Working papers series

WP ECON 21.02

Strategic justifications of the TAL-family of rules for bankruptcy problems

Juan D. Moreno-Ternero
Universidad Pablo de Olavide

Min-Hung Tsay
Department of Economics, National Chung Cheng University

Chun-Hsien Yeh
Institute of Economics, Academia Sinica; Chung-Hua Institute for Economic Research

Keywords: Nash program; bankruptcy problems; strategic justification; consistency; TAL-family of rules

JEL Classification: C71, C72, D63.

Department of Economics

Strategic justifications of the TAL-family of rules for bankruptcy problems*

Juan D. Moreno-Ternero† Min-Hung Tsay‡
Chun-Hsien Yeh§

February 6, 2021

Abstract

We follow the Nash program to provide strategic justifications of the TAL-family of rules for bankruptcy problems. The design of our game is inspired by an axiomatization of the TAL-family of rules exploiting the properties of consistency together with certain degrees of lower and upper bounds to all creditors. Bilateral negotiations of our game follow the spirit of those bounds. By means of consistency, we then extend the bilateral negotiations to an arbitrary number of creditors.

JEL Classification Numbers: C71; C72; D63.

Keywords: Nash program; bankruptcy problems; strategic justification; consistency; TAL-family of rules

*We thank Youngsub Chun and an anonymous referee for helpful comments and suggestions. Moreno-Ternero acknowledges financial support from the Spanish Ministry of Economy and Competitiveness (ECO2017-83069-P). Tsay acknowledges financial support from the grant of Ministry of Science and Technology, Taiwan (MOST-109-2628-H-194-001-MY3). Yeh acknowledges financial support from Career Development Award of Academia Sinica, Taiwan (AS-99-CDA-H01) and the grant of Ministry of Science and Technology, Taiwan (MOST-108-2410-H-001-027-MY3). We are responsible for any remaining deficiencies.

†Department of Economics, Universidad Pablo de Olavide, Seville, Spain.

‡Department of Economics, National Chung Cheng University, Chiayi, Taiwan.

§Corresponding author: Institute of Economics, Academia Sinica, Taipei, Taiwan; Chung-hua Institution for Economic Research, Taipei, Taiwan. E-mail: chyeh@econ.sinica.edu.tw

1 Introduction

How should we divide when there is not enough? Although this question can be traced back to ancient sources, the scientific literature dealing with it was initiated by O'Neill (1982). For an excellent, fully comprehensive, and recent survey on that fast-expanding literature, the reader is referred to Thomson (2019).

An obvious way to solve these so-called *bankruptcy problems*, or *claims problems*, is to implement the Aristotelian view that awards should be proportional to claims. But the literature went far beyond the basic principle of proportionality and many alternative rules have been suggested. The most salient are the so-called *Constrained Equal Awards* (CEA) rule (Aumann and Maschler, 1985; Dagan, 1996), *Constrained Equal Losses* (CEL) rule (Aumann and Maschler, 1985), and *Talmud* rule (Aumann and Maschler, 1985). Moreno-Ternero and Villar (2006) introduced a family of rules, called the *TAL-family*, which generalizes the latter and encompasses a wide variety of rules ranging, precisely, from the CEA rule to the CEL rule. More precisely, the family is defined by means of a parameter $\theta \in [0, 1]$ that captures a certain degree of the distributive power of the rule. If the amount to divide is below θ times the aggregate claim, then the corresponding rule will guarantee nobody gets more than θ times her claim. If, otherwise, the amount to divide is above θ times the aggregate claim, then the corresponding rule will guarantee nobody gets less than θ times her claim. Given θ, the corresponding rule is referred to as the T^θ rule. T^0 rule is the CEL rule; $T^{\frac{1}{2}}$ rule is the Talmud rule; T^1 rule is the CEA rule.

The TAL-family has mostly been explored from an axiomatic viewpoint (e.g., Moreno-Ternero and Villar, 2006; Moreno-Ternero, 2011a; Thomson, 2019). It has attracted less attention from a strategic viewpoint. Our goal in this paper is to deepen our understanding of the TAL-family of rules by establishing solid strategic justifications of this family of rules. To do so, we apply the *Nash program*, and construct a game with respect to the parameter θ in which there is one and only one equilibrium allocation, and this allocation corresponds to the one dictated by the corresponding T^θ rule.[1]

[1]The *Nash program* deals with studying strategic justifications of cooperative solutions, and it was initiated by Nash (1953) himself. Serrano (2020) refers to it as "bargaining design". It is a successful research field, which has flourished during almost seven decades by now. Recent instances are Hu et al. (2012, 2018), Tsay and Yeh (2019), and Moreno-Ternero et al. (2020). For excellent surveys on the related literature, readers are referred

The *Nash program* has been used for bankruptcy problems before. For instance, Dagan et al. (1997) introduced a game that captures a noncooperative dimension of the *bilateral consistency* property, defined below. Their game, together with a bilateral principle, yields the corresponding bilaterally consistent bankruptcy rule as a result of a unique outcome of Nash equilibria. Tsay and Yeh (2019) introduced games in which bilateral negotiations are resolved by non-cooperative bargaining procedures and show that these games strategically justify the CEA rule, the CEL rule, the proportional rule, and the Talmud rule. Moreno-Ternero et al. (2020) modified one of the previous games to provide an alternative strategic justification of the Talmud rule.[2]

Our starting point is the axiomatic characterization of the T^θ rule by Moreno-Ternero and Villar (2006). The authors remark (and indeed can be verified) that the axiomatization of the T^θ rule can be obtained by considering the following three properties. First is *bilateral consistency*, which says that if the rule chooses an awards vector for a bankruptcy problem, then for the associated "two-creditor reduced problem" derived by imagining that all the other creditors leave with their components of the vector, and reassessing the situation from the viewpoint of the two remaining creditors, it chooses the corresponding awards of the vector to that subgroup. The other two properties are related to lower and upper bounds to all creditors. Lower bounds protect those creditors with relatively small claims from receiving too little; whereas upper bounds protect those creditors with relatively big claims from receiving too little.[3] By specifying different degrees of the protection offered to small and large creditors, through the parameter θ, the TAL-family is generated. *Lower bound of degree* θ says that if each creditor receives at least either a fraction θ of her claim or an equal share of the endowment. *Upper bound of degree* θ says that each creditor receives at most either a fraction θ of her claim or an equal share of the shortfall (the difference between the sum of the claims of the creditors and the endowment).[4]

---

to Serrano (2005, 2020).

