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A Talmudic Approach to Bankruptcy Problems

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Keywords: Bankruptcy problems, Talmud rule, TAL-family, Equal awards, Equal losses

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A Talmudic Approach to Bankruptcy Problems*

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Abstract

Bankruptcy problems arise when agents hold claims against a certain (perfectly divisible) good, and the available amount is not enough to satisfy them all. A great source of inspiration to solve these problems emanates from the Talmud. We survey classical and recent contributions to the literature that constitute this Talmudic approach to bankruptcy problems.

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1 Introduction

The Babylonian Talmud is an ancient collection of writings that constitutes a central text of Rabbinic Judaism. Therein, several instances of what we call bankruptcy problems, and specific recommendations to solve them, are presented (see, for instance, Aumann (2002) for a leisurely discussion).

A canonical case is the so-called contested garment problem, in which two men disagree on the ownership of a garment. The first man claims half of it, and the other claims it all. Assuming both claims are made in good faith, the Talmud recommends that the first agent gets one fourth of the garment, whereas the second agent gets three fourths of the garment.

Another well-known case is the following. There are three creditors; the debts are 100, 200 and 300. When the estate is 100, it should be divided equally. If the estate is 300 it should be divided proportionally. Finally, if the estate is 200, the recommendation is to allocate the first creditor 50 and the other two creditors 75.

It was not until 30 years ago that a rationale for these, apparently unrelated, recommendations was provided. Aumann and Maschler (1985) presented what is now dubbed as the *Talmud* rule, which explains all those recommendations.

This survey is about the Talmud rule, and the ramifications that originated in the sizable literature on bankruptcy problems. For more general reviews and surveys of that literature, whose seminal work is O'Neill (1982), the reader is referred to Thomson (2003, 2014, 2015).

A bankruptcy problem refers to a situation in which one has to distribute a good whose available amount is not enough to cover all agents' demands (claims) on it. A variety of situations, like the bankruptcy of a firm (our running interpretation throughout this survey), the collection of a given amount of taxes, or the division of an insufficient estate fit this definition. Obvious ways to solve these problems amount to allocate awards proportionally to claims, or to impose equal awards or losses (subject to the condition that agents neither receive a negative amount not a higher amount than their claims). The Talmud rule proposes an alternative (and ingenious) procedure to solve these problems. More precisely, it applies equal division until the claimant with the smallest claim has obtained one half of her claim. Then, that agent stops receiving additional units and the remaining amount is divided equally among the other agents until the claimant with the second smallest claim gets one half of her claim. The process continues until every agent has received one half of her claim, or the available amount is distributed. If there is still something left after this process, agents are invited back to receive additional shares. Now agents receive additional amounts sequentially starting with those with larger claims and applying equal division of their losses.

There exist axiomatic as well as game-theoretical foundations for this rule and we shall survey the main ones here. We shall also be concerned with several alternatives and generalizations of this rule that have been considered in the literature.

2 The model

We study bankruptcy problems in a variable-population model. The set of potential claimants, or *agents*, is identified with the set of natural numbers $\mathbb{N}$. Let $\mathcal{N}$ be the class of finite subsets of $\mathbb{N}$, with generic element N. Let n denote the cardinality of N. For each $i \in N$, let $c_i \in \mathbb{R}_+$ be i's *claim* and $c \equiv (c_i)_{i \in N}$ the claims profile.[1] A *(bankruptcy) problem* is a triple consisting of a population $N \in \mathcal{N}$, a claims profile $c \in \mathbb{R}^n_+$, and an *endowment* $E \in \mathbb{R}_+$ such that $\sum_{i \in N} c_i \geq E$. Let $C \equiv \sum_{i \in N} c_i$. To avoid unnecessary complication, we assume $C > 0$. Let $\mathcal{D}^N$ be the domain of bankruptcy problems with population N and $\mathcal{D} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{D}^N$.

Given a problem $(N, c, E) \in \mathcal{D}^N$, an *allocation* is a vector $x \in \mathbb{R}^n$ satisfying the following two conditions: (i) for each $i \in N$, $0 \leq x_i \leq c_i$ and (ii) $\sum_{i \in N} x_i = E$. We refer to (i) as *boundedness* and (ii) as *balance*. A *rule* on $\mathcal{D}$, $R \colon \mathcal{D} \to \bigcup_{N \in \mathcal{N}} \mathbb{R}^n$, associates with each problem $(N, c, E) \in \mathcal{D}$ an allocation $R(N, c, E)$ for the problem. Each rule R has a *dual* rule R^d defined as $R^d(N, c, E) = c - R(N, c, C - E)$, for each $(N, c, E) \in \mathcal{D}$. A rule is *self-dual* if it coincides with its dual.

We now introduce some axioms that formalize standard properties of rules within the literature.

Equal Treatment of Equals is arguably the most basic axiom one could consider in this model. It states that agents with equal claims should receive equal amounts. Formally, for each $(N, c, E) \in \mathcal{D}$, and each pair $i, j \in N$, we have $R_i(N, c, E) = R_j(N, c, E)$, whenever $c_i = c_j$.

We now consider two independence properties, known as *Claims Truncation Invariance* and *Minimal Rights First* .[2] The former postulates that the part of a claim that is above the endowment should be ignored. That is,

$$R(N, c, E) = R(N, t(N, c, E), E),$$

[1]For each $N \in \mathcal{N}$, each $M \subseteq N$, and each $z \in \mathbb{R}^n$, let $z_M \equiv (z_i)_{i \in M}$.

[2]These two axioms were studied first by Curiel et al. (1987).

where $t_i(N, c, E) = \min\{E, c_i\}$ for each $i \in N$. The latter ensures each agent the portion of the endowment that is left to her when the claims of all other agents are fully honored (provided this amount is nonnegative) and divides the remainder according to revised claims. Formally,

$$R(N, c, E) = m(N, c, E) + R(N, c - m(N, c, E), E - M(N, c, E)) \ ,$$

where $m_i(N, c, E) = \max\{0, E - \sum_{j \in N \setminus \{i\}} c_j\}$, for each $i \in N$, and $M(N, c, E) = \sum_{i \in N} m_i(N, c, E)$.

