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A strategic justification of the Talmud rule based on lower and upper bounds

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A strategic justification of the Talmud rule based on lower and upper bounds*

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Abstract

We follow the Nash program to provide a new strategic justification of the Talmud rule in bankruptcy problems. The design of our game is based on a focal axiomatization of the rule, which combines consistency with meaningful lower and upper bounds to all creditors. Our game actually considers bilateral negotiations, inspired by those bounds, and consistently extended to an arbitrary number of creditors.

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1 Introduction

We consider the so-called bankruptcy problem in which a group of creditors have claims on the liquidation value of a (bankrupt) firm that is not enough to honor all claims. How should this value be allocated? A "rule" is a function that associates with each bankruptcy problem an allocation of the liquidation value, called an "awards vector".[1] A well-known rule is the Talmud rule (Aumann and Maschler, 1985), so-called because it rationalizes several numerical examples in the ancient Jewish document.[2] This rule has mostly been studied from an axiomatic perspective. However, little attention has been paid to it from a strategic perspective, in contrast with other bankruptcy rules.[3] Our goal in this paper is to further our understanding of the Talmud rule by offering another strategic justification of this rule. To do so, we apply the so-called *Nash program*,[4] and construct a game with the following property: it has a unique equilibrium allocation, and this allocation corresponds to the one dictated by the Talmud rule.

Our game relies on several properties that the Talmud rule satisfies.[5] First is "bilateral consistency", which says that if the rule chooses an awards vector for a bankruptcy problem, then for the associated "two-creditor reduced problem" derived by imagining that all the other creditors leave with their components of the vector, and reassessing the situation from the viewpoint of the two remaining creditors, it chooses the corresponding awards of the vector to that subgroup. Suppose that for each problem and each proper two-creditor subgroup, given an awards vector, the rule chooses the corresponding awards of the vector to this subgroup for the reduced problem it faces. "Converse consistency" says that the rule should choose the

[1] O'Neill (1982) initiates this literature. For surveys, see Thomson (2003, 2015, 2019).

[2] See Moreno-Ternero (2018) for a specific survey on the Talmud rule and its ramifications within the literature on bankruptcy problems.

[3] See, for instance, Chun (1989), Dagan et al. (1997), Chang and Hu (2008), or Hagiwara and Hanato (2019).

[4] Nash (1953) initiates the study on strategic justifications of cooperative solutions and is the first instance of this program. For references, see Maschler and Owen (1989), Serrano (1993, 1995), Krishna and Serrano (1995), Yan (2002), Vidal-Puga (2004), Hu et al. (2012, 2018), or Chun et al. (2017), among others. For surveys, see Serrano (2005, 2008).

[5] This is in line with Krishna and Serrano (1996), who suggest that the properties of a rule should play important roles to strategically justify the rule.

awards vector for the initial problem.[6] Finally, we exploit a lower bound on creditors' awards in designing our game. In the literature, Tsay and Yeh (2019) construct a game that strategically justifies the Talmud rule based on a lower bound on creditors' awards, called the minimal right of a creditor (Aumann and Maschler, 1985), and its dual. We consider another one, called the "average truncated claim lower bounds on awards" (Moreno-Ternero and Villar, 2004), and its dual. *Average truncated claim lower bounds on awards* says that no creditor should receive less than the amount obtained by dividing the minimum of her claim and the liquidation value by the number of creditors.[7]

Specifically, we consider the following three-stage extensive form game G.

***Stage 1*:** Each creditor announces an awards vector and a permutation (it is a function mapping from the set of creditors to itself). The composition of the permutations selects a creditor as the "coordinator". If all creditors, except possibly for the coordinator, announce the same awards vector, it is the "proposal"; otherwise, the awards vector announced by the coordinator is the proposal.

***Stage 2*:** The coordinator either accepts the proposal, in which case it is the outcome, or rejects the proposal, in which case she picks one creditor to negotiate their awards in the next stage, and all others receive their awards as specified in the proposal.

***Stage 3*:** *Nature* chooses one of the two remaining creditors as the "initiator". The initiator decides to adopt one of the two possible perspectives: *gain* or *loss*.

If the initiator chooses *gain*, the responder chooses an amount (to be interpreted as what she would like to divide with the initiator) out of these two numbers: the remaining liquidation value and the claim of the creditor with the smaller claim. The responder also proposes a

[6] For a survey on these two related properties, and the important roles they have played in the axiomatic literature, see Thomson (2016).

[7] This property is introduced by Moreno-Ternero and Villar (2004) under the name of "securement". It has also been used later by Dominguez and Thomson (2006), Moreno-Ternero (2006), Moreno-Ternero and Villar (2006a,b) and Yeh (2008), among others. More recently, Harless (2017) has connected it to the concept of *guarantees*, which measures the "worst case" scenario for (incumbent) creditors after adding new creditors to a problem, without increasing the liquidation value.

division of the chosen amount. Namely, she proposes to the initiator two numbers whose sum is equal to the chosen amount. The initiator then picks one of the two numbers as her award and the responder takes the residual.

If the initiator chooses *loss*, the responder chooses an amount out of these two numbers: the shortfall (the difference between the sum of the claims of the two creditors and the remaining liquidation value) and the claim of the creditor with the smaller claim. The responder also proposes a division of the chosen amount. Namely, she proposes to the initiator two numbers whose sum is equal to the chosen amount. The initiator then picks one of the two numbers and leaves the other to the responder. Finally, the responder receives the difference between her claim and the number left to her, and the initiator takes the residual.

We show that for each problem, there is a unique Nash Equilibrium outcome of the game G and that moreover, it is the Talmud awards vector. Namely, the game G strategically justifies the Talmud rule.

The paper is organized as follows. In Section 2, we introduce the model, the Talmud rule, and the properties. In Section 3, we formally introduce the above game and prove our results. We provide some concluding remarks in Section 4.

2 The model

There is an infinite set of "potential" creditors, indexed by the natural numbers $\mathbb{N}$. Let $\mathcal{N}$ be the class of non-empty and finite subsets of $\mathbb{N}$. Given $N \in \mathcal{N}$ and $i \in N$, let c_i be **creditor $\boldsymbol{i}$'s claim** and $c \equiv (c_i)_{i \in N}$ the claims vector. The **liquidation value** E of a bankrupt firm has to be divided among its creditors N. A bankruptcy problem for N, or simply a **problem for $\boldsymbol{N}$**, is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $\sum_{i \in N} c_i \geq E$. Let $\mathcal{B}^N$ be the class of all problems for N. An **awards vector** for $(c, E) \in \mathcal{B}^N$ is a vector $x \in \mathbb{R}^N$ such that $0 \leqq x \leqq c$ and $\sum_{i \in N} x_i = E$.[8] Let $X(c, E)$ be the set of awards vectors of (c, E). A **rule** is a function defined on $\bigcup_{N \in \mathcal{N}} \mathcal{B}^N$ that associates with each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{B}^N$ a vector in $X(c, E)$. Our generic

[8]By $\mathbb{R}^N_+$, we denote the Cartesian product of $|N|$ copies of $\mathbb{R}_+$, indexed by the elements of N. Vector inequalities: $x \geqq y$, $x \geq y$, and $x > y$.

notation for rules is φ. For each $N' \subset N$, we write $c_{N'}$ for $(c_i)_{i \in N'}$, $\varphi_{N'}(c, E)$ for $(\varphi_i(c, E))_{i \in N'}$, and so on.