[2]Aumann and Maschler (1985) introduced an orderly step-by-step procedure, according to which the creditors empower each other, leading towards the Talmud rule. Moreno-Ternero (2011b) generalizes it to lead to the whole TAL-family of rules. However, there is no strategic interaction among creditors in those designs.

[3]Bounds have a long tradition of use in fair allocation (e.g., Thomson, 2011). In bankruptcy problems, they were introduced by Moreno-Ternero and Villar (2004) and explored, among others, by Dominguez and Thomson (2006).

[4]When $\theta = \frac{1}{2}$, the properties become, respectively, *average truncated claim lower*

Krishna and Serrano (1996) suggest that the properties of a rule should play important roles to establish a strategic justification of the rule.[5] We follow the same tradition here. As a matter of fact, the main design of our game (which will provide strategic justifications of the TAL-family of rules) is in line with the above-mentioned axiomatization of the T^θ rule.

More precisely, for each $\theta \in [0, 1]$, we consider the following corresponding three-stage extensive form game, which is based on the properties introduced above.

***Stage 1*:** Each creditor announces an awards vector and a permutation. The composition of permutations selects the "coordinator". If all creditors, except possibly for the coordinator, announce the same awards vector, it is the "proposal", and then the game proceeds to the next stage; otherwise, the awards vector announced by the coordinator is the outcome.

***Stage 2*:** The coordinator picks another creditor to negotiate their awards in the next stage. All the others take their awards, as specified in the proposal, and leave the game.

***Stage 3*:** *Nature* chooses one of the two remaining creditors. We call "initiator" to the chosen creditor. The initiator adopts one of the two possible perspectives for bankruptcy problems: *gain* or *loss*.[6]

In the case of gain, the other creditor chooses either the remaining endowment or 2θ times the small claim (namely, 2θ times the minimum of the claims of the two creditors), and proposes to the initiator two numbers whose sum is equal to the chosen amount. The initiator then picks one of the two numbers as her award, and the other creditor receives the residual.

In the case of loss, the other creditor chooses either the remaining shortfall or $2(1-\theta)$ times the small claim, and proposes to the initiator two numbers whose sum is equal to the chosen amount. The initiator then picks one of the two numbers. The other creditor takes the remaining number as her loss and receives the amount obtained by subtracting her loss from her claim. Finally, the initiator receives the residual.

---

bounds, and *average truncated claim upper bounds*, which, together with bilateral consistency, characterize the Talmud rule (e.g., Moreno-Ternero and Villar, 2004).

[5]The use of axiomatizations of solutions in the *Nash program* is discussed at great length by Serrano (2005).

[6]Aumann and Maschler (1985) mention that a bankruptcy problem can be handled from either the gain perspective or the loss perspective. The gain perspective focuses on dividing the endowment, and the loss perspective on dividing the shortfall.

Stage 1 adopts a permutation mechanism, which is common and has a long tradition in the literature.[7] We consider this mechanism because it is endogenous and offers each creditor an equal force to influence the selection of the coordinator. This levels the playing field among creditors.

Stage 2 exploits *bilateral consistency*, moving the negotiation to a bilateral setting, which is a basic negotiation situation that takes place at Stage 3. Now, due to the nature of the game, the bilateral negotiation at Stage 3 plays the crucial role in our analysis. Stage 3 of our game closely relates to Stage 3 of Tsay and Yeh (2019) and Moreno-Ternero et al. (2020). More precisely, Stage 3 of the game in Tsay and Yeh (2019) is based on Dagan's (1996) axiomatization of the Talmud rule as it exploits two operational properties known as "invariance under claims truncation" and "minimal rights first". Stage 3 of the game in Moreno-Ternero et al. (2020) is based on Moreno-Ternero and Villar's (2004) axiomatization of the Talmud rule as it exploits "average truncated claim lower bounds on awards" and "average truncated claim upper bounds on awards". Our Stage 3 is based instead on Moreno-Ternero and Villar's (2006) axiomatization of the T^θ rule and exploits *lower bound of degree of* θ and *upper bound of degree of* θ, which are natural extensions of *average truncated claim lower bounds on awards* and *average truncated claim upper bounds on awards*.

A remarkable aspect of our game is that it does not invoke any rule to solve bilateral negotiations. Instead, we introduce strategic interaction in bilateral negotiations.[8] This is in contrast with previous strategic justifications of the rules for bankruptcy problems, which rely on exogenously given bankruptcy rules to solve bilateral negotiations.[9] From our viewpoint, that kind of design leaves room for improvement as the purpose of the *Nash program* is to strategically justify cooperative solutions through non-cooperative procedures, and, ideally, no cooperative solution should get involved in the details of non-cooperative procedures.

[7] See, for instance, Serrano and Vohra (2002), Thomson (2005), Tsay and Yeh (2019), Moreno-Ternero et al. (2020).

[8] This feature is also shared by Tsay and Yeh (2019) and Moreno-Ternero et al. (2020).

[9] See Serrano (1995), Dagan et al. (1997), and Chang and Hu (2017). In the case of Serrano (1995), the exogenous bankruptcy rule is actually obtained as the equilibrium outcome of a random dictator bargaining game, which could be considered as an *expected* strategic justification of the Talmud rule.

2 The model

There is an infinite set of "potential" creditors, indexed by the natural numbers $\mathbb{N}$. Let $\mathcal{N}$ be the class of non-empty and finite subsets of $\mathbb{N}$. Given $N \in \mathcal{N}$ and $i \in N$, let c_i be **creditor *i*'s claim** and $c \equiv (c_i)_{i \in N}$ the claims vector. The **endowment** E of a bankrupt firm has to be divided among its creditors in N. A bankruptcy problem for N, or simply a **problem for *N***, is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $\sum_{i \in N} c_i \geq E$. Let $\mathcal{B}^N$ be the class of all problems for N. Let $L \equiv \sum_{i \in N} c_i - E$ denote the **shortfall** of $(c, E) \in \mathcal{B}^N$. An **awards vector** for $(c, E) \in \mathcal{B}^N$ is a vector $x \in \mathbb{R}^N$ such that $0 \leqq x \leqq c$ and $\sum_{i \in N} x_i = E$. Let $X(c, E)$ be the set of awards vectors of (c, E). A **rule** is a function defined on $\bigcup_{N \in \mathcal{N}} \mathcal{B}^N$ that associates with each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{B}^N$ a vector in $X(c, E)$. Our generic notation for rules is φ. For each $N' \subset N$, we write $c_{N'}$ for $(c_i)_{i \in N'}$, $\varphi_{N'}(c, E)$ for $(\varphi_i(c, E))_{i \in N'}$, and so on. Without loss of generality, in the remainder of this paper, we assume that, for each $N \in \mathcal{N}$, and each $(c, E) \in \mathcal{B}^N$, $\min_{i \in N} c_i > 0$. For ease of exposition, we also assume that $N \equiv \{1, \cdots, n\}$ and $c_1 \leq \cdots \leq c_n$.