We now move to axioms modeling the concept of lower and upper bounds, which have a long tradition of use within the theory of fair allocation. A focal lower bound is the so-called *Average Truncated Lower Bound on Awards*, which is somewhat related to the *Claims Truncation Invariance* axiom considered above. It ensures each agent a minimal share of her individual claim, no matter what the other claims are. In particular, for a problem involving n agents, it establishes that any agent holding a feasible claim (a claim not larger than the endowment) will get at least one nth of her claim. And also that those agents whose individual claims are unfeasible will get at least one nth of the endowment.[3] Formally, a rule R satisfies *Average Truncated Lower Bound on Awards* if, for each $(N, c, E) \in \mathcal{D}$, $R_i(N, c, E) \geq \frac{1}{n} \min\{c_i, E\}$. Its dual property is also an interesting one. This property provides an upper bound to each claimant involved in the problem. Formally, a rule R satisfies *Average Truncated Lower Bound on Losses* if, for each $(N, c, E) \in \mathcal{D}$, $R_i(N, c, E) \leq c_i - \frac{1}{n} \min\{c_i, C - E\}$.

We conclude our inventory of axioms with a principle that has played a fundamental role in axiomatic analysis (e.g., Thomson, 2012). *Consistency* states that if some claimants leave with their awards and the problem of dividing among the remaining claimants what is left is considered, these claimants should receive the same awards as initially. Formally, a rule R is *consistent* if for each $(N, c, E) \in \mathcal{D}$, each $M \subset N$, and each $i \in M$, we have $R_i(N, c, E) = R_i(M, c_M, E_M)$, where $E_M = \sum_{i \in M} R_i(N, c, E)$.

3 The Talmud rule

The Talmud rule, introduced by Aumann and Maschler (1985), focusses on equal awards or equal losses depending on whether the endowment falls short or exceeds one half of the aggregate claim, using half-claims instead of claims. Formally,

[3] The property was introduced by Moreno-Ternero and Villar (2004) under the name of *Securement*.

Talmud rule, T: For each $(N, c, E) \in \mathcal{D}$, and each $i \in N$,

$$T_i(N, c, E) = \begin{cases} \min\left\{\frac{c_i}{2}, \lambda\right\} & \text{if } E \leq \frac{1}{2}C \\ \max\left\{\frac{c_i}{2}, c_i - \mu\right\} & \text{if } E \geq \frac{1}{2}C \end{cases}$$

where λ and μ are chosen so that $\sum_{i \in N} T_i(N, c, E) = E$.

The Talmud rule can also be given the following representation, which will be useful for the ensuing discussion. For each $(N, c, E) \in \mathcal{D}$,

$$T(N, c, E) = \begin{cases} A(N, \frac{1}{2}c, E) & \text{if } E \leq \frac{1}{2}C \\ \frac{1}{2}c + L(N, \frac{1}{2}c, E - \frac{1}{2}C) & \text{if } E \geq \frac{1}{2}C \end{cases}$$

That is, for "small" values of E the Talmud rule behaves as the constrained equal awards rule (A) and for "large" values of E as the constrained equal losses rule (L).[4]

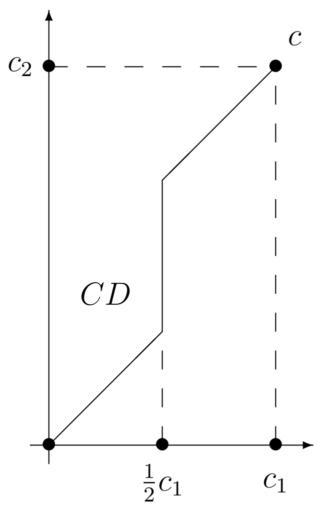

Figure 1: Concede-and-divide. This figure illustrates the "path of awards" of concede-and-divide. A point in the drawing corresponds to the awards that agents receive for a given endowment. The schedule relative to a typical claim c follows the 45^o line until it gives both agents half of the smallest claim, then it continues vertically until the endowment equals the highest claim, then again, it follows a line of slope 1 until it reaches the vector of claims.

Its two-agent version, also known as *concede-and-divide*, has a particularly appealing form, which we describe next.[5]

[4] The constrained equal-awards rule, A, selects, for each $(N, c, E) \in \mathcal{D}$, the vector $(\min\{c_i, \lambda\})_{i \in N}$, where $\lambda > 0$ is chosen so that $\sum_{i \in N} \min\{c_i, \lambda\} = E$. The constrained equal-losses rule, L, selects, for each $(N, c, E) \in \mathcal{D}$, the vector $(\max\{0, c_i - \lambda\})_{i \in N}$, where $\lambda > 0$ is chosen so that $\sum_{i \in N} \max\{0, c_i - \lambda\} = E$.

[5] The name was coined by Thomson (2003). To ease its presentation, we assume $N = \{1, 2\}$, but dismiss it from the definition.

Concede-and-divide, CD: For each $E \in \mathbb{R}_+$, and each $c = (c_1, c_2) \in \mathbb{R}^2_+$,

$$\begin{cases} CD_1(c,E) = \max\{0, E - c_2\} + \frac{1}{2}(E - \max\{0, E - c_1\} - \max\{0, E - c_2\}) \\ CD_2(c,E) = \max\{0, E - c_1\} + \frac{1}{2}(E - \max\{0, E - c_1\} - \max\{0, E - c_2\}) \end{cases}$$

That is, CD first concedes to each agent her *minimal rights* and then divides the remainder equally. Figure 1 illustrates the behavior of CD when the vector of claims is fixed and the endowment grows from zero to the aggregate claim (i.e., its "path of awards").

The following characterization results of concede-and-divide were proved by Dagan (1996) and Moreno-Ternero and Villar (2004, 2006c).[6]

Theorem 1 *Concede-and-divide is characterized by*

1. *Self-duality and Minimal Rights First.*
2. *Self-duality and Claims Truncation Invariance.*
3. *Equal treatment of equals, Minimal Rights First and Claims Truncation Invariance.*
4. *Self-duality and Average Truncated Lower Bound on Awards.*
5. *Self-duality and Average Truncated Lower Bound on Losses.*
6. *Average Truncated Lower Bound on Awards and Average Truncated Lower Bound on Losses.*
7. *Average Truncated Lower Bound on Awards and Minimal Rights First.*
8. *Average Truncated Lower Bound on Losses and Claims Truncation Invariance.*

Several rules coincide with concede-and-divide in the two-agent case. Among them, only the Talmud rule is consistent. Thus, by means of the so-called Elevator Lemma (e.g., Thomson, 2014), we can extend the previous characterizations to the Talmud rule, just appending each statement of Theorem 1 with the axiom of consistency.

