collectively feasible. They, however, disagree in the treatment of the opposite case; namely, when truncated baselines are not collectively feasible. The baselines-first extension operator proposes to adjust down the tentative allocation of those truncated baselines by means of the primitive rationing rule, applied to the resulting problem after replacing initial claims by (truncated) baselines. That is indeed equivalent to propose the allocation that the *dual* of the primitive rule yields for the problem in which initial claims are replaced by (the unfeasible) truncated baselines. The composition extension operator, however, advocates to use precisely the allocation proposed by the primitive rule to that latter problem. In doing so, the operator reflects the principle underlying the composition down property, as done with the composition up property for the other case (in which truncated baselines are collectively feasible).

The rest of the paper is organized as follows. In Section 2, we introduce the model and the operator we study. In Section 3, we present a characterization result for the operator. In Section 4, we study the robustness of the operator by means of the properties it preserves. We conclude in Section 5 placing our contribution in the related literature. Among other things, we state that the composition extension operator and the baselines first extension operator are essentially equally robust, but also that the former might be preferred when it comes to treat the case of uninformative baselines. For a smooth passage, we defer all the proofs and provide them in the appendix.

2 Preliminaries

2.1 The benchmark model

We study rationing problems in a variable-population model. The set of potential claimants, or *agents*, is identified with the set of natural numbers $\mathbb{N}$. Let $\mathcal{N}$ be the class of (non-empty) finite subsets of $\mathbb{N}$, with generic element N. Let n denote the cardinality of N. For each $i \in N$, let $c_i \in \mathbb{R}_+$ be i's *claim* and $c \equiv (c_i)_{i\in N}$ the claims profile.[4]

A *(standard rationing) problem* is a triple consisting of a population $N \in \mathcal{N}$, a claims profile $c \in \mathbb{R}^n_+$, and an *endowment* $E \in \mathbb{R}_+$ such that $\sum_{i\in N} c_i \geq E$. Let $C \equiv \sum_{i\in N} c_i$. To avoid unnecessary complication, we assume $C > 0$. Let $\mathcal{D}^N$ be the set of rationing problems with population N and $\mathcal{D} \equiv \bigcup_{N\in\mathcal{N}} \mathcal{D}^N$.

Given a problem $(N, c, E) \in \mathcal{D}^N$, an *allocation* is a vector $x \in \mathbb{R}^n$ satisfying the following two conditions: (i) for each $i \in N$, $0 \leq x_i \leq c_i$ and (ii) $\sum_{i\in N} x_i = E$. We refer to (i) as *boundedness* and (ii) as *balance*. A *(standard) rule* on $\mathcal{D}$, $R\colon \mathcal{D} \to \bigcup_{N\in\mathcal{N}} \mathbb{R}^n$, associates with each problem $(N, c, E) \in \mathcal{D}$ an allocation $R(N, c, E)$. The *proportional* rule, $P(N, c, E) = \frac{E}{C} \cdot c$, yielding allocations in proportion to claims, is an example of a classical rule in this context. Two other focal rules are the *constrained equal awards* rule, which distributes the endowment equally among all agents, subject to no agent receiving more than her claim, and the *constrained equal losses* rule, which makes losses as equal as possible, subject to no one receiving a negative amount. Formally, $A(N, c, E) = (\min\{c_i, \lambda\})_{i\in N}$ where $\lambda > 0$ is chosen so that $\sum_{i\in N} \min\{c_i, \lambda\} = E$, and $L(N, c, E) = (\max\{0, c_i - \lambda\})_{i\in N}$ where $\lambda > 0$ is chosen so that $\sum_{i\in N} \max\{0, c_i - \lambda\} = E$.

Each rule R has a *dual* rule R^d defined as $R^d(N, c, E) = c - R(N, c, C - E)$, for each $(N, c, E) \in \mathcal{D}$. The constrained equal awards rule and the constrained equal losses rule are dual rules. The proportional rule is self dual.

2.2 The general framework

As in Hougaard et al., (2013) we consider the extended framework of a *general (rationing) problem* (or problem with baselines) consisting of a population $N \in \mathcal{N}$, a *baselines* profile $b \in \mathbb{R}^n_+$, a claims profile $c \in \mathbb{R}^n_+$, and an endowment $E \in \mathbb{R}_+$ such that $\sum_{i\in N} c_i \geq E$. We denote by $\mathcal{E}^N$ the class of general problems with population N and $\mathcal{E} \equiv \bigcup_{N\in\mathcal{N}} \mathcal{E}^N$. For each general

[4] For each $N \in \mathcal{N}$, each $M \subseteq N$, and each $z \in \mathbb{R}^n$, let $z_M \equiv (z_i)_{i\in M}$. For each $i \in N$, let $z_{-i} \equiv z_{N\setminus\{i\}}$.

problem $(N, b, c, E) \in \mathcal{E}$, let $t_i(b, c) = \min\{b_i, c_i\}$, for each $i \in N$, and $t(b, c) = \{t_i(b, c)\}_{i \in N}$ denote the corresponding (baseline-claim) truncated vector. Let $T(b, c) = \sum_{i \in N} t_i(b, c)$.