We now formally introduce the rules and the central properties, whose verbal definitions were provided in Section 1.

2.1 Rules

First, the constrained equal awards rule (Aumann and Maschler, 1985), which makes awards as equal as possible subject to no one receiving more than her claim.

Constrained Equal Awards rule, *CEA*: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$, and each $i \in N$,

$$CEA_i(c, E) \equiv \min\{c_i, \lambda\},$$

where $\lambda \in \mathbb{R}_+$ is chosen such that $\sum_{i \in N} CEA_i(c, E) = E$.

The constrained equal losses rule (Aumann and Maschler, 1985) makes losses (a creditor's loss is the difference between her claim and her award) as equal as possible, subject to no one receiving a negative amount.

Constrained Equal Losses rule, *CEL*: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$, and each $i \in N$,

$$CEL_i(c, E) \equiv \max\{c_i - \lambda, 0\},$$

where $\lambda \in \mathbb{R}_+$ is chosen such that $\sum_{i\in N} CEL_i(c,E) = E$.

The Talmud rule (Aumann and Maschler, 1985), defined to rationalize some numerical examples in the Talmud, is a "hybrid" of the CEA and CEL rules, depending on whether the endowment falls below or above one half the aggregate claim.

Talmud rule, $\boldsymbol{T}$: For each $N \in \mathcal{N}$, each $(c,E) \in \mathcal{B}^N$, and each $i \in N$,

$$T_i(c,E) \equiv \begin{cases} \min\left\{\frac{1}{2}c_i, \lambda\right\} & \text{if } \sum_{i\in N} \frac{1}{2}c_i \geq E; \\ \frac{1}{2}c_i + \max\left\{\frac{1}{2}c_i - \lambda, 0\right\} & \text{otherwise,} \end{cases}$$

where $\lambda \in \mathbb{R}_+$ is chosen such that $\sum_{i\in N} T_i\left(c,E\right) = E$.

The TAL-family of rules was proposed by Moreno-Ternero and Villar (2006) to generalize the Talmud rule and include both the constrained equal awards rule and the constrained equal losses rule as two special cases. Each rule in this family is determined by a parameter $\theta \in [0,1]$, which can be described as a measure of the distributive power of the rule.

TAL-family of rules, $\{\boldsymbol{T^\theta} | \boldsymbol{\theta} \in [\mathbf{0},\mathbf{1}]\}$: For each $\theta \in [0,1]$, each $N \in \mathcal{N}$, each $(c,E) \in \mathcal{B}^N$, and each $i \in N$,

$$T_i^\theta(c,E) \equiv \begin{cases} \min\left\{\theta c_i, \lambda\right\} & \text{if } \sum_{i\in N} \theta c_i \geq E; \\ \theta c_i + \max\left\{(1-\theta)c_i - \lambda, 0\right\} & \text{otherwise,} \end{cases}$$

where $\lambda \in \mathbb{R}_+$ is chosen such that $\sum_{i\in N} T_i^\theta\left(c,E\right) = E$.

It will be interesting for the ensuing analysis to provide the two-agent expression of the family. Let $N \equiv \{1,2\}$ and $c_1 \leq c_2$. Then, we have the following:

$$T^\theta\left(c,E\right) = \begin{cases} \left(\frac{E}{2}, \frac{E}{2}\right) & \text{if } E \leq 2\theta c_1; \\ \left(\theta c_1, E - \theta c_1\right) & \text{if } 2\theta c_1 \leq E \leq c_2 - c_1 + 2\theta c_1; \\ \left(c_1 - \frac{(c_1+c_2-E)}{2}, c_2 - \frac{(c_1+c_2-E)}{2}\right) & \text{if } c_2 - c_1 + 2\theta c_1 \leq E. \end{cases}$$

2.2 Axioms

We now introduce the axioms for rules we shall consider in this paper. First, *bilateral consistency*, which says that if the rule chooses an awards vector for a problem, then for the associated "two-creditor reduced problem" derived by imagining that all the other creditors leave with their components of the vector, and reassessing the situation from the viewpoint of the two remaining creditors, it chooses the corresponding awards of the vector to that subgroup.

Bilateral consistency: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$, and each $N' \subset N$ with $|N'| = 2$, if $x = \varphi(c, E)$, then $x_{N'} = \varphi(c_{N'}, \sum_{N'} x_i)$.

Then, two properties related to lower and upper bounds. *Lower bound of degree* θ says that if each creditor receives at least either a fraction θ of her claim or an equal share of the endowment. *Upper bound of degree* θ says that each creditor receives at most either a fraction θ of her claim or an equal share of the shortfall (the difference between the sum of the claims of the creditors and the endowment).

Lower bound of degree of θ: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$, each $\theta \in [0, 1]$, and each $i \in N$, $\varphi_i(c, E) \geq \min\{\theta c_i, \frac{E}{|N|}\}$.

Upper bound of degree of θ: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$, each $\theta \in [0, 1]$, and each $i \in N$, $\varphi_i(c, E) \leq \min\{\theta c_i, c_i - \frac{L}{|N|}\}$.

3 The results

We first consider a strategic implementation for the TAL-family of rules in two-creditor problems. Let $N \in \mathcal{N}$ with $|N| = 2$, $(c, E) \in \mathcal{B}^N$ and $\theta \in [0, 1]$. Without loss of generality, assume that $N = \{1, 2\}$ and $c_1 \leq c_2$. Consider the following two-creditor negotiation procedure $\Gamma_2^\theta(c, E)$ (see Figure 1):

Stage 1. *Nature* randomly picks one of the two creditors, say i, as "initiator", who chooses a perspective $u \in \{gain, loss\}$ to divide E.