[6]See also Moreno-Ternero (2006).

4 The TAL-family

One natural way of generalizing the Talmud rule is obtained by moving the threshold in its definition from one half to any other possible fraction (of the aggregate and individual claims). In doing so, we would obtain a non-countable set of piece-wise linear rules ranging from the constrained equal-awards rule to the constrained equal-losses rule, and having the Talmud rule in the middle. Such a family, known as the TAL-family, was introduced by Moreno-Ternero and Villar (2006a). Formally:

TAL-family, R^{θ}: For each $\theta \in [0,1]$, each $(N,c,E) \in \mathcal{D}$, and each $i \in N$,

$$R_i^{\theta}(N,c,E) = \begin{cases} \min\{\theta c_i, \lambda\} & \text{if } E \leq \theta C \\ \max\{\theta c_i, c_i - \mu\} & \text{if } E \geq \theta C \end{cases}$$

where λ and μ are chosen so that $\sum_{i \in N} R_i^{\theta}(N,c,E) = E$.

A systematic analysis of the TAL-family was provided by Moreno-Ternero and Villar (2006a,b). Regarding the properties introduced in Section 2, all rules within this family satisfy equal treatment of equals and consistency. It is interesting to remark that the family behaves in a perfectly symmetric way regarding the remaining properties stated above. More precisely, $\theta = \frac{1}{2}$ is the precise value of the parameter that separates the rules in the family that satisfy Claims Truncation Invariance from those that satisfy Minimal Rights First. An analogous behavior occurs for the Average Truncated Lower Bounds.

Theorem 2 *The following statements hold:*

(i) For each $\theta \in [0, \frac{1}{2}]$, R^{θ} satisfies Minimal Rights First and Average Truncated Lower Bound on Losses.

(ii) For each $\theta \in [\frac{1}{2}, 1]$, R^{θ} satisfies Claims Truncation Invariance and Average Truncated Lower Bound on Awards.

The parameter θ that generates the TAL-family can actually be interpreted as an index of progressivity of the rules within the family. More precisely, given $x, y \in \mathbb{R}^n$ satisfying $x_1 \leq x_2 \leq ... \leq x_n$, $y_1 \leq y_2 \leq ... \leq y_n$, and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, we say that x *is greater than* y *in the Lorenz ordering* if $\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i$, for each $k = 1, ..., n-1$, with at least one strict inequality. This criterion induces a partial ordering on allocations which reflects their relative spread. When x is greater than y in the Lorenz ordering, the distribution x is unambiguously "more egalitarian" than the distribution y.

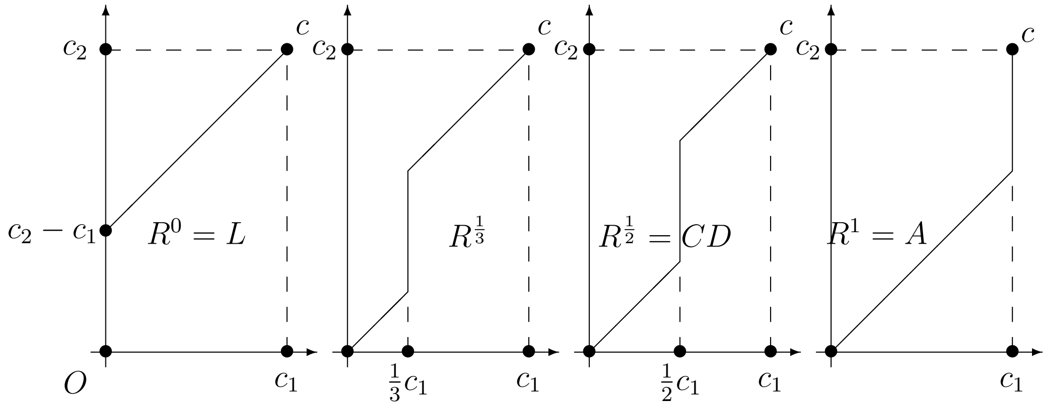

Figure 2: The TAL-family of rules. This figure illustrates the "path of awards" of some rules within the TAL-family for $N = \{1, 2\}$ and $c \in \mathbb{R}^N_+$ with $c_1 < c_2$. The path of awards for c of $R^0 = L$ follows the vertical axis until the average loss coincides with the lowest claim, i.e., until $E = c_2 - c_1$. After that, it follows the line of slope 1 until it reaches the vector of claims. The path of awards of $R^{1/3}$ follows the 45^o line until claimant 1 obtains one third of her claim. Then, it is a vertical line until $E = c_2 - \frac{1}{3}c_1$, from where it follows the line of slope 1 until it reaches the vector of claims. The path of awards of $R^{1/2}$ is that of CD. Finally, the path of awards of $R^1 = A$ follows the 45^o line until claimant 1 obtains her whole claim. Then, it is a vertical line until it reaches the vector of claims.

We say that a rule R *Lorenz dominates* a rule R', which we write as $R \succsim_L R'$, when for each $(N, c, E) \in \mathcal{D}$, $R(N, c, E)$ is greater than $R'(N, c, E)$ in the Lorenz ordering. The following result, which is due to Moreno-Ternero and Villar (2006b), says that all rules within the TAL-family are fully ranked in terms of the Lorenz dominance criterion.

Theorem 3 *For each pair $\theta_1, \theta_2 \in [0, 1]$ with $\theta_1 \geq \theta_2$, $R^{\theta_1} \succsim_L R^{\theta_2}$.*

5 The Generalized TAL-family

Bankruptcy rules can also be interpreted as taxation rules. In the usual parlance of taxation, the Talmud rule yields two possible types of tax schedules. If the aim is to collect a tax revenue below one half of the aggregate income, the tax rate is one half up to some income level (which is endogenously determined), and zero afterwards. If, on the contrary, the tax revenue is above one half of the aggregate income, the tax rate is one half first and then one. The rules within the TAL-family, interpreted as tax rules, would also yield two possible types of tax schedules that could be described similarly to those originating from the Talmud method. More precisely, for tax revenues below a fraction θ of the aggregate income, the tax rate would be θ up to some income level, and zero afterwards. For tax revenues above such a fraction, the tax rate would be θ first and then one.