Given a general problem $(N, b, c, E) \in \mathcal{E}$, an *allocation* is a vector $x \in \mathbb{R}^n$ satisfying the following two conditions: (i) for each $i \in N$, $0 \leq x_i \leq c_i$ and (ii) $\sum_{i \in N} x_i = E$. A *(general) rule* on $\mathcal{E}$, $S \colon \mathcal{E} \to \bigcup_{N \in \mathcal{N}} \mathbb{R}^n$, associates with each (general) problem $(N, b, c, E) \in \mathcal{E}$ an allocation $x = S\,(N, b, c, E)$.

An *extension operator* associates with each rule defined for the standard model a rule for the general model. We consider an operator that solves problems following a two-stage process. In the first stage, baselines (truncated by claims) either replace claims (if they are not collectively feasible) or are awarded to agents (if they are collectively feasible). In the second stage, the primitive rule is used to solve the standard problem that results after adjusting claims and endowment according to the process of the first stage. In the spirit of the properties of *composition up* and *composition down*, described in the introduction, we refer to it as the *composition extension operator*.

Formally,

$$R^c\,(N, b, c, E) = \begin{cases} R(N, t(b, c), E), & \text{if } \ E \leq T(b, c) \\ t(b, c) + R(N, c - t(b, c), E - T(b, c)), & \text{if } \ E \geq T(b, c). \end{cases} \tag{1}$$

The composition extension operator can be directly compared to the *baselines-first* operator defined in Hougaard et al., (2013). The difference lies where $E \leq T(b, c)$, in which case the baselines-first extension operator uses the dual rule of R.[5]

Formally,

$$R^{bf}\,(N, b, c, E) = \begin{cases} R^d(N, t(b, c), E) & \text{if } \ E \leq T(b, c) \\ t(b, c) + R(N, c - t(b, c), E - T(b, c)) & \text{if } \ E \geq T(b, c) \end{cases} \tag{2}$$

To use the dual rule, as in the baselines-first extension operator, is in many ways as natural as using the principle of composition down: Note that $R^d(N, t(b, c), E) = t(b, c) - R(N, t(b, c), T(b - c) - E)$. Thus, the baselines-first extension operator also implements a natural principle for the case in which truncated baselines are not collectively feasible; namely, subtracting the part of the loss found by rationing with claims as truncated baselines.

In what follows, we dwell on the similarities and differences between the two operators. We start out with a comparison based on axiomatic characterizations presented in the next

[5]For the case $E \geq T(b, c)$, both operators agree with the proposal made by Pulido et al., (2002) for bankruptcy problems with objective entitlements, which are a specific instance of our bankruptcy problems with baselines.

section. Before that, we provide some figures illustrating how the two operators perform in the two-agent case, for different baselines, when the primitive rules are the three focal rules in the benchmark rationing model; namely, the proportional, constrained equal awards, and constrained equal losses rules, described above. More precisely, let $N = \{1, 2\}$ and $c = (c_1, c_2)$, where $c_1 < c_2$, be a claims vector. The path of awards for the claims vector c is the locus of the awards vector chosen by a rule as the amount to divide E varies from 0 to $c_1 + c_2$. It is straightforward to see that the path of awards of the proportional rule follows the line joining the origin with the vector of claims. On the other hand, the path of awards for the constrained equal awards rule follows the 45^o line until it gives the whole claim to the lowest claimant, i.e., until $E = 2c_1$, from where it is vertical until it reaches the vector of claims. Similarly, the path of awards of the constrained equal losses rule follows the vertical line until the claimant with the highest claim loses an amount equal to the lowest claim, i.e., until $E = c_2 - c_1$, from where it follows a line of slope 1 until it reaches the vector of claims.

As the following figures illustrate, the path of awards of the extended versions of these three rules, via the two operators described, can be inferred from the above. In the case of the proportional rule, which is a self-dual rule and, hence, is extended identically by both operators, the path of awards is always piece-wise linear. The first piece is the straight line joining the origin with the vector of truncated baselines, whereas the second piece is the straight line joining the vector of truncated baselines with the vector of claims.

As for the constrained equal-awards rule, if it is extended via the baselines-first (respectively, composition) operator, then it follows the path of the constrained equal losses (respectively, awards) rule for the vector of truncated baselines. From such a vector, it follows the translation of the path of awards of the constrained equal awards rule for the resulting vector from substracting truncated baselines to claims.

Finally, as for the constrained equal-losses rule, if it is extended via the baselines-first (respectively, composition) operator, then it follows the path of the constrained equal awards (respectively, losses) rule for the vector of truncated baselines. From such vector, it follows the translation of the path of awards of the constrained equal losses for the resulting vector from substracting truncated baselines to claims.