Stage 2. Given the perspective u chosen in Stage 1:

If $u = gain$, then the other creditor r, as "responder" chooses $q \in \{2\theta \min\{c_1, c_2\}, E\}$ and proposes a division $D^q = \{a, b\}$ such that $a, b \in \mathbb{R}_+$ and $a + b = q$. Then, i picks $x_i \in D^q$ as her award, and r receives $E - x_i$.

If $u = loss$, then r chooses $q \in \{2(1 - \theta) \min\{c_1, c_2\}, c_1 + c_2 - E\}$ and proposes a division $D^q = \{a, b\}$ such that $a, b \in \mathbb{R}_+$ and $a + b = q$. Then, i picks $x_i \in D^q$. Finally, r gets the remainder $(q - x_i)$ as her loss (namely, her award is $c_r - (q - x_i)$), and i receives the remainder, i.e., $E - c_r + (q - x_i)$.

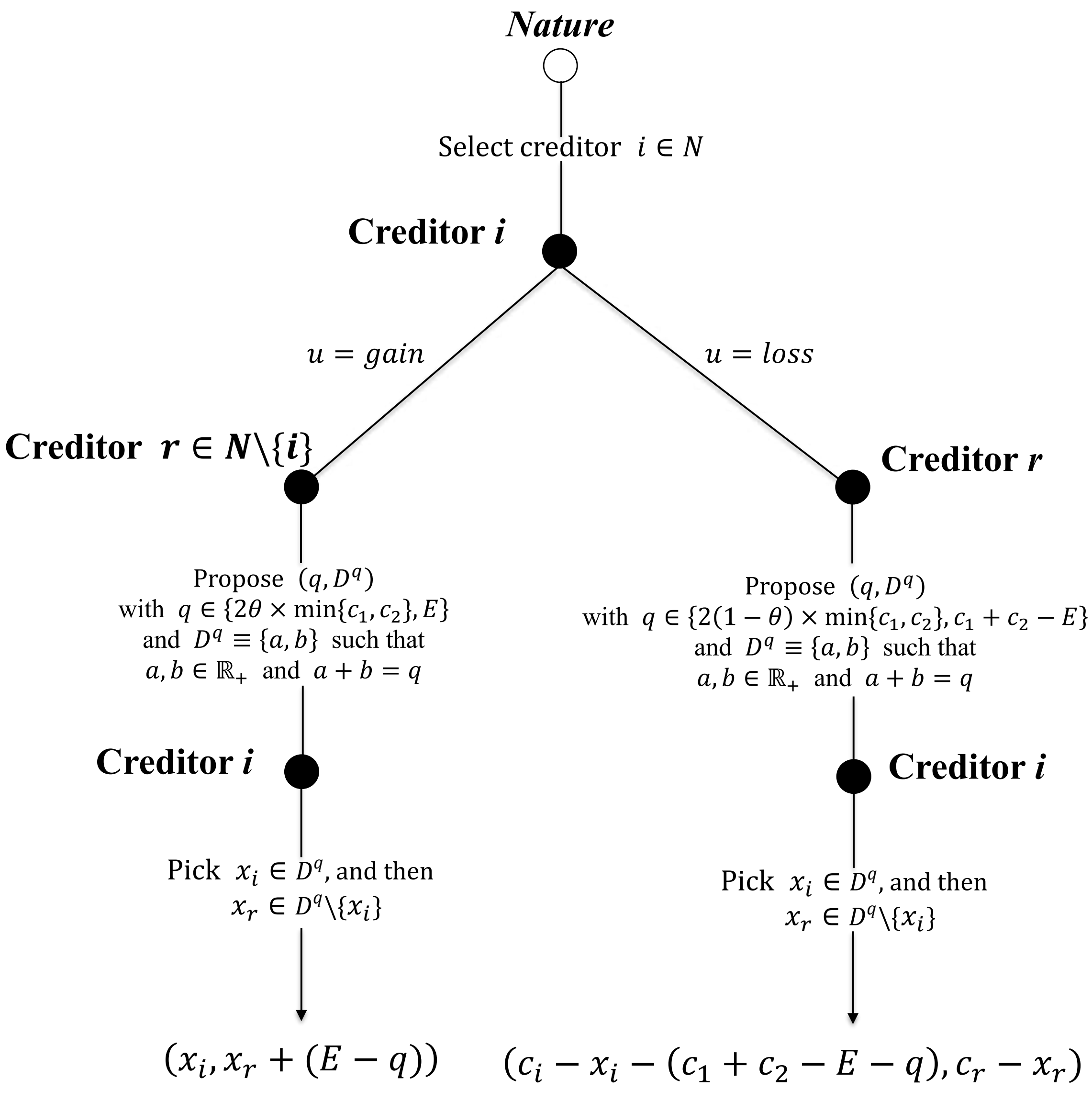

Figure 1: The game tree of $\Gamma_2^\theta(N, c)$ for the TAL-family

We show that for each $\theta \in [0,1]$, the two-creditor game Γ_2^θ strategically implements the rule T^θ.

Proposition 1. For each $N \in \mathcal{N}$ with $|N| = 2$, each $(c, E) \in \mathcal{B}^N$ and each $\theta \in [0,1]$, the unique Nash Equilibrium (NE) outcome of the game $\Gamma_2^\theta(c, E)$ is $T^\theta(c, E)$. Moreover, it can be supported by a pure strategy Subgame-Perfect Nash Equilibrium (SPE).

Proof. Let $N \in \mathcal{N}$ with $|N| = 2$, $(c, E) \in \mathcal{B}^N$ and $\theta \in [0,1]$. We first prove the existence part. Let $G^\theta \equiv \min\{2\theta \min\{c_1, c_2\}, E\}$ and $L^\theta \equiv \min\{2(1-\theta)\min\{c_1, c_2\}, c_1 + c_2 - E\}$. Consider the following strategy profile for the game $\Gamma_2^\theta(c, E)$, $\bar{\sigma}^\theta(c, E) = (\bar{\sigma}_j^\theta(c, E))_{j \in N}$.

In Stage 1, if creditor 1 is the initiator, then she chooses $u_1^\theta = gain$ if $E \leq c_2 + (2\theta - 1)c_1$; $u_1^\theta = loss$, otherwise. Similarly, if creditor 2 is the initiator, then she chooses $u_2^\theta = gain$ if $E \leq 2\theta c_1$; $u_2^\theta = loss$, otherwise.