We now introduce the rule and the properties that are central to our analysis. The Talmud rule was proposed by Aumann and Maschler (1985) to rationalize the suggestions made in the Talmud for numerical examples. Formally,

Talmud rule, *T*: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$, and each $i \in N$,

$$T_i(c, E) \equiv \begin{cases} \min\left\{\frac{c_i}{2}, \lambda\right\} & \text{if } \sum_{i \in N} \frac{c_i}{2} \geq E; \\ \frac{c_i}{2} + \max\left\{\frac{c_i}{2} - \lambda, 0\right\} & \text{otherwise,} \end{cases}$$

where $\lambda \in \mathbb{R}_+$ is chosen so that $\sum_{i \in N} T_i(c, E) = E$.

It will be useful to have an explicit expression of the Talmud rule in the two-creditor case.[9]

For each $N \equiv \{i, j\} \in \mathcal{N}$ and each $(c, E) \in \mathcal{B}^N$ with $c_i \leq c_j$,

$$T(c, E) \equiv \begin{cases} \left(\frac{E}{2}, \frac{E}{2}\right) & \text{if } E \leq c_i \\ \left(\frac{c_i}{2}, E - \frac{c_i}{2}\right) & \text{if } c_i \leq E \leq c_j \\ \left(c_i - \frac{(c_i + c_j - E)}{2}, c_j - \frac{(c_i + c_j - E)}{2}\right) & \text{if } c_j \leq E \end{cases}.$$

Consider a problem and the awards vector x chosen by a rule for it. The rule is "bilaterally consistent" if for each two-creditor reduced problem, it chooses the restriction of x to this subgroup.

Bilateral consistency: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$, and each $N' \subset N$ with $|N'| = 2$, if $x = \varphi(c, E)$, then $x_{N'} = \varphi\left(c_{N'}, \sum_{N'} x_i\right)$.

Suppose that an awards vector x is such that for each two-creditor reduced problem, a rule chooses the restriction of x to this subgroup. The rule is "conversely consistent" if it chooses x for the initial problem.

[9]Thomson (2003) coined the indicative name of *concede-and-divide* for the rule in this case.

Converse consistency: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$, and each $x \in X(c, E)$, if for each $N' \subset N$ with $|N'| = 2$, we have $x_{N'} = \varphi(c_{N'}, \sum_{N'} x_i)$, then $x = \varphi(c, E)$.

We next introduce a property related to lower bounds. A creditor can never get an amount above her *effective claim*, by which we refer to the minimum of her claim and the liquidation value. The following lower bound property states that each creditor should receive at least one n-th of her effective claim.

Average truncated claim lower bounds on awards: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$, and each $i \in N$, $\varphi_i(c, E) \geq \frac{\min\{c_i, E\}}{|N|}$.

We can also consider a "dual" property referring to losses:[10]

Average truncated claim upper bounds on awards: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$, and each $i \in N$, $\varphi_i(c, E) \leq c_i - \frac{\min\{c_i, \sum_{j \in N} c_j - E\}}{|N|}$.

Note that, for each creditor i, $\boldsymbol{\min\{c_i, E\}}$ can be interpreted as **her truncated claim from gains** and $\boldsymbol{\frac{\min\{c_i, E\}}{|N|}}$ can be seen as **her minimal award**; similarly, $\boldsymbol{\min\{c_i, \sum_{j \in N} c_j - E\}}$ can be interpreted as **her truncated claim from losses** and $\boldsymbol{\frac{\min\{c_i, \sum_{j \in N} c_j - E\}}{|N|}}$ can be regarded as **her minimal loss**. Thus, $\boldsymbol{c_i - \frac{\min\{c_i, \sum_{j \in N} c_j - E\}}{|N|}}$ can be interpreted as creditor i's maximal award after experiencing her minimal loss just defined.

It is well known that the Talmud rule satisfies all the properties introduced in this section (e.g., Thomson, 2019).

[10] The dual of a property can be defined as follows. Given a rule φ, its dual φ^d treats the problem of dividing what is available in the same way as φ treats the problem of dividing what is missing. Formally, for each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{B}^N$, $\varphi^d(c, E) \equiv c - \varphi\left(c, \sum_{i \in N} c_i - E\right)$. We say that two properties are dual whenever a rule φ satisfies one of them, φ^d satisfies the other. Clearly, the *average truncated claim lower bounds on awards* and the *average truncated claim upper bounds on awards* are *dual*. For a study of duality relations among rules and among properties, see Thomson and Yeh (2008).

3 A strategic justification of the Talmud rule

Our objective is to offer another strategic justification of the Talmud rule. As suggested by Krishna and Serrano (1996), the property of a rule plays an important role in strategically justifying the rule. Recently, Tsay and Yeh (2019) constructed a game that strategically justifies the Talmud rule based on a lower bound on creditors' awards, called the minimal right of a creditor (Aumann and Maschler, 1985), and its dual. Here, we consider instead the dual pair of bounds introduced above to construct a game that, as we will show, also justifies the Talmud rule strategically.

3.1 Preliminaries in the two-agent case

We first show that the Talmud rule can be represented by the following formula in two-creditor cases, which is interesting by itself, and highlights the connection to the dual pair of bounds we shall consider.

Proposition 1. For each $N \equiv \{i, j\} \in \mathcal{N}$ with $i \neq j$, and each $(c, E) \in \mathcal{B}^N$ with $c_i \leq c_j$,

$$T_i(c,E) = \max\left\{\frac{\min\{c_i, E\}}{2}, c_i - \frac{\min\{c_i, \sum_{k\in N} c_k - E\}}{2} - \left(\sum_{k\in N} c_k - E - \min\left\{c_i, \sum_{k\in N} c_k - E\right\}\right)\right\}, \quad (1)$$

and $T_j(c, E) = E - T_i(c, E)$.

Proof. Let $N \equiv \{i, j\} \in \mathcal{N}$ with $i \neq j$ and $(c, E) \in \mathcal{B}^N$ with $c_i \leq c_j$. Without loss of generality, assume that $i = 1$ and $j = 2$. As $T_1(c, E) + T_2(c, E) = E$, it suffices to show that Equation (1) holds for $i = 1$. We consider three cases.

Case 1: $\boldsymbol{E \leq c_1}$. In this case, $T_1(c, E) = T_2(c, E) = \frac{E}{2}$. As $\min\{c_1, E\} = E$ and $\min\{c_1, \sum_{k\in N} c_k - E\} = c_1$ it follows that

$$\begin{aligned} & c_1 - \frac{\min\{c_1, \sum_{k\in N} c_k - E\}}{2} - \left(\sum_{k\in N} c_k - E - \min\left\{c_1, \sum_{k\in N} c_k - E\right\}\right) \\ &= \frac{c_1}{2} - (c_2 - E) \\ &\leq \frac{E}{2}. \end{aligned}$$

Thus, the right-hand side of Equation (1) is $\frac{E}{2}$, which is equal to $T_1(c, E)$.