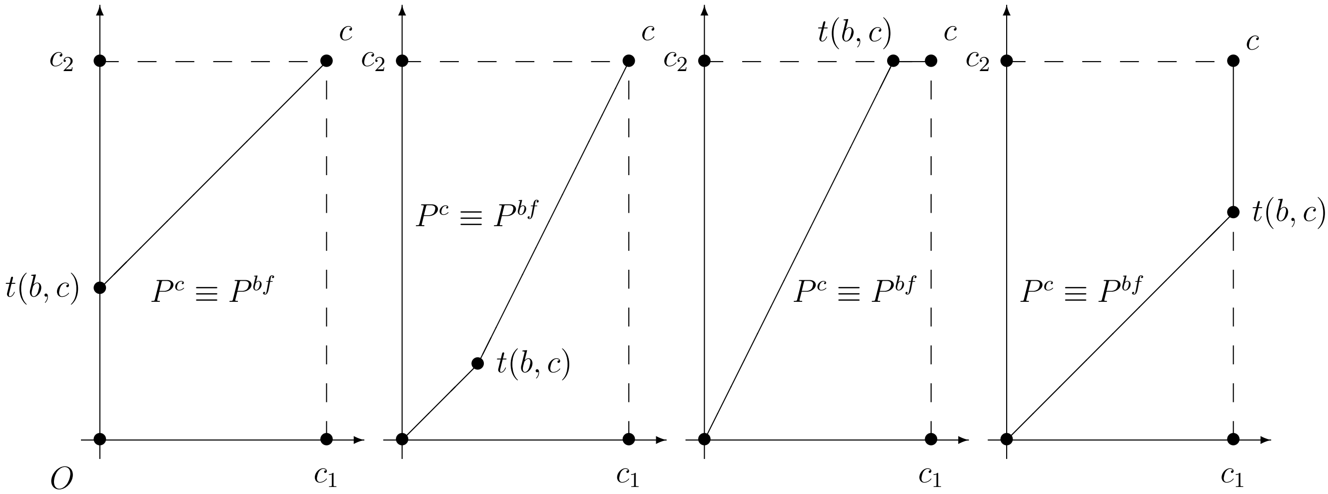

Figure 1: The extended proportional rule in the two-claimant case.

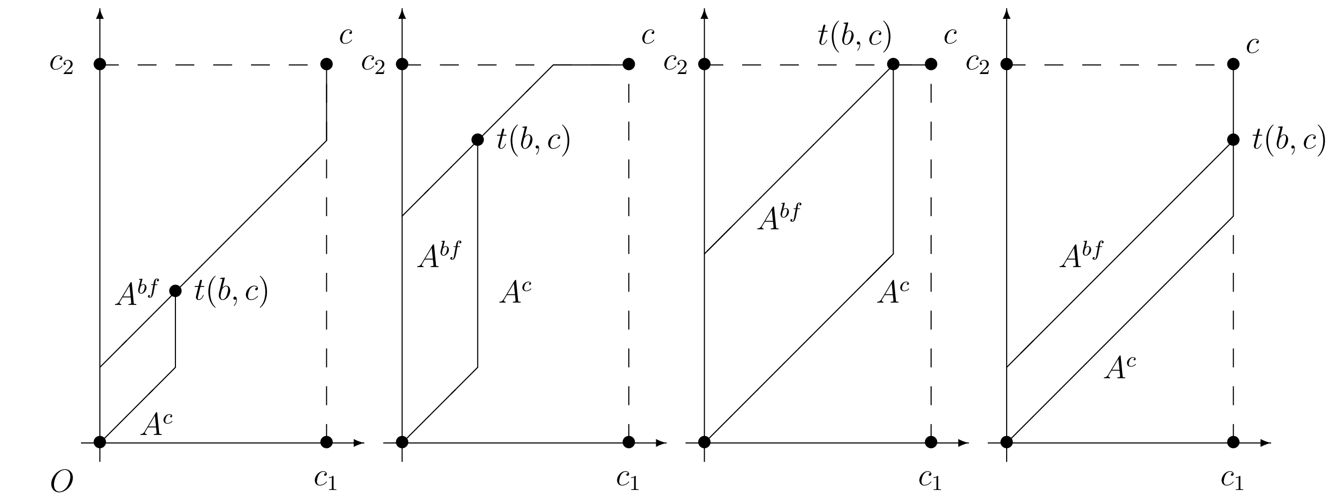

Figure 2: The extended constrained equal awards rule in the two-claimant case.