In Stage 2, let $i, r \in N$ with $i \neq r$. If creditor i is the initiator, then given creditor r's proposal (q, D^q), she picks $\max D^q$ if $u = gain$ is chosen in Stage 1; $\min D^q$, otherwise. Now, if creditor r is the responder, then she proposes

$$(q^\theta, D^{q,\theta}) = \begin{cases} \left(G^\theta, \left\{\frac{G^\theta}{2}, \frac{G^\theta}{2}\right\}\right) & \text{if } u = gain; \\ \left(L^\theta, \left\{\frac{L^\theta}{2}, \frac{L^\theta}{2}\right\}\right) & \text{otherwise.} \end{cases}$$

We claim that $\bar{\sigma}^\theta(c, E)$ is a SPE of $\Gamma_2^\theta(c, E)$ with outcome $T^\theta(c, E)$. First, it is easy to see that by following $\bar{\sigma}^\theta(c, E)$, the outcome of $\Gamma_2^\theta(c, E)$ is $T^\theta(c, E)$.

Next, we show $\bar{\sigma}^\theta(c, E)$ is a SPE of $\Gamma_2^\theta(c, E)$. In Stage 2, suppose that creditor $i \in N$ is the initiator. Then, given $\bar{\sigma}_i^\theta(c, E)$ by the game rule, it is clear that picking $\max D^q$ when $u = gain$ and $\min D^q$ when $u = loss$, respectively, is a best response of creditor i to (q, D^q).

Next, suppose that creditor $r \in N$ is the responder. In the case of $u = gain$, creditor r follows $\bar{\sigma}_r^\theta(c, E)$ to choose $(q^\theta, D^{q,\theta}) = (G^\theta, \{\frac{G^\theta}{2}, \frac{G^\theta}{2}\})$, which grants her $\frac{G^\theta}{2} + (E - G^\theta) = E - \frac{G^\theta}{2}$. If creditor r deviates to $(q, D^q) \neq (q^\theta, D^{q,\theta})$, then as $\min D^q + \max D^q = q$, she would end up with $\min D^q + (E - q) \leq \frac{q}{2} + (E - q) = E - \frac{q}{2} \leq E - \frac{G^\theta}{2}$, which implies that she is not better off. In the other case of $u = loss$, creditor r follows $\bar{\sigma}_r^\theta(c, E)$ to choose $(q^\theta, D^{q,\theta}) = (L^\theta, \{\frac{L^\theta}{2}, \frac{L^\theta}{2}\})$, which grants her $c_r - \frac{L^\theta}{2}$. If creditor r deviates to $(q, D^q) \neq (q^\theta, D^{q,\theta})$, then as $\min D^q + \max D^q = q$, she ends up with $c_r - \max D^q \leq c_r - \frac{L^\theta}{2}$, which implies that she is not better off.

Finally, we show that the initiator can not benefit by deviating from $\bar{\sigma}^\theta(c, E)$ in Stage 1. The following two cases complete the proof.

Case 1: credtior 1 is the initiator. We consider two subcases.

Subcase 1.1: $\boldsymbol{E \leq c_2 + (2\theta - 1)c_1}$. By following $\bar{\sigma}_1^\theta(c, E)$ to pick $u_1^\theta = gain$, subgame perfection implies that creditor 1 obtains $\frac{G^\theta}{2}$. Suppose that creditor 1 deviates to pick $u = loss$, then subgame perfection implies that she obtains:

$$\begin{aligned}
c_1 - \frac{L^\theta}{2} - (c_1 + c_2 - E - L^\theta) &= c_1 - (1-\theta)c_1 - \{(c_1 + c_2 - E) - 2(1-\theta)c_1\} \\
&= \theta c_1 + \{E - c_2 - (2\theta - 1)c_1\} \\
&\leq \theta c_1 \\
&\leq \frac{G^\theta}{2},
\end{aligned}$$

where the first equality and the inequalities follow from the hypothesis $E \leq c_2 + (2\theta - 1)c_1$. This implies that she is not better off deviating.

Subcase 1.2: $\boldsymbol{E > c_2 + (2\theta - 1)c_1}$. By following $\bar{\sigma}_1^\theta(c, E)$, creditor 1 picks $u_1^\theta = loss$. Subgame perfection implies that creditor 1 obtains $c_1 - \frac{L^\theta}{2} - \left\{(c_1 + c_2 - E) - L^\theta\right\} = c_1 - \frac{c_1 + c_2 - E}{2}$. Suppose that creditor 1 deviates to pick $u = gain$, then subgame perfection implies that she obtains

$$\frac{G^\theta}{2} = \theta c_1 \leq c_1 - \frac{c_1 + c_2 - E}{2},$$

where the equality and the inequality follow from the hypothesis $E > c_2 + (2\theta - 1)c_1$. This implies that she is not better off deviating.

Case 2: credtior 2 is the initiator. We consider two subcases.

Subcase 2.1: $\boldsymbol{E \leq 2\theta c_1}$. By following $\bar{\sigma}_2^\theta(c, E)$, creditor 2 picks $u_2^\theta = gain$. Subgame perfection implies that creditor 2 obtains $\frac{G^\theta}{2} = \frac{E}{2}$. Suppose that creditor 2 deviates to pick $u = loss$, then subgame perfection implies that she obtains

$$\begin{aligned}
c_2 - \frac{L^\theta}{2} - (c_1 + c_2 - E - L^\theta) &= c_1 - (1-\theta)c_1 - \{(c_1 + c_2 - E) - 2(1-\theta)c_1\} \\
&= \theta c_1 + \{E - c_2 - (2\theta - 1)c_1\} \\
&\leq \frac{E}{2},
\end{aligned}$$

where the first equality and the inequality follow from the hypothesis $E \leq 2\theta c_1$. This implies that she is not better off deviating.