In order to accommodate less restrictive methods too, while preserving the principle behind the Talmud method, Moreno-Ternero (2011a) allowed for other minimum and maximum tax rates, instead of always imposing zero and one for those values. More precisely, tax methods yielding two possible types of tax schedules; namely, for tax revenues below a fraction θ of the aggregate income, the tax rate would be θ up to some income level, and $\theta_{\min}$ afterwards. For tax revenues above such a fraction, the tax rate would be θ first and then $\theta_{\max}$. Formally,

Generalized TAL-family, GR^{θ}: For each $\theta_{\min}, \theta_{\max} \in [0,1]$ with $\theta_{\min} < \theta_{\max}$, each $\theta \in [\theta_{\min}, \theta_{\max}]$, each $(N, c, E) \in \mathcal{D}$, and each $i \in N$,

$$GR_i^{\theta}(N, c, E) = \begin{cases} \min\{\theta c_i, \max\{\theta_{\min} c_i + \lambda, 0\}\} & \textit{if } E \leq \theta C \\ \max\{\theta c_i, \min\{c_i, \theta_{\max} c_i - \mu\}\} & \textit{if } E \geq \theta C \end{cases}$$

where λ and μ are chosen so that $\sum_{i \in N} GR_i^{\theta}(N, c, E) = E$.

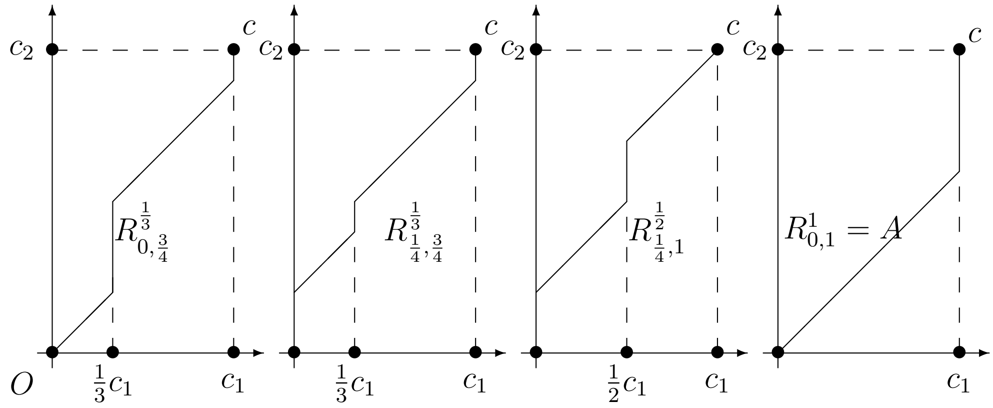

Figure 3: The Generalized TAL-family of rules. The path of awards for c of $R^{\frac{1}{3}}_{0,\frac{3}{4}}$ follows the 45^o line until claimant 1 obtains one third of her claim. Then, it is a vertical line until $E = \frac{3}{4}(c_1 + c_2) - 2(\frac{3}{4} - \frac{1}{3})c_1$. After that, it follows the line of slope 1 until claimant 1 obtains the whole of her claim. Then, it follows a vertical line until it reaches the vector of claims. The path of awards of $R^{\frac{1}{3}}_{\frac{1}{4},\frac{3}{4}}$ follows the vertical axis until claimant 2 obtains one fourth of the difference between claims. After that, it follows a line of slope 1, until claimant 1 obtains one third of her claim. Then, it is a vertical line until $E = \frac{3}{4}(c_1 + c_2) - 2(\frac{3}{4} - \frac{1}{3})c_1$, from where it follows the line of slope 1 until claimant 1 obtains the whole of her claim. Then, it follows a vertical line until it reaches the vector of claims. The path of awards of $R^{\frac{1}{2}}_{\frac{1}{4},1}$ follows the vertical axis until claimant 2 obtains one fourth of the difference between claims. After that, it follows a line of slope 1, until claimant 1 obtains one half of her claim. Then, it is a vertical line until $E = (c_1 + c_2) - 2(1 - \frac{1}{2})c_1$, from where it follows the line of slope 1 until it reaches the vector of claims.

As shown by Moreno-Ternero (2011a), rules within this family satisfy the so-called *single-crossing* property, which allows one to separate those agents who benefit from the application of one rule or the other, depending on the rank of their claims. More precisely, let $0 \leq \theta_{\min} \leq \theta_1 \leq \theta_2 \leq \theta_{\max} \leq 1$, with $\theta_{\min} < \theta_{\max}$, and $(N, c, E) \in \mathcal{D}$ be given. For ease of exposition, assume that $N = \{1, \ldots, n\}$ and $c_1 \leq c_2 \leq \cdots \leq c_n$. Then, there exists $i^* \in N$ such that:

(i) $R_i^{\theta_1}(N, c, E) \leq R_i^{\theta_2}(N, c, E)$ for each $i = 1, ..., i^*$ and

(ii) $R_i^{\theta_1}(N, c, E) \geq R_i^{\theta_2}(N, c, E)$ for each $i = i^* + 1, ..., n$.

This property has strong implications for the decentralization of the choice of rules. More precisely, suppose agents vote for rules according to majority rule. Suppose too that voters are self-interested: given a pair of alternatives, an agent votes for the alternative that gives her the greatest award. We say that a rule R is a majority voting equilibrium for a domain of rules $\mathcal{R}$ if, for each $(N, c, E) \in \mathcal{D}$, there is no other rule $R' \in \mathcal{R}$ such that, $R'_i(N, c, E) > R_i(N, c, E)$ for the majority of voters.

Theorem 4 *There is a majority voting equilibrium for the Generalized TAL-family.*

Another important implication of the single-crossing property is to guarantee that rules within the Generalized TAL-family are completely ranked according to the Lorenz dominance criterion, as stated in Theorem 3 for the case in which $\theta_{min} = 0$ and $\theta_{max} = 1$.