Case 2: $c_1 < E \leq c_2$. In this case, $T_1(c,E) = \frac{c_1}{2}$ and $T_2(c,E) = E - \frac{c_1}{2}$. As $\min\{c_1, E\} = c_1$ and $\min\left\{c_1, \sum_{k\in N} c_k - E\right\} = c_1$, it follows that

$$\begin{aligned}
& c_1 - \frac{\min\left\{c_1, \sum_{k\in N} c_k - E\right\}}{2} - \left(\sum_{k\in N} c_k - E - \min\left\{c_1, \sum_{k\in N} c_k - E\right\}\right) \\
&= \frac{c_1}{2} - (c_2 - E) \\
&\leq \frac{c_1}{2}.
\end{aligned}$$

Thus, the right-hand side of Equation (1) is $\frac{c_1}{2}$, which is equal to $T_1(c,E)$.

Case 3: $c_2 < E$. In this case, $T_1(c,E) = c_1 - \frac{c_1+c_2-E}{2}$ and $T_2(c,E) = c_2 - \frac{c_1+c_2-E}{2}$. As $\min\{c_1, E\} = c_1$ and $\min\left\{c_1, \sum_{k\in N} c_k - E\right\} = \sum_{k\in N} c_k - E$, it follows that

$$\begin{aligned}
& c_1 - \frac{\min\left\{c_1, \sum_{k\in N} c_k - E\right\}}{2} - \left(\sum_{k\in N} c_k - E - \min\left\{c_1, \sum_{k\in N} c_k - E\right\}\right) \\
&= c_1 - \frac{c_1 + c_2 - E}{2} \\
&\geq \frac{c_1}{2}.
\end{aligned}$$

It follows that the right-hand side of Equation (1) is $c_1 - \frac{c_1+c_2-E}{2}$, which is equal to $T_1(c,E)$. *Q.E.D.*

Proposition 1 says that in two-creditor cases, the awards vector prescribed by the Talmud rule can be obtained as follows. First, the creditor with the smaller claim, called the small creditor, is asked to compare her average truncated claim lower bound with the difference between her average truncated claim upper bound and the residual loss (the difference between the total loss and her truncated claim from losses). The small creditor picks the maximum of the two amounts as her award. The other creditor, called the big creditor, then receives the residual.

Interestingly, Proposition 1 suggests the following non-cooperative procedure. First, the small creditor is designed to choose a perspective between gains and losses. Next, the big creditor is called up. If the perspective of gains (losses) is chosen, then the big creditor chooses an amount between the small creditors claim and the liquidation value (the total loss), and in addition, proposes a division of her chosen amount, namely she proposes two numbers such that the sum of the proposed numbers is equal to her chosen amount. Then, the small creditor chooses a number from the proposed division as her award (chooses a number from the proposed division and leaves the last number to the big creditor as the big creditor's loss[11]), and the big (small) creditor receives the residual.

It can be shown that in two-creditor cases, the Talmud rule can be strategically justified by this non-cooperative procedure. However, by doing so, the strategy spaces of the two creditors are different. Thus, the creditors are not treated symmetrically. To avoid such an asymmetric treatment, we introduce *Nature* in the following bilateral negotiation game in which the two creditors negotiate on how to divide the liquidation value. Formally, let $N \equiv \{k, l\}$ and $(c, E) \in \mathcal{B}^N$ with $c_k \leq c_l$.

$\bar{\boldsymbol{G}}(\boldsymbol{c}, \boldsymbol{E})$ **:** First, *Nature* selects one of the two creditors as "initiator", say creditor $i \in \{k, l\}$, who chooses a perspective $u \in \{gain, loss\}$.

If $u = gain$, then the other creditor $(-i)$ as "responder" chooses $q \in \{c_k, E\}$ and proposes a division $D^q = \{a, b\}$ such that $a, b \in \mathbb{R}_+$ and $a + b = q$. Creditor i then picks $x_i \in D^q$ as her award, and creditor $-i$ receives the remainder, i.e., $E - x_i$.

If $u = loss$, then creditor $-i$ chooses $q \in \{c_k, c_k + c_l - E\}$ and proposes a division $D^q = \{a, b\}$ such that $a, b \in \mathbb{R}_+$ and $a + b = q$. Creditor i then picks $x_i \in D^q$. Finally, creditor $-i$ gets the remainder $(q - x_i)$ as her loss (namely, her award is $c_{-i} - (q - x_i)$) and creditor i receives the remainder, i.e., $E - c_{-i} + (q - x_i)$.

3.2 Our mechanism

We now consider, for the general case of $n \geq 3$ agents, the following three-stage extensive form game G of which $\bar{G}$ is the final stage. Its tree is depicted in Figure 1. As we shall show, G strategically justifies the Talmud rule.

Stage 1: Each creditor $j \in N$ announces an awards vector y^j and a permutation $\pi^j : N \to N$. For ease of exposition, assume $N = \{1, 2, \ldots, n\}$ and $c_1 \leq c_2 \leq \cdots \leq c_n$. Let π be the

[11] The big creditor receives the amount obtained by subtracting her loss from her claim.

composition of these permutations according to the order of agents' labels, i.e., $\pi \equiv \pi^1 \circ \cdots \circ \pi^n$ and let $\pi(1) = k \in N$ be the **coordinator**.[12] If each creditor $j \in N \setminus \{k\}$ announces the same awards vector, say y^*, then the proposal y is y^*; otherwise, $y = y^k$.

Stage 2: The coordinator decides either to Accept y (which we refer as taking action A) in which case y is the outcome, or Reject y. In the case of a rejection, the coordinator takes a creditor, say creditor $l \in N \setminus \{k\}$ (which we refer as taking action (R, l)), to negotiate the awards of her and her chosen creditor in the next stage. Each creditor $j \in N \setminus \{k, l\}$ receives y_j.

Stage 3: Creditors l and k play the game $\bar{G}((c_k, c_l), y_k + y_l)$.

3.3 Existence

Our first result is that, for each problem, the awards vector provided by the Talmud rule can be obtained as a Nash Equilibrium (NE) of game G.

Theorem 1. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{B}^N$. There exists a Nash Equilibrium of $G(c, E)$ with outcome $T(c, E)$.

The following lemma, which is independently interesting, is necessary to prove the theorem.

Lemma 1. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{B}^N$. Suppose that $y \in X(c, E)$ is the proposal in Stage 1 of $G(c, E)$, and creditor $k \in N$ is the coordinator. For each $l \in N \setminus \{k\}$, if creditor k takes (R, l) in Stage 2 of $G(c, E)$, then $\left(y_{N\setminus\{k,l\}}, T\left((c_k, c_l), y_k + y_l\right)\right)$ is a Subgame Perfect Equilibrium (SPE) outcome of the subgame $\bar{G}\left((c_k, c_l), y_k + y_l\right)$.