Subcase 2.2: $\boldsymbol{E > 2\theta c_1}$. By following $\bar{\sigma}_2^\theta(c,E)$ to pick $u_2^\theta = loss$, subgame perfection implies that creditor 2 obtains $c_2 - \frac{L^\theta}{2} - \left\{(c_1 + c_2 - E) - L^\theta\right\}$. Suppose that creditor 2 deviates to pick $u = gain$, then subgame perfection implies that she obtains $\frac{G^\theta}{2} = \theta c_1$. As

$$
\begin{aligned}
& c_2 - \frac{L^\theta}{2} - \left\{(c_1 + c_2 - E) - L^\theta\right\} \\
&= \begin{cases} c_2 - (1-\theta)c_1 - \{(c_1 + c_2 - E) - 2(1-\theta)c_1\} & \text{if } c_1 + c_2 - E \geq 2(1-\theta)c_1; \\ c_2 - \frac{c_1+c_2-E}{2} & \text{if } c_1 + c_2 - E < 2(1-\theta)c_1, \end{cases} \\
&= \begin{cases} E - \theta c_1 & \text{if } c_1 + c_2 - E \geq 2(1-\theta)c_1; \\ c_2 - \frac{c_1+c_2-E}{2} & \text{if } c_1 + c_2 - E < 2(1-\theta)c_1, \end{cases} \\
&\geq \theta c_1,
\end{aligned}
$$

where the inequality follows from the hypothesis $E > 2\theta c_1$, she is not better off deviating.

Thus, $\bar{\sigma}^\theta(c,E)$ is a SPE of the game $\Gamma_2^\theta(c,E)$, and generates $T^\theta(c,E)$ as outcome.

We next show the uniqueness part. First, as a SPE is a NE, the above implies that $\bar{\sigma}^\theta(c,E)$ is a NE of the game $\Gamma_2^\theta(c,E)$ with outcome $T^\theta(c,E)$. Next, for each $N \in \mathcal{N}$ with $|N| = 2$, each $(c,E) \in \mathcal{B}^N$, and each $\theta \in [0,1]$, it is clear that the sum of each possible outcome of $\Gamma_2^\theta(c,E)$ is always the endowment E. Thus, the proof can be completed in the same way as the proof of the uniqueness part at Proposition 1 in Tsay and Yeh (2019). $Q.E.D.$

We next extend Proposition 1 to the case of more than two creditors. To do so, we consider a three-stage extensive form game Γ^θ of which Γ_2^θ is the final stage. We then resort to the analysis in Tsay and Yeh (2019) to show that the corresponding game strategically justifies the T^θ rule in the general case of n agents. Key to this result is the fact that all rules within the TAL-family satisfy *consistency* (and not just the bilateral version of the axiom stated above) as well as "endowment monotonicity".[10]

[10] *Endowment monotonicity* says that if E increases, no creditor receives a smaller amount.

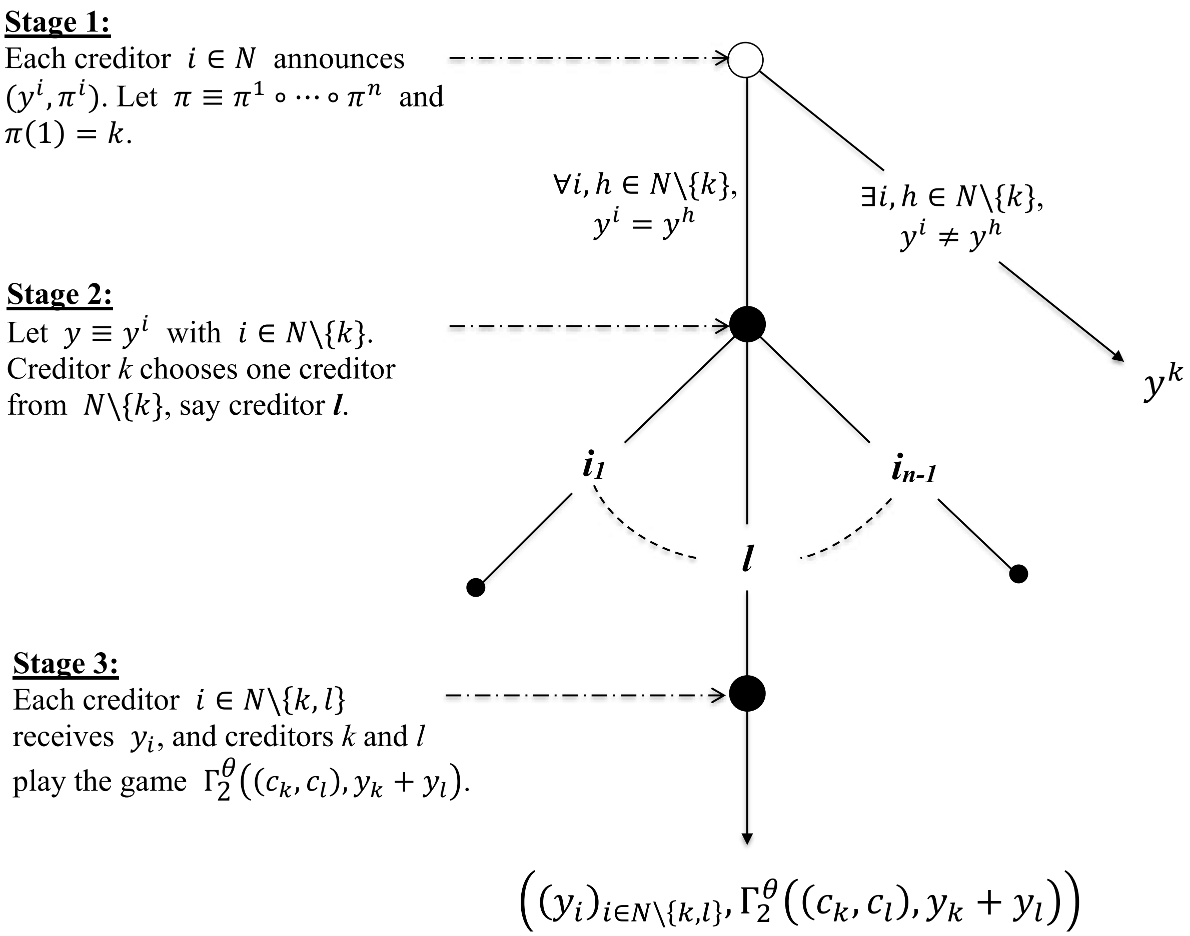

Figure 2: The game tree of $\Gamma^\theta(N, c)$ for the TAL-family

Formally, the game is described as follows (see Figure 2).[11] Let $N \in \mathcal{N}$, $(c, E) \in \mathcal{B}^N$ and $\theta \in [0, 1]$.

Stage 1: Each creditor $i \in N$ simultaneously announces an awards vector y^i and a permutation $\pi^i : N \to N$. For simplicity, let $\pi \equiv \pi^1 \circ \cdots \circ \pi^n$ and $\pi(1) = k \in N$ be the **coordinator**. If the creditors except the coordinator do not have a consensus, that is, there exist $h, j \in N \setminus \{k\}$ such that $y^h \neq y^j$, then the game ends up with the allocation y^k, announced by the coordinator; otherwise, the game moves forward to the next stage.