6 The Reverse Talmud

The Talmud rule has a natural counterpart rule in which the equal awards and equal losses principles are applied in the reverse order. More precisely, the so-called *reverse Talmud* rule (e.g., Chun et al., 2001) originates when, for each claims vector, we apply the equal losses principle in the lower half of the range of the endowment, and the equal awards principle to the upper half. As with the Talmud rule, half-claims are used instead of the claims themselves.

Reverse Talmud, RT: For each $(N, c, E) \in \mathcal{D}$, and each $i \in N$,

$$RT_i(N, c, E) = \begin{cases} \max\left\{\frac{c_i}{2} - \lambda, 0\right\} & \text{if } E \leq \frac{1}{2}C \\ \frac{1}{2}c_i + \min\left\{\frac{c_i}{2}, \mu\right\} & \text{if } E \geq \frac{1}{2}C \end{cases}$$

where λ and μ are chosen so that $\sum_{i \in N} RT_i(N, c, E) = E$.

Alternatively, the Reverse Talmud rule can also be given the following representation. For each $(N, c, E) \in \mathcal{D}$,

$$RT(N, c, E) = \begin{cases} L(N, \frac{1}{2}c, E) & \text{if } E \leq \frac{1}{2}C \\ \frac{1}{2}c + A(N, \frac{1}{2}c, E - \frac{1}{2}C) & \text{if } E \geq \frac{1}{2}C \end{cases}$$

The same natural idea considered above to generalize the Talmud rule could be considered to generalize the reverse Talmud rule, as was suggested by van den Brink et al., (2013).[7] That process gives rise to a new family of rules, the *reverse TAL-family*. Such a family also comprises a non-countable set of piece-wise linear rules, ranging from the constrained equal-awards rule

[7] See also van de Brink and Moreno-Ternero (2016).

to the constrained equal-losses rule, but this time having the reverse Talmud rule in the middle. Formally,

Reverse TAL-family, RT^{θ}: For each $\theta \in [0,1]$, each $(N,c,E) \in \mathcal{D}$, and each $i \in N$,

$$RT_i^{\theta}(N,c,E) = \begin{cases} \max\{\theta c_i - \lambda, 0\} & \text{if } E \leq \theta C \\ \theta c_i + \min\{(1-\theta)c_i, \mu\} & \text{if } E \geq \theta C \end{cases}$$

where λ and μ are chosen so that $\sum_{i \in N} RT_i^{\theta}(N,c,E) = E$.

7 The Talmudic operator

Thomson and Yeh (2008) introduced the concept of operators on the space of bankruptcy rules. They focussed on three operators in order to uncover the structure of such a space. The above discussion partly inspires the following definition of a different operator from the cartesian product of the space of rules onto itself. More precisely, for a given $\theta \in [0,1]$, the *talmudic operator* T^{θ} is the operator assigning to each pair of rules (R,S), the rule $T^{\theta}(R,S)$ defined as follows. For each $(N,c,E) \in \mathcal{D}$,

$$T^{\theta}(R,S)(N,c,E) = \begin{cases} R(N,\theta c,E) & \text{if } E \leq \theta C \\ \theta c + S(N,(1-\theta)c, E-\theta C) & \text{if } E \geq \theta C \end{cases} \quad (1)$$

A consequence of (1) is that $T^{\theta}(R,S)$ yields allocations satisfying $x_i \leq \theta c_i$ for each $i \in N$ if and only if $E \leq \theta C$, and $x_i \geq \theta c_i$ for each $i \in N$ if and only if $E \geq \theta C$. In words, the operator T^{θ} imposes a rationing of the same sort for each individual and the whole society, which somehow reflects the Talmudic dictum for these problems.

It is straightforward to see that, for each $\theta \in [0,1]$, $T^{\theta}(A,L)$ yields the corresponding member of the TAL-family of rules, whereas $T^{\theta}(L,A)$ yields the corresponding member of the reverse TAL-family of rules. Similarly, $T^{\frac{1}{2}}(A,A)$ is the so-called Piniless' rule (e.g., Thomson, 2014), whereas $T^{\theta}(A,A)$ gives rise to a sort of generalized Piniless' rules (in the same form as the TAL-family does with respect to the Talmud rule). Likewise, $T^{\frac{1}{2}}(L,L)$ is the dual of the Piniless' rule, whereas $T^{\theta}(L,L)$ gives rise to the corresponding generalized rules. Finally, $T^{\theta}(P,P) = P$, for each θ, i.e., the proportional rule is a fixed point for such an operator. Note that if the operator T^{θ} applies to the same rule (or to a rule and its dual), then it simply becomes a member of the family of composition operators studied by Hougaard et al., (2012, 2013a, 2013b).

This operator has interesting properties when combined with the so-called duality operator in specific ways, as stated in the next result.

Proposition 1 *The following statements hold:*

$(T^{\theta}(R, R^d))^d = T^{1-\theta}(R, R^d)$, *for each rule* R *and* $\theta \in [0, 1]$.

$(T^{\theta}(R, R))^d = T^{1-\theta}(R^d, R^d)$, *for each rule* R *and* $\theta \in [0, 1]$.

In words, the first statement of the proposition says that the dual of the rule the operator associated to θ yields for a pair of dual rules is the rule the operator associated to $1 - \theta$ yields for the same pair of rules. Likewise, the second statement of the proposition says that the dual of the rule the operator associated to θ yields for a pair made of a replicated rule is the rule the operator associated to $1 - \theta$ yields for the pair made of the replicated dual rule.

8 Game-theoretical foundations

In the seminal paper on bankruptcy problems, O'Neill (1982) not only explored the axiomatic approach to these problems, but also a game-theoretical approach. He suggested to use solutions to (transferable utility) coalitional games to generate rules for bankruptcy problems, by means of a natural procedure. More precisely, for each $(N, c, E) \in \mathcal{D}$, its associated (transferable utility) coalitional game is the one defined by the characteristic function $v(S) = \max\{0, E - \sum_{j \notin S} c_j\}$, for each $S \subset N$, with $v(N) = E$ and $v(\emptyset) = 0$. In words, the worth of each coalition $S \subset N$ is the difference between the endowment and the sum of the claims of the members of the complementary coalition, if this difference is non-negative, and 0 otherwise.[8] In this context, individual rationality is given by $x_i \geq v(\{i\}) = m_i(N, c, E)$. Moreover, for each pair $i, j \in N$ such that $c_i, c_j \geq E$ it follows that $v(\{i\}) = v(\{j\}) = 0$ (that is, players i and j are "permuted players" whenever they claim more than there is). Therefore, the limits given by the properties of *Minimal Rights First* and *Claims Truncation Invariance* turn out to be the natural bounds to the characteristic function of the associated game. Note that the core of this game is given by all allocations $x \in \mathbb{R}^n$ such that $\sum_{i \in N} x_i = E$ and $m_i(N, c, E) \leq x_i \leq t_i(N, c, E)$.