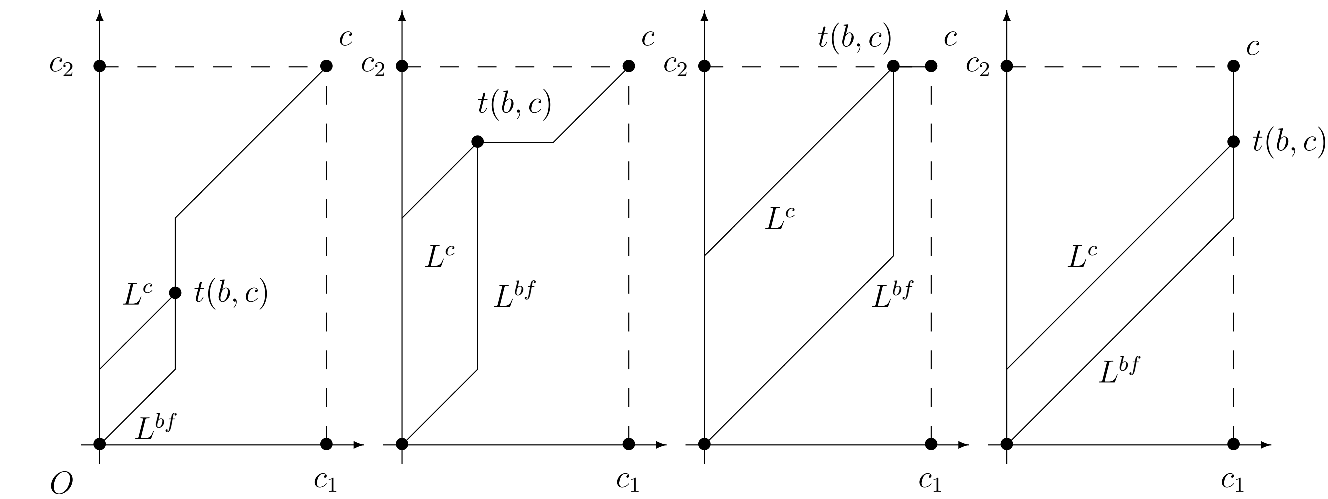

Figure 3: The extended constrained equal losses rule in the two-claimant case.

3 An axiomatic characterization

We provide in this section a set of axioms characterizing the composition extension operator. These axioms are closely related to the characterization of the baselines-first extension operator and hence the reader is referred to Hougaard et al., (2013) for further details about most of them.

The first axiom requires a baseline to be disregarded if it is above the corresponding claim. Formally,

Baseline truncation: for each $(N, b, c, E) \in \mathcal{E}$, $S(N, b, c, E) = S(N, t(b, c), c, E)$.

The second axiom requires to disregard the amount of a claim exceeding its corresponding baseline, whenever truncated baselines cannot be jointly covered. Formally,

Truncation of excessive claims: for each $(N, b, c, E) \in \mathcal{E}$ such that $E \leq T(b, c)$, $S(N, b, c, E) = S(N, b, t(b, c), E)$.

The third axiom complements the previous one as it refers to a situation where all truncated baselines can be covered. It states that, in such a case, if an individual's claim and baseline are reduced by an amount k_i, then such reduction in the endowment should be taken from that individual, while the others remain unaffected. Formally,

Baseline invariance: for each $(N, b, c, E) \in \mathcal{E}$ such that $E \geq T(b, c)$, and $k \in \mathbb{R}^n_+$ such that $k_j \leq t_j(b, c)$, for each $j \in N$, then $S(N, b, c, E) = k + S(N, b - k, c - k, E - \sum_{i \in N} k_i)$.

The fourth axiom is new and thus deserves further explanation. It deals with the two polar cases of *non-informative* baselines and argues that both of them should be treated equally. More precisely, it states that a problem with null baselines and the corresponding problem in which baselines are equal to claims should be allocated identically. Formally,

Polar baseline equivalence: for each $(N, c, E) \in \mathcal{D}$, $S(N, 0, c, E) = S(N, c, c, E)$.

These four axioms together characterize the range of the composition extension operator, introduced above.

Theorem 1 *A rule satisfies baseline truncation, truncation of excessive claims, baseline invariance and polar baseline equivalence if and only if it is the image of a standard rule via the composition extension operator.*

The axiom polar baseline equivalence is a plausible alternative to the following axiom, also introduced in Hougaard et al., (2013), which treats non-informative baselines in a different way, and that characterizes the baseline first operator when combined with the first three axioms described above (e.g., Hougaard et al., 2013; Theorem 5).

Baseline self-duality: for each $(N, c, E) \in \mathcal{D}$, $S(N, 0, c, E) = c - S(N, c, c, C - E)$.

One might argue that polar baseline equivalence has a higher appeal to deal with non-informative baselines, especially when there is not enough information to determine whether (non-informative) baselines are null or equal to claims. This speaks in favor of the composition extension operator when baselines are not clearly established.

4 Preservation of axioms

The composition extension operator associates with each standard rule a new rule to solve (rationing) problems with baselines. A natural question is whether the rule so constructed inherits the properties of the original rule.[6] To explore this question, we consider several standard axioms in the literature on rationing problems, formulating in different ways the principle of solidarity, which has a long tradition in the theory of justice.

The first group comprises fixed-population axioms.

Resource monotonicity (e.g., Roemer, 1986): for each $(N, c, E) \in \mathcal{D}$ and each $E' > E$, with $E' \leq \sum c_i$, we have $R(N, c, E) \leq R(N, c, E')$.

Claims monotonicity (e.g., Thomson, 2003): for each $(N, c, E) \in \mathcal{D}$, each $i \in N$, and each $c'_i > c_i$, we have $R_i(N, (c'_i, c_{N\setminus\{i\}}), E) \geq R_i(N, (c_i, c_{N\setminus\{i\}}), E)$.