Proof. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{B}^N$. Let $y \in X(c, E)$ be the proposal in Stage 1 of $G(c, E)$ and creditor $k \in N$ be the coordinator. Suppose that creditor k takes creditor $l \in N \setminus \{k\}$ to Stage 3. Without loss of generality, let $c_k \leq c_l$. Let $\eta^{gain} \equiv \min\{c_k, y_k + y_l\}$ and $\eta^{loss} \equiv \min\{c_k, c_k + c_l - y_k - y_l\}$. We show that the following strategy profile constitutes an SPE

[12]The permutation mechanism has been considered by Serrano and Vohra (2002) and Thomson (2005), among others. Our results hold even when a different order of the compositions is considered.

of $\bar{G}\left((c_k, c_l), y_k + y_l\right)$ with outcome $\left(y_{N\setminus\{k,l\}}, T\left((c_k, c_l), y_k + y_l\right)\right)$. Formally, let $i \in \{k, l\}$ and $j \equiv \{k, l\} \setminus \{i\}$.

• ***i* is chosen as the initiator.** If $y_k + y_l \le c_j$ then she chooses $\bar{u}^i = gain$; otherwise, she chooses $\bar{u}^i = loss$. Moreover, given a chosen aspect u and the responder's (creditor j) proposal (q, D^q), she chooses $\max D^q$ if $u = gain$ and $\min D^q$ otherwise.

• ***i* is chosen as the responder.** She proposes $(\bar{q}, \bar{D}^{\bar{q}}) = (\eta^u, \{\frac{\eta^u}{2}, \frac{\eta^u}{2}\})$, for each $u \in \{gain, loss\}$.

We denote the above strategy by

$$\bar{\sigma} = (\bar{\sigma}_k, \bar{\sigma}_l).$$

By definition of the game Ω_T, each creditor $h \in N\setminus\{k, l\}$ receives y_h. It is not difficult to see that the above strategy guarantees creditors k and l receive the components of $T\left((c_k, c_l), y_k + y_l\right)$. We next show $\bar{\sigma}$ is an SPE of $\bar{G}\left((c_k, c_l), y_k + y_l\right)$.

To do so, we note first that the last part of the initiator's strategy (namely, picking $\max D^q$ if $u = gain$ and $\min D^q$ otherwise) is a best response to the responder's proposal (q, D^q).

We next show that the responder's strategy is also a best response. Let $i \in \{k, l\}$ and $j \equiv \{k, l\} \setminus \{i\}$. We consider three cases.

Case 1: $y_k + y_l \le c_k$. Suppose that creditors i and j are the initiator and the responder, respectively. In this case, the initiator proposes $\bar{u}^i = gain$ and the responder selects $(\bar{q}, D^{\bar{q}}) = (\eta^{gain}, \{\frac{\eta^{gain}}{2}, \frac{\eta^{gain}}{2}\}) = (y_k + y_l, \{\frac{y_k+y_l}{2}, \frac{y_k+y_l}{2}\})$, which grants her $\frac{y_k+y_l}{2}$. Suppose that the responder deviates to $(q, D^q) \neq (\bar{q}, D^{\bar{q}})$. Then, as $\min D^q + \max D^q = q$, she would end up with $\min D^q + ((y_k + y_l) - q) \le \frac{q}{2} + (y_k + y_l) - q = (y_k + y_l) - \frac{q}{2} \le \frac{y_k+y_l}{2}$, which implies that she would not be better off.[13]

Case 2: $c_k < y_k + y_l \le c_l$. Suppose first that l is the initiator. Then, $\bar{u}^l = loss$ and the responder (k) selects $(\bar{q}, D^{\bar{q}}) = (\eta^{loss}, \{\frac{\eta^{loss}}{2}, \frac{\eta^{loss}}{2}\}) = (c_k, \{\frac{c_k}{2}, \frac{c_k}{2}\})$, which grants her $\frac{c_k}{2}$.

[13] As $y_k + y_l \le c_k$, it follows that $y_k + y_l \le q$.

Suppose that k deviates to propose $(q, D^q) \neq (\bar{q}, D^{\bar{q}})$. Then, as $\min D^q + \max D^q = q$, she would end up with $c_k - \max D^q \leq c_k - \frac{q}{2} \leq \frac{c_k}{2}$, which implies that she is not better off.[14]

Next, suppose that k is the initiator. Then, $\bar{u}^k = gain$ and the responder (l) selects $(\bar{q}, D^{\bar{q}}) = (\eta^{gain}, \{\frac{\eta^{gain}}{2}, \frac{\eta^{gain}}{2}\}) = (c_k, \{\frac{c_k}{2}, \frac{c_k}{2}\})$, which grants her $\frac{c_k}{2} + (y_k + y_l - c_k)$. Suppose that l deviates to propose $(q, D^q) \neq (\bar{q}, D^{\bar{q}})$. Then, as $\min D^q + \max D^q = q$, she would end up with $\min D^q + ((y_k + y_l) - q) \leq \frac{q}{2} + (y_k + y_l) - q \leq \frac{c_k}{2} + (y_k + y_l - c_k)$, which implies that she is not better off.

Case 3: $y_k + y_l > c_l$**.** Suppose that creditors i and j are the initiator and the responder, respectively. In this case, the initiator proposes $\bar{u}^i = loss$ and the responder selects $(\bar{q}, D^{\bar{q}}) = (\eta^{loss}, \{\frac{\eta^{loss}}{2}, \frac{\eta^{loss}}{2}\})$, where $\eta^{loss} = c_k + c_l - (y_k + y_l)$, which grants her $c_j - \frac{c_k + c_l - (y_k + y_l)}{2}$. Suppose that the responder deviates to $(q, D^q) \neq (\bar{q}, D^{\bar{q}})$. Then, as $\min D^q + \max D^q = q$, she would end up with $c_j - \max D^q \leq c_j - \frac{q}{2} \leq c_j - \frac{c_k + c_l - (y_k + y_l)}{2}$, which implies that she is not better off.[15]

Finally, we show that the first part of the initiator's strategy is also a best response. We distinguish the same three cases as above.

Case I: $y_k + y_l \leq c_k$**.** Suppose that creditors i and j are the initiator and the responder, respectively. In this case, the initiator proposes $\bar{u}^i = gain$, and the responder selects $(\bar{q}, D^{\bar{q}}) = (\eta^{gain}, \{\frac{\eta^{gain}}{2}, \frac{\eta^{gain}}{2}\}) = (y_k + y_l, \{\frac{y_k + y_l}{2}, \frac{y_k + y_l}{2}\})$, which grants the initiator $\frac{y_k + y_l}{2}$. Suppose that the initiator deviates and chooses $u = loss$. Then, as the responder proposes $(\eta^{loss}, \{\frac{\eta^{loss}}{2}, \frac{\eta^{loss}}{2}\}) = \left(c_k, \{\frac{c_k}{2}, \frac{c_k}{2}\}\right)$, the initiator picks $\frac{c_k}{2}$, which grants her $\frac{c_k}{2} - c_j + y_k + y_l \leq \frac{y_k + y_l}{2}$. This implies that she is not better off.