Stage 2: Let $y = y^j$ for some $j \neq k$. The coordinator takes a creditor, say creditor $l \neq k$ (i.e., takes action l), to negotiate their awards in the next stage, and each creditor $i \in N \setminus \{k, l\}$ receives y_i.

Stage 3: Creditors k and l play the game $\Gamma_2^\theta(c_{\{k,l\}}, y_k + y_l)$.

We first show the outcome existence result.

Proposition 2. For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{B}^N$ and each $\theta \in [0, 1]$, the game $\Gamma^\theta(c, E)$ has a SPE with outcome $T^\theta(c, E)$.

Proof. Given $N \in \mathcal{N}$, $(c, E) \in \mathcal{B}^N$ and $\theta \in [0, 1]$, we show that the following strategy profile $\sigma^\theta(c, E) = (\sigma_i^\theta(c, E))_{i \in N}$ constitutes a SPE of $\Gamma^\theta(c, E)$ with outcome $T^\theta(c, E)$.

Stage 1: Each creditor $i \in N$ proposes $(\pi^{i\theta}, y^{i\theta}) = (\pi^{Id}, T^\theta(c, E))$, where $\pi^{Id} : N \to N$ is the identity permutation, that is, for each $i \in N$, $\pi^{Id}(i) = i$.

Stage 2: Suppose that $\pi(1) = k$ and there is y such that $y^i = y$ for each $i \neq k$. Creditor k chooses one creditor, say creditor l, from $N \setminus \{k\}$ (i.e., takes action l), where $l \in \arg\max_{i \in N \setminus \{k\}} T_k^\theta \left(c_{\{i,k\}}, y_i + y_k\right)$.

Stage 3: Suppose that $\pi(1) = k$. Then, there exists y such that $y^i = y$ for each $i \neq k$, and the coordinator chooses creditor $l \neq k$ in Stage 2. Creditors k and l adopt $\bar{\sigma}^\theta(c_{\{k,l\}}, y_k + y_l) = (\bar{\sigma}_i^\theta(c_{\{k,l\}}, y_k + y_l))_{i \in \{k,l\}}$ defined in the proof of Proposition 1.

[11] The main difference between our game and Tsay and Yeh's (2019) is the following. When the proposal in Stage 1 is the awards vector announced by the coordinator, it would be reasonable for her just to accept it as the outcome of the game. This is exactly Stage 1 of our game. However, in their game, to make the proposal the outcome, the coordinator has to take an extra action by accepting the proposal in Stage 2. It can be shown that our Theorem 1 also holds if we replace our design of Stages 1 and 2 with theirs.

It is easy to see that following $\sigma^\theta(c,E)$, the game ends with outcome $T^\theta(c,E)$. We now show that $\sigma^\theta(c,E)$ is a SPE of the game $\Gamma^\theta(c,E)$. Suppose that $\pi(1)=k$, and there exists y such that for each $i\neq k$, $y^i=y$. Moreover, suppose that the coordinator chooses creditor $l\neq k$ in Stage 2. By Proposition 1, creditors k and l are not better off deviating from $\sigma^\theta_k(c,E)$ and $\sigma^\theta_l(c,E)$ in Stage 3, respectively. Moreover, the outcome is $\left(T^\theta\left(c_{\{k,l\}}, y_k+y_l\right), y_{N\setminus\{k,l\}}\right)$ in this case. Thus, by subgame perfection, creditor k takes action l, where $l\in\arg\max_{i\in N\setminus\{k\}} T^\theta_k\left(c_{\{i,k\}}, y_i+y_k\right)$, that is, $\sigma^\theta_k(c,E)$ is a best response for her in Stage 2. Finally, we complete the proof by showing that no creditor $i\in N$ is better off deviating from announcing $(\pi^{Id}, T^\theta(c,E))$. Note that following $\sigma^\theta(c,E)$, creditor 1 is the coordinator and all creditors have a consensus on $T^\theta(c,E)$ in Stage 1.

Suppose that creditor $i\in N$ deviates from $(\pi^{Id}, T^\theta(c,E))$ to (π^i, y^i). As after deviation the creditors $N\setminus\{i\}$ have a consensus on $T^\theta(c,E)$, by subgame perfection and *bilateral consistency* of the T^θ rule, the game ends up with the outcome $T^\theta(c,E)$. This implies that creditor i is not better off deviating. Thus, $\sigma^\theta(c,E)$ is a SPE of the game $\Gamma^\theta(c,E)$. *Q.E.D.*

We now show the outcome uniqueness result.

Proposition 3. For each $N\in\mathcal{N}$, each $(c,E)\in\mathcal{B}^N$ and each $\theta\in[0,1]$, $T^\theta(c,E)$ is the unique NE outcome of the game $\Gamma^\theta(c,E)$.

Proof. Given $N\in\mathcal{N}$, $(c,E)\in\mathcal{B}^N$ and $\theta\in[0,1]$, suppose that $\sigma^*(c,E) = (\sigma^*_i(c,E))_{i\in N}$ is a NE of $\Gamma^\theta(c,E)$ with outcome $x^*=(x^*_i)_{i\in N}$. For each $i\in N$, let y^{i*} be her announced awards vector in Stage 1 by following $\sigma^*_i(c,E)$.

We first claim that by following $\sigma^*(c,E)$, each creditor $i\in N$ receives an award no less than her award prescribed by $T^\theta(c,E)$ in the game, namely $x^*_i\geq T^\theta_i(c,E)$. We consider the following two cases.

Case 1: $|N|=2$. By the game rule and Proposition 1, we conclude that for each $i\in N$, $x^*_i\geq T^\theta_i(c,E)$.

Case 2: $|N|>2$. Suppose, by contradiction, that there is $i\in N$ such that $x^*_i< T^\theta_i(c,E)$. We complete this case by the following two subcases.

Subcase 2.1: There exists a pair $j,h\in N\setminus\{i\}$ such that $y^{j*}\neq y^{h*}$. Suppose that creditor i deviates from $\sigma^*_i(c,E)$ to a strategy such that, after deviation, she is chosen as the coordinator and announces an awards vector y^i such that $y^i_i=\min\{c_i,E\}$. Then, by the game rule, creditor i receives

$y_i^i = \min\{c_i, E\} \geq T_i^\theta(c, E) > x_i^*$ in the game, which violates that $\sigma^*(c, E)$ is a NE.