We say that a bankruptcy rule is associated to a coalitional game solution if the recommendation made by the rule coincides with the recommendation made by the solution when

[8]This is a rather pessimistic assessment of what a coalition can achieve. Other more optimistic proposals have been considered in the literature (e.g., Driessen, 1998).

applied to the coalitional game associated with the problem.[9]

The following result was proved by Aumann and Maschler (1985).

Theorem 5 *The Talmud rule is associated to the prenucleolus solution.*[10]

Dagan and Volij (1993) showed how to associate each bankruptcy problem with a bargaining problem and, actually, they derived some rules proceeding accordingly, as others did later. Thomson (2003) summarized some of the existing results along those lines. No result connecting the Talmud rule with a known bargaining solution exists. Except for the domain of two-agent problems, for which the Talmud rule (i.e., concede-and-divide) is associated to the so-called Perles-Maschler bargaining solution, which, in the two-player case, when the undominated boundary of the problem is a segment, selects the middle of the segment (e.g., Perles and Maschler, 1981).

Aumann and Maschler (1985) also considered a coalitional procedure to explain the awards obtained by the Talmud rule. We present here a generalization of their procedure (originally introduced by Moreno-Ternero, 2011b) which explains the awards obtained by each of the rules within the TAL-family.[11]

Fix some $\theta \in [0,1]$, and consider the following procedure. First, in the case of a two-agent problem, we apply the two-agent version of the corresponding rule within the family. Formally,

$$R^{\theta}(N,c,E) = \begin{cases} \left(\frac{E}{2}, \frac{E}{2}\right) & \text{if } E \leq 2\theta c_1 \\ (\theta c_1, E - \theta c_1) & \text{if } 2\theta c_1 \leq E \leq c_2 - c_1 + 2\theta c_1 \\ \left(c_1 - \frac{C-E}{2}, c_2 - \frac{C-E}{2}\right) & \text{if } c_2 - c_1 + 2\theta c_1 \leq E \end{cases} \tag{2}$$

Suppose now that we have a problem with three creditors. Then, we proceed in the following way. First, creditors 2 and 3 pool their claims an act as a single agent vis-a-vis 1. The solution (2) of the resulting problem yields awards to agent 1, and to the coalition of agents 2 and 3; to divide its award among its members, the coalition again applies solution (2). The result is order preserving if and only if $3\theta c_1 \leq E \leq C - 3(1-\theta)c_1$. To see this, note that if $3\theta c_1 > E$, then the award of creditor 1, θc_1, would be strictly greater than the one of creditor 2, which is $\frac{E-\theta c_1}{2}$, as a result of the awards sharing in the coalition of creditors 2 and 3. Analogously, if $E > C - 3(1-\theta)c_1$, then the loss of creditor 1, $(1-\theta)c_1$, would be greater than $\frac{c_2+c_3-E+\theta c_1}{2}$,

[9]Curiel et al., (1987) showed that the necessary and sufficient condition for a bankruptcy rule to be associated with a coalitional game is precisely *Claims Truncation Invariance*.

[10]The prenucleolus (e.g., Schemeidler, 1969) is the set of payoff vectors at which the vector of dissatisfactions is minimized in the lexicographic (maximin) order among all efficient payoff vectors.

[11]Quant and Borm (2010) proposed a different generalization of Aumann and Maschler's procedure.

the resulting loss associated to creditor 2, after dividing the awards in the coalition. If one divides the awards equally when $E \leq 3\theta c_1$, and the losses equally when $E \geq C - 3(1-\theta)c_1$, it is obtained, precisely, the solution provided by the rule R^θ, over the entire range $0 \leq E \leq C$.

By using induction, one may generalize this in a natural way to an arbitrary n. Suppose we already know the solution for $(n-1)$-agent problems. Depending on the values of the endowment and the vector of claims, we treat a given $n-$person problem in one of the following three ways:

(i) Divide E between $\{1\}$ and $M = \{2, ..., n\}$, in accordance with the solution (2) to the two-agent problem $(\{1, M\}, (c_1, c_2 + ... + c_n), E)$, and then use the $(n-1)$-agent rule, which we know by induction, to divide the amount assigned to the coalition M between its members.

(ii) Assign equal awards to all creditors.

(iii) Assign equal losses to all creditors.

Specifically, (i) is applied whenever it yields an order-preserving result, which is precisely when $n\theta c_1 \leq E \leq C - n(1-\theta)c_1$. We apply (ii) when $E \leq n\theta c_1$. Finally, we apply (iii) when $E \geq C - n(1-\theta)c_1$. We call this generalization the $\theta-$*coalitional procedure*. In the particular case of $\theta = \frac{1}{2}$, the θ-*coalitional procedure* corresponds to the coalitional procedure stated by Aumann and Maschler (1985). To summarize the previous discussion, we can state the following result:

Theorem 6 *For each $\theta \in [0,1]$, and for each bankruptcy problem, the $\theta-$coalitional procedure and the rule R^θ in the TAL-family yield the same solution to the problem.*

Theorem 6 describes an orderly step-by-step process, which by its very definition must lead to a unique result, therefore characterizing the TAL-family of rules.