Case II: $c_k < y_k + y_l \leq c_l$**.** Suppose first that l is the initiator. Then, $\bar{u}^l = loss$ and the responder (k) selects $(\bar{q}, D^{\bar{q}}) = (\eta^{loss}, \{\frac{\eta^{loss}}{2}, \frac{\eta^{loss}}{2}\}) = (c_k, \{\frac{c_k}{2}, \frac{c_k}{2}\})$, which grants l (the initiator) $y_k + y_l - \frac{c_k}{2}$. Suppose that the initiator deviates to choose $u^l = gain$. Then, as the responder would propose $(\eta^{gain}, \{\frac{\eta^{gain}}{2}, \frac{\eta^{gain}}{2}\}) = (c_k, \{\frac{c_k}{2}, \frac{c_k}{2}\})$, the initiator would pick $\frac{c_k}{2}$, which would grant her $\frac{c_k}{2} \leq y_k + y_l - \frac{c_k}{2}$. This implies that she would not be better off.

Next, suppose first that k is the initiator. Then, $\bar{u}^k = gain$ and the responder (l) se-

[14] As $c_k < y_k + y_l \leq c_l$, it follows that $c_k \leq c_k + c_l - (y_k + y_l) \leq c_l$ and, therefore, $q \geq c_k$.

[15] As $y_k + y_l > c_l$, it follows that $c_l + c_k - (y_k + y_l) < c_k$, and, therefore, $q \geq c_l + c_k - (y_k + y_l)$.

lects $(\bar{q}, D^{\bar{q}}) = (\eta^{gain}, \{\frac{\eta^{gain}}{2}, \frac{\eta^{gain}}{2}\}) = (c_k, \{\frac{c_k}{2}, \frac{c_k}{2}\})$, which grants k (the initiator) $\frac{c_k}{2}$. Suppose that the initiator deviates to choose $u^k = loss$. Then, as the responder would propose $(\eta^{loss}, \{\frac{\eta^{loss}}{2}, \frac{\eta^{loss}}{2}\}) = (c_k, \{\frac{c_k}{2}, \frac{c_k}{2}\})$, the initiator would pick $\frac{c_k}{2}$, which would grant her $y_k + y_l - (c_l - \frac{c_k}{2}) \leq \frac{c_k}{2}$. This implies that she would not be better off.

Case III: $y_k + y_l > c_l$**.** Suppose that creditors i and j are the initiator and the responder, respectively. In this case, the initiator proposes $\bar{u}^i = loss$ and the responder selects $(\bar{q}, D^{\bar{q}}) = (\eta^{loss}, \{\frac{\eta^{loss}}{2}, \frac{\eta^{loss}}{2}\})$, where $\eta^{loss} = c_k + c_l - (y_k + y_l)$. This would grant the initiator $c_i - \frac{c_k + c_l - (y_k + y_l)}{2}$. Suppose that the initiator deviates to choose $u = gain$. Then, as the responder would propose $(\eta^{gain}, \{\frac{\eta^{gain}}{2}, \frac{\eta^{gain}}{2}\}) = (c_k, \{\frac{c_k}{2}, \frac{c_k}{2}\})$, the initiator would pick $\frac{c_k}{2}$, which would grant her $\frac{c_k}{2} \leq c_i - \frac{c_k + c_l - (y_k + y_l)}{2}$. This implies that she would not be better off. *Q.E.D.*

With the help of Lemma 1, we are now ready to prove Theorem 1.

Proof of Theorem 1. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{B}^N$. Without loss of generality, let $N \equiv \{1, \cdots, n\}$ and $c_1 \leq \cdots \leq c_n$. We show that the following strategy profile $\sigma^* = (\sigma_1^*, \cdots, \sigma_n^*)$ constitutes a NE of $G(c, E)$ with outcome $T(c, E)$.

Stage 1: Each creditor $i \in N$ proposes $(y^{i*}, \pi^{i*}) = (T(c, E), \pi^{Id})$, where $\pi^{Id} : N \to N$ is the identity permutation (i.e., for each $i \in N$, $\pi^{Id}(i) = i$).

Stage 2: Suppose that $\pi(1) = i$ and y is the proposal in Stage 1. Creditor i accepts y (takes action A) if $y_i \geq \max_{k \in N \setminus \{i\}} T_i\left((c_i, c_k), y_i + y_k\right)$; otherwise, i rejects y and chooses one creditor, say creditor $j \in N \setminus \{i\}$ (i takes action (R, j)), where $j \in \arg\max_{k \in N \setminus \{i\}} T_i\left((c_i, c_k), y_i + y_k\right)$.

Stage 3: Suppose that y is the proposal in Stage 1, and creditor i is the coordinator and chooses creditor $j \neq i$ in Stage 2. Creditors i and j adopt the strategy profile $\bar{\sigma}$ defined in the proof of Lemma 1 with $k = i$ and $l = j$ ($k = j$ and $l = i$) if $i \leq j$ ($j < i$).

It is not difficult to see that the strategy profile σ^* guarantees that the game ends with the outcome $T(c, E)$. We now show that σ^* is an SPE of $G(c, E)$, which implies that σ^* is a NE of $G(c, E)$. Suppose that y is the proposal in Stage 1, and creditor i is the coordinator and chooses creditor $j \neq i$ in Stage 2. By the proof of Lemma 1, it is clear that σ^* is an SPE of the subgame $\bar{G}(\left(c_i, c_j\right), y_i + y_j))$ and the outcome is $\left(y_{N \setminus \{i,j\}}, T\left((c_i, c_j), y_i + y_j\right)\right)$. By subgame perfection, if $y_i \geq \max_{k \in N \setminus \{i\}} T_i\left((c_i, c_k), y_i + y_k\right)$, creditor i takes action A; otherwise, she takes action (R, j), where $j \in \arg\max_{k \in N \setminus \{i\}} T_i\left((c_i, c_k), y_i + y_k\right)$. We now claim that no creditor $i \in N$ is better off by deviating from announcing $(T(c, E), \pi^{Id})$. Note that following σ^*, creditor 1 is the coordinator and $T(c, E)$ is the proposal in Stage 1. Suppose that creditor i deviates to announce (y^i, π^i). We consider two cases.

Case 1: Creditor i is the coordinator. As for each $k \in N \setminus \{i\}$, $y^k = T(c, E)$, then $y = T(c, E)$. By *bilateral consistency* of the Talmud rule, for each $j \in N \setminus \{i\}$, $T_i(c, E) = T_i\left((c_i, c_j), T_i(c, E) + T_j(c, E)\right)$. By subgame perfection, creditor i receives $T_i(c, E)$, which implies that she is not better off.