Linked claims-resource monotonicity (e.g., Moreno-Ternero and Villar 2006; Thomson and Yeh 2008): for each $(N, c, E) \in \mathcal{D}$ and $i \in N$, $R_i(N, (c_i + \varepsilon, c_{N\setminus\{i\}}), E + \varepsilon) \leq R_i(N, c, E) + \varepsilon$.[7]

The second group comprises variable-population axioms.

Population monotonicity (e.g., Thomson, 1983): for each $(N, c, E) \in \mathcal{D}$ and $(N', c', E) \in \mathcal{D}$ such that $N \subseteq N'$ and $c'_N = c$, then $R_i(N', c', E) \leq R_i(N, c, E)$, for each $i \in N$.

Linked resource-population monotonicity (e.g., Thomson and Yeh, 2008): for each $(N, c, E) \in \mathcal{D}$ and each $(N', c', E') \in \mathcal{D}$ such that $N \subseteq N'$, $c'_N = c$, and $E = E'$, then $R_i(N, c, E) \leq R_i\left(N', c', E + \sum_{N'\setminus N} c'_j\right)$, for each $i \in N$.

Resource-population uniformity (e.g., Chun 1999; Moreno-Ternero and Roemer 2006): for each $(N, c, E) \in \mathcal{D}$ and $(N', c', E') \in \mathcal{D}$ such that $N \subseteq N'$ and $c'_N = c$, then, either $R_i(N', c', E') \leq R_i(N, c, E)$, for each $i \in N$, or $R_i(N', c', E') \geq R_i(N, c, E)$, for each $i \in N$.

[6]Such a question was initially posed by Thomson and Yeh (2008) with respect to the operators they study.
[7]Note that the last two properties are dual, whereas the first one is self-dual.

Consistency (e.g., Thomson, 2007): for each $(N, c, E) \in \mathcal{D}$, each $M \subset N$, and each $i \in M$, we have $R_i(N, c, E) = R_i(M, c_M, E_M)$, where $E_M = \sum_{i \in M} R_i(N, c, E)$.[8]

We say that an axiom is *preserved* by our composition extension operator if, whenever a rule R satisfies it, then the induced rule R^c satisfies the corresponding *general* version of the axiom.[9]

The following three theorems are counterpart results to the Theorems 1, 3 and 4 in Hougaard et al., (2013), which state an identical behavior of the baselines first operator.

Our first preservation result refers to the properties that are individually preserved.

Theorem 2 *The properties of resource monotonicity, consistency, and resource-population uniformity are preserved by the composition extension operator.*

The second preservation result refers to the properties that are preserved in pairs, either by themselves or "with the assistance" of an additional property.[10]

Theorem 3 *The following pairs are preserved by the composition extension operator: claims monotonicity and linked claims-resource monotonicity; population monotonicity and linked resource-population monotonicity, if assisted by resource monotonicity.*

In other words, if a rule R satisfies claims monotonicity and linked claims-resource monotonicity, then the induced rule R^c satisfies the corresponding two general properties. Also, if a rule R satisfies population monotonicity and linked resource-population monotonicity, and in addition is resource monotonic, then the induced rule R^c satisfies the corresponding three general properties.

To conclude with this section, we consider two other standard properties in the literature on rationing. On the one hand, the requirement of allotting equal amounts to agents with equal claims. Formally, a rule satisfies *equal treatment of equals* if, for each $(N, c, E) \in \mathcal{D}$, and each $i, j \in N$, we have $R_i(N, c, E) = R_j(N, c, E)$, whenever $c_i = c_j$.

A strengthening of the previous notion says that agents with larger claims receive and lose at least as much as agents with smaller claims. That is, a rule is *order preserving* if, for each $(N, c, E) \in \mathcal{D}$ and $i, j \in N$, $c_i \geq c_j$ implies that $R_i(N, c, E) \geq R_j(N, c, E)$ and

[8]Note that the first two properties in this block are dual, whereas the last two are self-dual.

[9]For ease of exposition, we skip the straightforward definitions of the general versions of each axiom introduced above.

[10]The terminology is borrowed from Hokari and Thomson (2008).

$c_i - R_i(N, c, E) \geq c_j - R_j(N, c, E)$. The first inequality is referred to as *order preservation in gains*. The second inequality is referred to as *order preservation in losses*.

Equal treatment of equals should not be expected to be preserved because the direct extension of this property to the enriched model does not include hypotheses about baselines. Nevertheless, it is straightforward to see that, on the subdomain of problems in which baselines are equals, for agents with equal claims, equal treatment of agents with equal claims is preserved by the composition extension operator (as it happens with the baselines first extension operator).

As for order preservation, we impose additional conditions on baselines to guarantee its preservation. We say that *baselines are ordered like claims* if, whenever $c_i \leq c_j$, then $b_i \leq b_j$. Similarly, we say that *claim-baseline differences are ordered like claims* if, whenever $c_i \leq c_j$, then $c_i - b_i \leq c_j - b_j$.