9 Coalitional manipulation

An important strategic aspect of the bankruptcy model is the manipulation by merging and splitting agents' claims. We say that a rule is merging-proof, if there is no incentive for a coalition $Q \subset N$ to consolidate their claims $(c_i)_{i \in Q}$ into a single one $c_Q = \sum_{i \in Q} c_i$. Similarly, we say that a rule is splitting-proof if no single agent $i \in N$ has incentives to represent her claim c_i as a collection of several claims, $c_{i1}, c_{i2}, ..., c_{iK}$. Formally,

*A rule R is **merging-proof** on $\widehat{\mathcal{D}} \subseteq \mathcal{D}$, if for all (M, c', E) and $(N, c, E) \in \widehat{\mathcal{D}}$, with $M \subset N$, and such that there is some $i \in M$ such that $c'_i = c_i + \sum_{j \in N \setminus M} c_j$ and for each $j \in M \setminus \{i\}$,*

$c'_j = c_j$, then $R_i(M, c', E) \le R_i(N, c, E) + \sum_{j \in N \setminus M} R_j(N, c, E)$.

A rule R is ***splitting-proof*** *on $\widehat{\mathcal{D}} \subseteq \mathcal{D}$, if for each (M, c', E) and $(N, c, E) \in \widehat{\mathcal{D}}$, with $M \subset N$, and such that there is some $i \in M$ such that $c'_i = c_i + \sum_{j \in N \setminus M} c_j$ and for each $j \in M \setminus \{i\}$, $c'_j = c_j$ then $R_i(M, c', E) \ge R_i(N, c, E) + \sum_{j \in N \setminus M} R_j(N, c, E)$.*

A rule R is ***non manipulable*** *on $\widehat{\mathcal{D}} \subseteq \mathcal{D}$, if it is simultaneously merging-proof on $\widehat{\mathcal{D}}$ and splitting-proof on $\widehat{\mathcal{D}}$.*

Let $\tau(N, c, E) = \frac{E}{C}$ stand for the share of the endowment in the aggregate claim of a given problem, and define

$$\mathcal{D}^{\delta} = \{(N, c, E) \in \mathcal{D} : \tau(N, c, E) = \delta\},$$

for each $\delta \in (0, 1)$. In other words, $\mathcal{D}^{\delta}$ is the domain of problems whose ratio between the endowment and the aggregate claim is δ.

The following result was proved by Moreno-Ternero (2007):

Theorem 7 *Let $\{R^{\theta}\}_{\theta \in [0,1]}$ denote the TAL-family, and let $\delta \in (0, 1)$ be given. The following statements hold:*

(i) If $\theta < \delta$ then R^{θ} is splitting-proof on $\mathcal{D}^{\delta}$.

(ii) If $\theta > \delta$ then R^{θ} is merging-proof on $\mathcal{D}^{\delta}$.

(iii) If $\theta = \delta$ then R^{θ} is non manipulable on $\mathcal{D}^{\delta}$.

Besides determining whether a rule is non-manipulable or not, one could also be interested in comparing the relative non-manipulability of different rules in terms of their outcomes. This can be done by introducing an index of non-manipulability that measures the difference between the resulting and primitive outcomes of the claimants who incurred in the manipulation. Such a difference can be contemplated as the magnitude of the incentive against the manipulation.

Formally, we say that a rule F is **more merging-proof than** G on $\widehat{\mathcal{D}}$ (which we write $\mathcal{M}^{\widehat{\mathcal{D}}}(F) \ge \mathcal{M}^{\widehat{\mathcal{D}}}(G)$) if, for each $(N, c, E) \in \widehat{\mathcal{D}}$ and (M, c', E), with $M \subset N$, and such that there is some $i \in M$ such that $c'_i = c_i + \sum_{j \in N \setminus M} c_j$ and for each $j \in M \setminus \{i\}$, $c'_j = c_j$, then

$$\left(\sum_{j \in (N \setminus M) \cup \{i\}} F_j(N, c, E)\right) - F_i(M, c', E) \ge \left(\sum_{j \in (N \setminus M) \cup \{i\}} G_j(N, c, E)\right) - G_i(M, c', E).$$

Similarly, we say that a rule F is **more splitting-proof than** G on $\widehat{\mathcal{D}}$ (which we write $\mathcal{S}^{\widehat{\mathcal{D}}}(F) \ge \mathcal{S}^{\widehat{\mathcal{D}}}(G)$) if, for each $(N, c, E) \in \widehat{\mathcal{D}}$ and (M, c', E) in the above conditions,

$$\left(\sum_{j \in (N \setminus M) \cup \{i\}} F_j(N, c, E)\right) - F_i(M, c', E) \le \left(\sum_{j \in (N \setminus M) \cup \{i\}} G_j(N, c, E)\right) - G_i(M, c', E).$$

The following result, also proved by Moreno-Ternero (2007), is obtained:

Theorem 8 *Let $\{R^\theta\}_{\theta\in[0,1]}$ denote the TAL-family, and let θ_1 and $\theta_2 \in [0,1]$, such that $\theta_1 \geq \theta_2$, be given. Let $\delta \in (0,1)$ be fixed. The following statements hold:*

(i) If $\theta_1 \geq \theta_2 \geq \delta$ then $\mathcal{M}^{\mathcal{D}^\delta}(R^{\theta_1}) \geq \mathcal{M}^{\mathcal{D}^\delta}(R^{\theta_2}) \geq \mathcal{M}^{\mathcal{D}^\delta}(R^{\delta})$.

(ii) If $\theta_2 \leq \theta_1 \leq \delta$ then $\mathcal{S}^{\mathcal{D}^\delta}(R^{\theta_2}) \geq \mathcal{S}^{\mathcal{D}^\delta}(R^{\theta_1}) \geq \mathcal{S}^{\mathcal{D}^\delta}(R^{\delta})$.

(iii) If $\theta_2 \leq \delta \leq \theta_1$ then $\mathcal{M}^{\mathcal{D}^\delta}(R^{\theta_1}) \geq \mathcal{M}^{\mathcal{D}^\delta}(R^{\delta})$ and $\mathcal{S}^{\mathcal{D}^\delta}(R^{\theta_2}) \geq \mathcal{S}^{\mathcal{D}^\delta}(R^{\delta})$.