Case 2: Creditor $k \in N \setminus \{i\}$ is the coordinator. Suppose that $y^i = T(c, E)$. Thus, $y = T(c, E)$. By *bilateral consistency* of the Talmud rule and subgame perfection, creditor i receives $T_i(c, E)$, which implies that she is not better off. Suppose now that $y^i \neq T(c, E)$. Then, $y = y^k = T(c, E)$. By *bilateral consistency* of the Talmud rule and subgame perfection, creditor i receives $T_i(c, E)$, which implies that she is not better off. *Q.E.D.*

3.4 Uniqueness

Our second result states that, for each problem, the awards vector provided by the Talmud rule is the only Nash equilibrium outcome of our game.

Theorem 2. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{B}^N$. $T(c, E)$ is the unique Nash Equilibrium outcome of the game $G(c, E)$.

The following lemmata, which are independently interesting, are used to prove the theorem.

Lemma 2. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{B}^N$. Suppose that σ is a NE of $G(c, E)$ with outcome z^σ, and y^σ is the proposal following σ. Then, for each $i, j \in N$ with $i \neq j$, $z_i^\sigma \geq T_i\left((c_i, c_j), y_i^\sigma + y_j^\sigma\right)$.

Proof. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{B}^N$. Suppose that σ is a NE of $G(c, E)$ with outcome z^σ, and y^σ is the proposal following σ. We show that for each $i, j \in N$ with $i \neq j$, $z_i^\sigma \geq T_i\left((c_i, c_j), y_i^\sigma + y_j^\sigma\right)$. Suppose, by contradiction, that there are $i, j \in N$ with $i \neq j$ such that $z_i^\sigma < T_i\left((c_i, c_j), y_i^\sigma + y_j^\sigma\right)$. We show that creditor i is better off by deviating to the following strategy σ_i'.

Stage 1: Creditor i announces $(y^{\sigma'_i}, \pi^{\sigma'_i})$ such that $y^{\sigma'_i} = y^{\sigma}$ and $\pi^{\sigma'}(1) \equiv \left(\pi^{\sigma_1} \circ \cdots \circ \pi^{\sigma_{i-1}} \circ \pi^{\sigma'_i} \circ \pi^{\sigma_{i+1}} \circ \cdots\right.$ i.

Stage 2: If creditor i is the coordinator and the proposal is y^{σ}, then she takes (R, j); otherwise, she follows σ_i.

Stage 3: Suppose that y is the proposal. If creditors i and j are involved in Stage 3, then creditor i adopts the strategy $\bar{\sigma}_i$ defined in the proof of Lemma 1 with $k = i$ $(l = i)$ if $i \leq j$ $(j < i)$; otherwise, she follows σ_i.

Let σ' be the strategy profile after creditor i's deviation. Namely, for each $l \in N \backslash \{i\}$, $\sigma'_l = \sigma_l$. Let $z^{\sigma'}$ be the outcome of $G(c, E)$ by following σ'. Note that by following σ', the proposal $y^{\sigma'}$ in Stage 1 is still y^{σ}.[16] As $y^{\sigma'} = y^{\sigma}$, by following σ', each creditor $l \in N \setminus \{i, j\}$ receives $z^{\sigma'}_l = y^{\sigma}_l$, and creditors i and j play $\bar{G}\left((c_i, c_j), y^{\sigma}_i + y^{\sigma}_j\right)$ in Stage 3. By the game rule, $z^{\sigma'}_i + z^{\sigma'}_j = y^{\sigma}_i + y^{\sigma}_j$. As creditor i adopts $\bar{\sigma}_i$, by the proof of Lemma 1, $z^{\sigma'}_j \leq T_j\left((c_i, c_j), y^{\sigma}_i + y^{\sigma}_j\right)$. As $T_i\left((c_i, c_j), y^{\sigma}_i + y^{\sigma}_j\right) + T_j\left((c_i, c_j), y^{\sigma}_i + y^{\sigma}_j\right) = y^{\sigma}_i + y^{\sigma}_j$, it follows that $z^{\sigma'}_i \geq T_i\left((c_i, c_j), y^{\sigma}_i + y^{\sigma}_j\right) > z^{\sigma}_i$, which implies that creditor i is better off. This violates that σ is a NE of $G(c, E)$. $Q.E.D.$

Lemma 3. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{B}^N$. Suppose that σ is a NE of $G(c, E)$ with outcome z^{σ} and y^{σ} is the proposal following σ. Then, for each $i \in N$, $z^{\sigma}_i \geq y^{\sigma}_i$.

Proof. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{B}^N$. Suppose that σ is a NE of $G(c, E)$ with outcome z^{σ} and y^{σ} is the proposal following σ. We show that for each $i \in N$, $z^{\sigma}_i \geq y^{\sigma}_i$. Suppose, by contradiction, that there is $i \in N$ such that $z^{\sigma}_i < y^{\sigma}_i$. We claim that creditor i is better off by deviating to the following strategy σ'_i.

Stage 1: Creditor i announces $(y^{\sigma'_i}, \pi^{\sigma'_i})$ such that $y^{\sigma'_i} = y^{\sigma}$ and $\pi^{\sigma'}(1) \equiv \left(\pi^{\sigma_1} \circ \cdots \circ \pi^{\sigma_{i-1}} \circ \pi^{\sigma'_i} \circ \pi^{\sigma_{i+1}} \circ \cdots\right.$ i.

Stage 2: If creditor i is the coordinator and $y = y^{\sigma}$, then she takes action A (accepts y); otherwise, she follows σ_i.

[16] To see this, if $\pi^{\sigma}(1) = i$, then obviously $y^{\sigma'} = y^{\sigma}$. If $\pi^{\sigma}(1) = k \neq i$, then by the game rule, either $y^{\sigma} = y^{\sigma_i} \neq y^{\sigma_k}$ or $y^{\sigma} = y^{\sigma_k}$. In the former case, since for each $l \in N \setminus \{k\}$, $y^{\sigma'_l} = y^{\sigma} \neq y^{\sigma_k} = y^{\sigma'_k}$, then $y^{\sigma'} = y^{\sigma'_i} = y^{\sigma}$. In the latter case, if there is $l \in N \setminus \{k\}$ such that $y^{\sigma_l} \neq y^{\sigma_k}$, then $y^{\sigma'} = y^{\sigma'_i} = y^{\sigma}$; otherwise, $y^{\sigma'} = y^{\sigma_k} = y^{\sigma}$. Thus, $y^{\sigma'} = y^{\sigma}$.

Stage 3: Creditor i follows σ_i.

Let σ' be the strategy profile of $G(c, E)$ after creditor i's deviation. Namely, for each $j \in N \setminus \{i\}$, $\sigma'_j = \sigma_j$. After deviation, creditor i is the coordinator and the proposal is still y^σ.[17] Thus, following σ'_i, creditor i receives $y_i^\sigma > z_i^\sigma$, which implies that creditor i is better off. This violates that σ is a NE of $G(c, E)$. *Q.E.D.*

With the help of the previous lemmata, we are now ready to prove Theorem 2.