Theorem 4 *If baselines are ordered like claims, and R is order preserving, then R^c is order preserving in gains. If baselines and claim-baseline differences are ordered like claims then order preservation is preserved by the composition extension operator.*

To conclude, it is worth mentioning that the three focal rules in the standard rationing model obey all the principles described above. Thus, their extensions to both operators (as illustrated in Figures 1-3 above for the two-agent case) will obey them too, if the corresponding provisos (regarding equal treatment of equals and order preservation) hold.

Properties	**Preservation**
Resource monotonicity	+
Consistency	+
Resource-population uniformity	+
Claims monotonicity + its dual	+
Population monotonicity + its dual	+ (RM)
Equal treatment of equals	+ (*)
Order preservation	+ (**)

Table 1: Preservation of axioms. This table summarizes the behavior of the composition extension operator with respect to the properties we have considered. A plus sign indicates that the property (or the pair of properties) is preserved. A parenthesis after the plus sign indicates that the preservation

occurs with a proviso. For instance, under the presence of resource monotonicity (RM), the pair of properties formed by population monotonicity and its dual is preserved. Similarly, if baselines are equals for agents with equal claims (*), equal treatment of equals is preserved. Finally, if baselines and claim-baseline differences are ordered like claims (**), then order preservation is preserved.

5 Concluding remarks

We have introduced the composition extension operator within the framework of extended rationing problems with exogenous baselines. The composition extension operator was compared to a natural counterpart - the baselines-first extension operator defined and analyzed in Hougaard et al., (2013). Both operators are in many ways equally natural. For example, concerning preservation of properties of the associated standard rationing rules, the results are basically identical. Yet, comparing the axiomatic foundation it appears that the way in which the composition extension operator treats the case of uninformative baselines is somewhat more desirable than that of the baselines-first operator.

Finally, we note that the composition extension operator we introduce here is reminiscent of the approach we take in Hougaard et al., (2012) to analyze rationing problems. Therein, we assume that, rather than being exogenously given, baselines associate with each rationing problem some (not necessarily feasible) allocation. For each baseline, we impose the two-stage *composition* process described above to associate a new rule with each rule. The resulting family of operators encompasses (and generalizes) two focal operators for the domain of rules in the standard rationing problem, known as the attribution of minimal rights and truncation operators, respectively (e.g., Thomson and Yeh, 2008).

6 Appendix: Proof of the results

Proof of Theorem 1

It is straightforward to see that, for any (standard) rule R, R^c satisfies the four axioms. Thus, we focus on the converse implication. Let S be a rule satisfying the four axioms and let (N, b, c, E) be a problem with baselines. If $E \geq T(b,c)$, then, by baseline invariance, and baseline truncation, $S(N,b,c,E) = S(N,0,c-t(b,c),E-T(b,c))$. If $E \leq T(b,c)$, by baseline truncation and truncation of excessive claims, $S(N,b,c,E) = t(b,c) + S(N,t(b,c),t(b,c),E)$. Moreover, by polar baseline equivalence, $S(N,t(b,c),t(b,c),E) = S(N,0,t(b,c),E)$.

Let $R : \mathcal{D} \to \bigcup_{N \in \mathcal{N}} \mathbb{R}^n$ be such that, for any $(N,c,E) \in \mathcal{D}$,

$$R(N,c,E) = S(N,0,c,E).$$

In other words, R assigns to each problem the allocation that S yields for the corresponding problem with baselines in which baselines are null. Then,

$$S(N,b,c,E)=\begin{cases} R(N,t(b,c),E) & \text{if } \ E\leq T(b,c) \\ t(b,c)+R(N,c-t(b,c),E-T(b,c)) & \text{if } \ E>T(b,c) \end{cases},$$

which implies that $S\equiv R^c$. ■

Proof of Theorem 2

Note first that, by Theorem 1 in Hougaard et al., (2013), the three properties are preserved by the baselines-first extension operator. As this operator coincides with the composition extension operator for the case in which truncated baselines are collectively feasible, we only need to focus on the opposite case to prove the statements of the theorem.

Let R be a rule satisfying resource monotonicity. Let $(N,c,E)\in\mathcal{D}$ and each $E'>E$, with $E'\leq\sum c_i$. Let $b\in\mathbb{R}^n$ be a baseline profile and let $i\in N$ be a given agent. If $E<E'\leq T(b,c)$, then $R_i^c(N,b,c,E)=R_i(N,t(b,c),E)$ and $R_i^c(N,b,c,E')=R_i(N,t(b,c),E')$. Now, as R satisfies resource monotonicity, the desired inequality follows.