Without loss of generality, we may assume that if R is a non-manipulable rule on a domain $\widehat{\mathcal{D}}$, then $\mathcal{M}^{\widehat{\mathcal{D}}}(R) = \mathcal{S}^{\widehat{\mathcal{D}}}(R) = 0$. Consequently, we may also assume that $\mathcal{M}^{\widehat{\mathcal{D}}}(R) < 0$ for each manipulable-by-merging rule R on $\widehat{\mathcal{D}}$ and $\mathcal{S}^{\widehat{\mathcal{D}}}(R) < 0$ for each manipulable-by-splitting rule R on $\widehat{\mathcal{D}}$. This convention and Theorem 8 provide us with a precise interpretation of the parameter θ that generates the TAL-family as an index of relative non-manipulability. More precisely, fix some $\delta \in (0,1)$ and consider its corresponding domain of problems $\mathcal{D}^\delta$. Up to affine transformations, the indexes can be expressed as follows:

- $\mathcal{M}^{\mathcal{D}^\delta}(R^\theta) = \theta - \delta$ for all $\theta \in [0,1]$.
- $\mathcal{S}^{\mathcal{D}^\delta}(R^\theta) = \delta - \theta$ for all $\theta \in [0,1]$.

Thus, $\mathcal{M}^{\mathcal{D}^\delta}(R^\theta) < 0$ for each $\theta \in [0,\delta)$, which means that they are all manipulable (by merging) rules on $\mathcal{D}^\delta$. Furthermore, the rules corresponding to the remaining values of the parameter θ increase the degree of merging-proofness from R^δ, which coincides with the proportional rule on $\mathcal{D}^\delta$, to $R^1 = A$. Similarly, $\mathcal{S}^{\mathcal{D}^\delta}(R^\theta) < 0$ for each $\theta \in (\delta, 1]$, which means that they are all manipulable (by splitting) rules on $\mathcal{D}^\delta$. Furthermore, the rules corresponding to the remaining values of the parameter θ increase the degree of splitting-proofness from R^δ, which coincides with the proportional rule on $\mathcal{D}^\delta$, to $R^0 = L$.

10 Final remarks

To conclude, we report on some other families of rules that have emerged in the literature, while extending the Talmud rule in other directions.

Hokari and Thomson (2003) introduced a family of consistent rules meeting the two characteristic properties of the Talmud rule described above (*Minimal Rights First* and *Claims Truncation Invariance*), while dismissing *Equal Treatment of Equals*. The resulting rules are *weighted* versions of the Talmud rule, defined by partitioning the set of potential claimants into priority classes, and selecting reference weights for all potential claimants. The rules, which could also be seen as hybrid rules between weighted versions of the constrained equal awards and constrained equal losses rules, consistently extend a one-parameter family in the two-agent case, dubbed as *weighted concede-and-divide* rules. These rules also endorse the Talmudic dictum of conceding minimal rights to each claimant, but then divide the remainder unequally (and according to the weights). Formally, let $\alpha \in (0,1)$.[12]

Weighted concede-and-divide, CD^{α}: For each $E \in \mathbb{R}_+$, and each $c = (c_1, c_2) \in \mathbb{R}^2_+$,

$$\begin{cases} CD_1^{\alpha}(c,E) = \max\{0, E - c_2\} + \alpha(E - \max\{0, E - c_1\} - \max\{0, E - c_2\}) \\ CD_2^{\alpha}(c,E) = \max\{0, E - c_1\} + (1-\alpha)(E - \max\{0, E - c_1\} - \max\{0, E - c_2\}) \end{cases}$$

Thomson (2008) introduced the so-called ICI family, which constitutes a further generalization of the TAL-family. Rules within the ICI-family impose that the evolution of each claimant's award, as a function of the endowment, is increasing first, constant next and finally increasing again.[13]

Formally, let $\mathcal{G}^N$ be the family of lists $G \equiv \{E_k, F_k\}_{k=1}^{n-1}$, where $n = |N|$, of real-valued functions of the claims vector, satisfying for each $c \in \mathbb{R}^N_+$, the following relations:

$$\begin{aligned} \frac{E_1(c)}{n} + \frac{C - F_1(c)}{n} &= c_1 \\ c_1 + \frac{E_2(c) - E_1(c)}{n-1} + \frac{F_1(c) - F_2(c)}{n-1} &= c_2 \\ &\vdots \\ c_{k-1} + \frac{E_k(c) - E_{k-1}(c)}{n-k+1} + \frac{F_{k-1}(c) - F_k(c)}{n-k+1} &= c_k \\ &\vdots \\ c_{n-1} + \frac{-E_{n-1}(c)}{1} + \frac{F_{n-1}(c)}{1} &= c_n \end{aligned}$$

The ICI rule relative to $G \equiv \{E_k, F_k\}_{k=1}^{n-1} \in \mathcal{G}^N$, is defined as follows. For each $c \in \mathbb{R}^N_+$, the awards vector is given as the following function of the amount available E, as it varies from 0 to C. As E increases from 0 to $E_1(c)$, equal division prevails; as it increases from $E_1(c)$ to $E_2(c)$, claimant 1's award remains constant, and equal division of each new unit prevails among

[12]To ease its presentation, we assume $N = \{1, 2\}$, but dismiss it from the definition.

[13]More recently, Huijink et al., (2015) have identified the rules in such a family as *claim-and-right* rules, which give a specific interpretation to the concept of *baselines* formalized earlier by Hougaard et al., (2012, 2013a, 2013b). See also Timoner and Izquierdo (2016) for a related notion.

the other claimants. As E increases from $E_2(c)$ to $E_3(c)$, claimants 1 and 2's awards remain constant, and equal division of each new unit prevails among the other claimants, and so on. This process goes on until E reaches $E_{n-1}(c)$. The next units go to claimant n until E reaches $F_{n-1}(c)$, at which point equal division of each new unit prevails among claimants n and $n-1$. This goes on until E reaches $F_{n-2}(c)$, at which point equal division of each new unit prevails among claimants n through $n-2$. The process continues until E reaches $F_1(c)$, at which point claimant 1 re-enters the scene and equal division of each new unit prevails among all claimants.

This is a large family encompassing many rules. Nevertheless, if we impose the property of *consistency* introduced above, as well as the innocuous condition of *scale invariance*, then the family shrinks precisely to the TAL-family, which we have thoroughly described here.

Thomson (2008) also introduced the so-called CIC-family, which imposes that the evolution of each claimant's award, as a function of the endowment, is constant first, increasing next and finally constant again. Its formal definition is a *reverse* parallel of that of the ICI-family just described. Imposing *consistency* and *scale invariance* to the rules within the family, it shrinks to the reverse TAL-family (also described above).

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