Proof of Theorem 2. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{B}^N$. Without loss of generality, let $N \equiv \{1, \cdots, n\}$ and $c_1 \leq \cdots \leq c_n$. Suppose that $\sigma = (\sigma_1, \cdots, \sigma_n)$ is a NE of $G(c, E)$ whose outcome is denoted by $z^\sigma = (z_1^\sigma, \cdots, z_n^\sigma)$. For each $i \in N$, let

$$\delta_i^T \equiv \max\left\{ y_i^\sigma, \max_{j \in N \setminus \{i\}} T_i((c_i, c_j), y_i^\sigma + y_j^\sigma) \right\}.$$

By Lemma 2 and Lemma 3, we conclude that, for each $i \in N$, $z_i^\sigma \geq \delta_i^T$. We next claim that, for each $i, j \in N$ with $i \neq j$,

$$z_i^\sigma = y_i^\sigma = T_i((c_i, c_j), y_i^\sigma + y_j^\sigma).$$

To show the first equality, note that, for each $i \in N$, $z_i^\sigma \geq \delta_i^T$. It follows that, for each $i \in N$, $z_i^\sigma \geq y_i^\sigma$. As $\sum_{h \in N} z_h^\sigma = E = \sum_{h \in N} y_h^\sigma$, then, for each $i \in N$, $z_i^\sigma = y_i^\sigma$.

To show the second equality, suppose, by contradiction, that there exists a pair of agents $i, j \in N$ with $i \neq j$ such that $z_i^\sigma > T_i((c_i, c_j), y_i^\sigma + y_j^\sigma)$. As, for each $h \in N$, $z_h^\sigma = y_h^\sigma$, then $y_i^\sigma > T_i((c_i, c_j), y_i^\sigma + y_j^\sigma)$. Thus, $z_j^\sigma = y_j^\sigma < T_j((c_i, c_j), y_i^\sigma + y_j^\sigma)$, which contradicts the fact that for each $h \in N \setminus \{j\}$, $z_j^\sigma \geq T_j((c_h, c_j), y_h^\sigma + y_j^\sigma)$. It follows that for each $i, j \in N$ with $i \neq j$, $z_i^\sigma = y_i^\sigma = T_i((c_i, c_j), y_i^\sigma + y_j^\sigma)$. By *converse consistency* of the Talmud rule, $z^\sigma = T(c, E)$. *Q.E.D.*

[17] To see this, suppose that by following σ, creditor $k \in N$ is the coordinator. If for each $l, h \in N$, $y^{\sigma_l} = y^{\sigma_h}$, then the original proposal (y^σ) coincides with the new proposal ($y^{\sigma'} = y^{\sigma'_i} = y^\sigma$). If for each $l, h \in N \setminus \{k\}$, $y^{\sigma_l} = y^{\sigma_h}$ and $y^{\sigma_k} \neq y^{\sigma_i}$, then the original proposal (y^σ) still coincides with the new proposal ($y^{\sigma'} = y^{\sigma'_i} = y^\sigma$). If for each $l, h \in N \setminus \{i\}$, $y^{\sigma_l} = y^{\sigma_h}$ and $y^{\sigma_k} \neq y^{\sigma_i}$, then the original proposal (y^σ) again coincides with the new proposal ($y^{\sigma'} = y^{\sigma'_i} = y^\sigma$). If for some $l, h \in N \setminus \{i, k\}$, $y^{\sigma_l} \neq y^{\sigma_h}$, then the original proposal (y^σ) coincides with the new proposal ($y^{\sigma'} = y^{\sigma'_i} = y^\sigma$).

4 Concluding remarks

We have introduced in this paper a game (G) that strategically justifies the Talmud rule, one of the classical rules to solve bankruptcy problems. As G exploits *bilateral consistency* and *converse consistency*, the bilateral negotiation game ($\bar{G}$) plays an important role in our analysis. The design of $\bar{G}$ exploits the *average truncated claim lower bounds on awards* and the *average truncated claim upper bounds on awards*. To see this, let $N \equiv \{k, l\} \in \mathcal{N}$ with $k \neq l$ and $(c, E) \in \mathcal{B}^N$ with $c_k \leq c_l$. As $|N| = 2$, the *average truncated claim lower bounds on awards* requires that each creditor $i \in N$ should receive at least $\frac{\min\{c_i, E\}}{2}$. To embed this lower bound requirement into $\bar{G}$, when the initiator chooses the *gain* perspective, the responder is allowed to pick a number q within $\{c_k, E\}$ and propose a division of q (namely, $D^q = \{a, b\}$ such that $a + b = q$). The number q is interpreted as the amount the responder would like to divide with the initiator. After the initiator picks a number x from D^q, the responder receives $q - x$ as well as $E - q$, i.e., $E - x$ overall. Note that the responder proposes the division but is the last creditor to pick. This is the so-called divide-and-choose mechanism.[18] Therefore, the responder will select $q = \min\{c_k, E\}$ and propose $D^q = \{\frac{q}{2}, \frac{q}{2}\}$. Analogously, in the case when the initiator chooses the *loss* perspective, it can be seen that $\bar{G}$ exploits the *average truncated claim upper bounds on awards*.

Our analysis closely relates to Tsay and Yeh (2019). The two papers exploit *bilateral consistency* and *converse consistency* of the Talmud rule and obtain (exact) strategic justifications of the rule. More precisely, the game proposed in this paper is based on the general game Ω_φ introduced by Tsay and Yeh (2019). We keep Stages 1 and 2 of Ω_φ unchanged, but replace Stage 3 of Ω_φ by $\bar{G}$. Stage 3 of Tsay and Yeh (2019) is based on Dagan's (1996) characterization of the Talmud rule, as it exploits the fact that the rule satisfies "equal treatment of equals", "invariance under claims truncation", and "minimal rights first". Our Stage 3 instead is based on Moreno-Ternero and Villar's (2004) characterization of the Talmud rule and thus exploits the two lower bounds conditions mentioned above. These different properties lead to

[18]See Brams and Taylor (1996) for a survey and Tsay and Yeh (2019) or Li and Ju (2016) for recent instances in this context.

different bilateral negotiations in Stage 3 of both games. In Tsay and Yeh's (2019) game, the responder picks one process between the "minimal awards first" process and the "minimal losses first" process rather than an amount within the liquidation value (the shortfall) and the smaller claim between two creditors in Stage 3 if the divider picks the gain aspect (the loss aspect).

Serrano (1995), Dagan et al. (1997), and Chang and Hu (2008) also offer strategic justifications of the Talmud rule. However, their results rely on exogenously given bankruptcy rules to solve bilateral negotiation games.[19] From our viewpoint, this design leaves some room for improvement as the purpose of the Nash program is to strategically justify cooperative solutions through non-cooperative procedures, and ideally, no cooperative solution should get involved in the details of non-cooperative procedures. Our result and Tsay and Yeh's (2019) result of the Talmud rule do not have such a design.

We stress that, somewhat in contrast with the literature (see, for instance, Serrano (2005)), we do not invoke any rule to solve bilateral negotiations. Instead, we introduce strategic interaction in bilateral negotiations.[20] Thus, the design of our game enhances our understanding of the non-cooperative features of the Talmud rule.