Let R be a rule satisfying consistency. Let $(N,c,E)\in\mathcal{D}$ and $b\in\mathbb{R}^n_+$. Let $x=R^c(N,b,c,E)$. The aim is to show that, for any $M\subset N$,

$$R^c(M,b_M,c_M,\sum_{i\in M}x_i)=x_M.$$

Fix $M\subset N$ and let $E'=\sum_{j\in M}x_j$ and $T'(b,c)=\sum_{j\in M}t_j(b,c)$. Then, it is straightforward to show that $E\leq T(b,c)$ if and only if $E'\leq T'(b,c)$. If $E\leq T(b,c)$, then $x_i=R_i(N,t(b,c),E)$ for each $i\in N$, and thus $E'=\sum_{i\in M}R_i(N,t(b,c),E)$. Therefore, $R_i^c(M,b_M,c_M,E')=R_i(M,t_M(b,c),E')$ for each $i\in M$. Now, as R is consistent, it follows that $R_i(N,t(b,c),E)=R_i(M,t_M(b,c),E')$, for each $i\in M$, as desired.

To conclude, the statement on resource-population uniformity follows from the fact that such axiom is equivalent to the combination of resource monotonicity and consistency (e.g., Hougaard et al., 2013). ■

Proof of Theorem 3

Again, by Theorem 3 in Hougaard et al., (2013), we only need to focus on the case in which truncated baselines are collectively unfeasible to prove the statements of the theorem.

Let R be a rule satisfying claims monotonicity and linked claims-resource monotonicity. Our aim is to show that R^c satisfies the general versions of the two properties.

Claims monotonicity. Let $(N,c,E)\in\mathcal{D}$ and $i\in N$, such that $c_i\leq c_i'$. Let $b\in\mathbb{R}^n_+$ be a baseline profile, $T(b,c)=\sum_{j\in N}t_j(b,c)$, and $T'(b,c)=\sum_{j\in N}t_j(b,c')$.[11]

[11]Note that $t_{N\setminus\{i\}}(b,c')\equiv t_{N\setminus\{i\}}(b,c)$, $t_i(b,c')\geq t_i(b,c)$ and thus, $T'(b,c)\geq T(b,c)$.

If $E \leq T(b,c)$, then, $R_i^c(N,b,c,E) = R_i(N,t(b,c),E)$ and

$$R_i^c(N,b,c',E) = R_i(N,t(b,c'),E) = R_i(N,(t_{N\setminus\{i\}}(b,c),t_i(b,c')),E).$$

As R satisfies claims monotonicity, the desired inequality follows.

If $T(b,c) < E < T'(b,c)$, then, $R_i^c(N,b,c,E) = t_i(b,c) + R_i(N,c-t(b,c),E-T(b,c))$, and $R_i^c(N,b,c',E) = R_i(N,t(b,c'),E)$. Now, this case implies that $t_i(b,c) = c_i$ (otherwise, $t_i(b,c) = b_i$ and hence $T(b,c) = T'(b,c)$). Thus, by boundedness, $R_i^c(N,b,c,E) = c_i$. Now, by resource monotonicity and claims monotonicity of R, $R_i(N,t(b,c'),E) \geq R_i(N,t(b,c),T(b,c)) = t_i(b,c) = c_i$, from where the desired inequality follows.

Linked claims-resource monotonicity. Let $(N,c,E) \in \mathcal{D}$ and $i \in N$. Let $b \in \mathbb{R}_+^n$ be a baseline profile, $\varepsilon > 0$ and $c' = (c_i+\varepsilon, c_{N\setminus\{i\}})$. Let $T'(b,c) = T(b,c) + t_i(b,c') - t_i(b,c)$. Then, $t_i(b,c') \leq t_i(b,c)+\varepsilon$ and $T(b,c) \leq T'(b,c) \leq T(b,c)+\varepsilon$.

If $E \leq T'(b,c)-\varepsilon$, then $R_i^c(N,b,c,E) = R_i(N,t(b,c),E)$ and $R_i^c(N,b,c',E+\varepsilon) = R_i(N,t(b,c'),E+\varepsilon) = R_i(N,(t_i(b,c'),t_{-i}(b,c)),E+\varepsilon)$. By claims monotonicity of R, $R_i(N,t(b,c'),E+\varepsilon) \leq R_i(N,(t_i(b,c)+\varepsilon,t_{-i}(b,c)),E+\varepsilon)$. By linked claims-resource monotonicity of R, $R_i(N,(t_i(b,c)+\varepsilon,t_{-i}(b,c)),E+\varepsilon) \leq R_i(N,t(b,c),E)+\varepsilon$, from where the desired inequality follows.

If $T'(b,c)-\varepsilon < E < T(b,c)$, then $R_i^c(N,b,(c_i+\varepsilon,c_{N\setminus\{i\}}),E+\varepsilon) = t_i(b,c') + R_i(N,(c_i+\varepsilon-t_i(b,c'),(c-t(b,c))_{-i}),E+\varepsilon-T'(b,c))$, and $R_i^c(N,b,c,E) = t_i(b,c') - R_i^d(N,t(b,c),T(b,c)-E)$. Thus, the desired inequality becomes

$$\varepsilon - t_i(b,c') + t_i(b,c) \geq R_i^d(N,t(b,c),T(b,c)-E) + R_i(N,(c_i+\varepsilon-t_i(b,c'),(c-t(b,c))_{-i}),E+\varepsilon-T'(b,c)) \tag{3}$$

Now, by balance and boundedness, the right hand side of (3) is bounded above by $T(b,c)-E+E+\varepsilon-T'(b,c)$, which is precisely the left hand side of (3).

As for the second statement of the theorem, let R be a rule satisfying resource monotonicity, population monotonicity and linked resource-population monotonicity. By Theorem 2, R^c satisfies the general property of resource monotonicity. Our aim is to show that R^c also satisfies the general properties of population monotonicity and linked resource-population monotonicity.

Population monotonicity. Let $(N,c,E) \in \mathcal{D}$ and $(N',c',E') \in \mathcal{D}$ be such that $N \subseteq N'$, $c'_N = c$ and $E = E'$. Let $b \in \mathbb{R}_+^n$ and $b' \in \mathbb{R}_+^{n'}$ be two baseline profiles such that $b'_N = b$. Note that $t_j(b',c') = t_j(b,c)$ for each $j \in N$. Finally, let $T(b,c) = \sum_{j\in N} t_j(b,c)$ and $T'(b,c) = \sum_{j \in N'} t_j(b',c')$. If $E \leq T(b,c)$, $R_i^c(N',b',c',E) = R_i(N',t(b',c'),E)$ and $R_i^c(N,b,c,E) = R_i(N,t(b,c),E)$. As R satisfies population monotonicity, the desired inequality follows.

Linked resource-population monotonicity. Let $(N,c,E) \in \mathcal{D}$ and $(N',c',E) \in \mathcal{D}$ be such that $N \subseteq N'$ and $c'_N = c$. Let $b \in \mathbb{R}_+^n$ and $b' \in \mathbb{R}_+^{n'}$ be two baseline profiles such that $b'_N = b$.

Note that $t_j(b',c') = t_j(b,c)$ for each $j \in N$. Finally, let $T(b,c) = \sum_{j\in N} t_j(b,c)$, and $T'(b,c) = \sum_{j\in N'} t_j(b',c')$. If $E \le T(b,c) - \sum_{N'\setminus N}\left(c'_j - t_j(b',c')\right)$, then $E \le T(b,c)$ and $E' \le T'(b,c)$ and, therefore, $R^c_i(N',b',c',E') = R_i(N',t(b',c'),E')$ and $R^c_i(N,b,c,E) = R_i(N,t(b,c),E)$. By resource monotonicity and population monotonicity of R, the desired inequality follows. ■

Proof of Theorem 4

In order to prove the first statement, let R be a rule satisfying order preservation and let (N,b,c,E) be an extended problem for which baselines are ordered like claims. Let $i,j \in N$ be such that $c_i \le c_j$. As $b_i \le b_j$ it follows that $t_i(b,c) \le t_j(b,c)$. Now, if $E \le T(b,c)$, $R^c_i(N,b,c,E) = R_i(N,t(b,c),E)$ and $R^c_j(N,b,c,E) = R_j(N,t(b,c),E)$. As R is order preserving, and $t_i(b,c) \le t_j(b,c)$, it follows that $R^c_i(N,b,c,E) \le R^c_j(N,b,c,E)$, as desired. If, on the other hand, $E \ge T(b,c)$, the result follows from the proof in Hougaard et al., (2013).

As for the second statement, let R be a rule satisfying order preservation and let (N,b,c,E) be an extended problem for which baselines, and claim-baseline differences, are ordered like claims. Let $i,j \in N$ be such that $c_i \le c_j$. It then follows that $t_i(b,c) \le t_j(b,c)$ and that $c_i - t_i(b,c) \le c_j - t_j(b,c)$.

Now, if $E \le T(b,c)$, $R^c_i(N,b,c,E) = R_i(N,t(b,c),E)$ and $R^c_j(N,b,c,E) = R_j(N,t(b,c),E)$.

As R is order preserving, and $t_i(b,c) \le t_j(b,c)$, it follows that $R^c_i(N,b,c,E) \le R^c_j(N,b,c,E)$.

As R^d is order preserving, and $c_i - t_i(b,c) \le c_j - t_j(b,c)$, it follows that

$$c_i - t_i(b,c) + R^d_i(N,t(b,c),T(b,c)-E) \le c_j - t_j(b,c) + R^d_j(N,t(b,c),T(b,c)-E),$$

i.e., $c_i - R^c_i(N,b,c,E) \le c_j - R^c_j(N,b,c,E)$, as desired.

If, on the other hand, $E \ge T(b,c)$, $R^c_i(N,b,c,E) = t_i(b,c) + R_i(N,c-t(b,c),E-T(b,c))$ and $R^c_j(N,b,c,E) = t_j(b,c) + R_j(N,c-t(b,c),E-T(b,c))$. As $t_i(b,c) \le t_j(b,c)$, $c_i - t_i(b,c) \le c_j - t_j(b,c)$, and R is order preserving, it follows that $R^c_i(N,b,c,E) \le R^c_j(N,b,c,E)$ and $c_i - R^c_i(N,b,c,E) \le c_j - R^c_j(N,b,c,E)$, as desired. ■

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