To conclude, the Talmud rule is a hybrid between equal awards and equal losses. It focusses on one or the other notion, depending on whether the endowment falls short or exceeds one half of the aggregate claim, using half-claims instead of claims. A natural generalization is obtained by considering any possible fraction (of the aggregate and individual claims). The resulting family of rules, known as the TAL-family, was introduced by Moreno-Ternero and Villar (2006a). In the taxation jargon, the rules within the TAL-family would yield two types of tax schedules: for tax revenues below a fraction θ of the aggregate income, the tax rate would be θ up to some income level, and zero afterwards. For tax revenues above such a fraction, the tax rate would be θ first and then one. A more general family can be obtained by allowing for other minimum and maximum tax rates, instead of always imposing zero and one for those

[19] Serrano (1995) mentions that the exogenous bankruptcy rule in his game can be obtained as the equilibrium outcome of a random dictator bargaining game. In this case, he obtains an expected strategic justification of the Talmud rule. In contrast, our result and Tsay and Yeh's (2019) result are exact strategic justifications of the Talmud rule.

[20] This feature is also shared by Tsay and Yeh (2019).

values (e.g., Moreno-Ternero (2011)). It is left for further research to explore whether the mechanism introduced in this paper could be extended to provide strategic justifications for such families of *generalized Talmud rules*.

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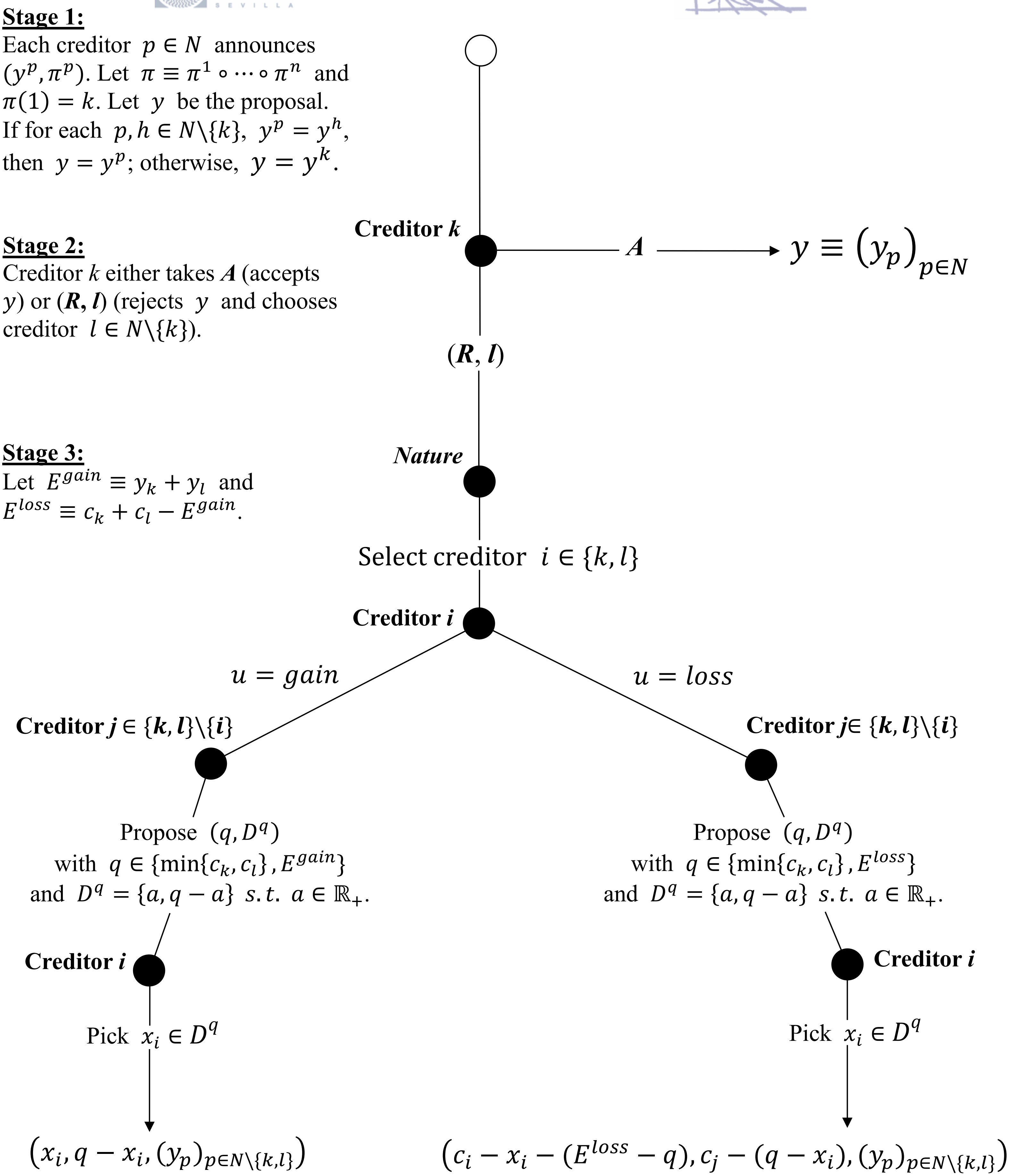
Stage 1:
Each creditor $p \in N$ announces (y^p, π^p). Let $\pi \equiv \pi^1 \circ \cdots \circ \pi^n$ and $\pi(1) = k$. Let y be the proposal. If for each $p, h \in N \backslash \{k\}$, $y^p = y^h$, then $y = y^p$; otherwise, $y = y^k$.
Stage 2:
Creditor k either takes ***A*** (accepts y) or $(\boldsymbol{R}, \boldsymbol{l})$ (rejects y and chooses creditor $l \in N \backslash \{k\}$).
Stage 3:
Let $E^{gain} \equiv y_k + y_l$ and $E^{loss} \equiv c_k + c_l - E^{gain}$.
Creditor *k*
A
$y \equiv (y_p)_{p \in N}$
$(\boldsymbol{R}, \boldsymbol{l})$
Nature
Select creditor $i \in \{k, l\}$
Creditor *i*
$u = gain$
$u = loss$
Creditor $\boldsymbol{j} \in \{\boldsymbol{k}, \boldsymbol{l}\} \backslash \{\boldsymbol{i}\}$
Creditor $\boldsymbol{j} \in \{\boldsymbol{k}, \boldsymbol{l}\} \backslash \{\boldsymbol{i}\}$
Propose (q, D^q) with $q \in \{\min\{c_k, c_l\}, E^{gain}\}$ and $D^q = \{a, q - a\}$ $s.t.$ $a \in \mathbb{R}_+$.
Propose (q, D^q) with $q \in \{\min\{c_k, c_l\}, E^{loss}\}$ and $D^q = \{a, q - a\}$ $s.t.$ $a \in \mathbb{R}_+$.
Creditor *i*
Creditor *i*
Pick $x_i \in D^q$
Pick $x_i \in D^q$
$(x_i, q - x_i, (y_p)_{p \in N \backslash \{k,l\}})$
$(c_i - x_i - (E^{loss} - q), c_j - (q - x_i), (y_p)_{p \in N \backslash \{k,l\}})$