Subcase 2.2: For each pair $j, h \in N \setminus \{i\}$, $y^{j*} = y^{h*}$.
Let $y^* \equiv y^{j*}$ for some $j \neq i$. We first claim that for each $j \in N$,

$$x_j^* \geq \max\left\{y_j^*, \arg\max_{h \in N \setminus \{j\}} T_j^\theta\left(c_{\{j,h\}}, y_j^* + y_h^*\right)\right\}.$$

Suppose, by contradiction, that there is $j \in N$ such that either (i) $x_j^* < y_j^*$, or (ii) $x_j^* < \arg\max_{h \in N \setminus \{j\}} T_j^\theta(c_{\{j,h\}}, y_j^* + y_h^*)$. First, suppose that (i) holds. If for each $h \neq j$, $y^{h*} = y^*$, then consider that creditor j deviates from $\sigma_j^*(c, E)$ to a strategy such that after deviation, she is not the coordinator and announces an awards vector $y^j \neq y^*$. By the game rule, after deviation, creditor j receives $y_j^* > x_j^*$ at the end of the game, a contradiction. Otherwise, consider that creditor j deviates from $\sigma_j^*(c, E)$ to a strategy such that after deviation, she is chosen as the coordinator and announces the awards vector y^*. Then, by the game rule, after deviation, creditor j receives $y_j^* > x_j^*$ in the end of the game, a contradiction. Next, suppose that (ii) holds. If for each $h \neq j$, $y^{h*} = y^*$, then consider that creditor j deviates from $\sigma_j^*(c, E)$ to a strategy such that after deviation, she is chosen as the coordinator, and chooses $l \in \arg\max_{h \in N \setminus \{j\}} T_j^\theta\left(c_{\{j,h\}}, y_j^* + y_h^*\right)$ in Stage 2. Moreover, she follows $\bar{\sigma}_j^\theta(c_{\{j,l\}}, y_j^* + y_l^*)$ defined in the proof of Proposition 1 in Stage 3. By the game rule and the proof of Proposition 1, after deviation creditor j receives $T_j^\theta\left(c_{\{j,l\}}, y_j^* + y_l^*\right) > x_j^*$ in the end of the game, a contradiction. Otherwise, consider that creditor j deviates from $\sigma_j^*(c, E)$ to a strategy such that after deviation, she is chosen as the coordinator and announces an awards vector y^j in which $y_j^j = \min\{c_j, E\}$. Then, by the game rule, after deviation, creditor j receives $y_j^j = \min\{c_j, E\} \geq \arg\max_{h \in N \setminus \{j\}} T_j^\theta(c_{\{j,h\}}, y_j^* + y_h^*) > x_j^*$ at the end of the game, a contradiction. Thus, we conclude that for each $j \in N$,

$$x_j^* \geq \max\left\{y_j^*, \arg\max_{h \in N \setminus \{j\}} T_j^\theta\left(c_{\{j,h\}}, y_j^* + y_h^*\right)\right\}.$$

We next claim that for each pair $j, h \in N$ with $j \neq h$,

$$x_j^* = y_j^* = T_j^\theta(c_{\{j,h\}}, y_j^* + y_h^*).$$

First, for each $j \in N$, as the game rule implies that both x^* and y^* are awards vectors, $x_j^* = y_j^*$. Next, for each pair $j, h \in N$ with $j \neq h$, as

$x_j^* \geq T_j^\theta(c_{\{j,h\}}, y_j^* + y_h^*)$ and $y_j^* = x_j^*$, $y_j^* \geq T_j^\theta(c_{\{j,h\}}, y_j^* + y_h^*)$. Hence, $x_j^* = y_j^* = T_j^\theta(c_{\{j,h\}}, y_j^* + y_h^*)$. Therefore, by *bilateral consistency* and "endowment monotonicity" of the T^θ rule, it follows from Proposition 5 of Chun (1999) that $x^* = T^\theta(c, E)$.[12] This contradicts with the hypothesis $x_i^* < T_i^\theta(c, E)$.

Thus, for each $i \in N$, $x_i^* \geq T_i^\theta(c, E)$. As both x^* and $T^\theta(c, E)$ are awards vectors, $x^* = T^\theta(c, E)$. *Q.E.D.*

4 Concluding remarks

We have presented in this paper strategic justifications of the TAL-family of rules for bankruptcy problems. Such a family arises as a generalization of the Talmud rule, which is a hybrid between the CEA and CEL rules. It considers one or the other rule, depending on whether the endowment falls short or exceeds one half of the aggregate claim, using half-claims instead of claims. A sort of reverse protocol to the one provided by the Talmud rule, switching the roles between the equal awards and equal losses principles, has been proposed in the literature, giving rise to the so-called "Reverse Talmud rule" (e.g., Chun et al., 2001). Formally, for each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{B}^N$, the **Reverse Talmud rule**, $\boldsymbol{RT}$, selects the vector

$$RT\,(c,E) = \begin{cases} CEL(\frac{1}{2}c, E) & \text{if } E \leq \sum_{i\in N} \frac{1}{2}c_i; \\ \frac{1}{2}c + CEA(\frac{1}{2}c, E - \sum_{i\in N} \frac{1}{2}c_i) & \text{if } E \geq \sum_{i\in N} \frac{1}{2}c_i. \end{cases}$$

Thus, the same natural idea mentioned above to generalize the Talmud rule has been adopted to generalize the reverse Talmud rule, giving rise to a new family of rules: the *reverse TAL-family* (e.g., van den Brink and Moreno-Ternero, 2017). Such a family also consists of a one-parameter set of piece-wise linear rules, ranging from the CEA rule to the CEL rule, but this time having the reverse Talmud rule in the middle. Formally, for each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{B}^N$, the **Reverse TAL-family of rules**, $\{\boldsymbol{RT^\theta} | \boldsymbol{\theta} \in [\mathbf{0}, \mathbf{1}]\}$ selects the vector such that for each $i \in N$,

$$RT_i^\theta\,(c,E) = \begin{cases} \max\{\theta c_i - \lambda, 0\} & \text{if } E \leq \sum_{i\in N} \theta c_i; \\ \theta c_i + \min\{(1-\theta)c_i, \mu\} & \text{if } E \geq \sum_{i\in N} \theta c_i, \end{cases}$$

[12] *Endowment monotonicity* says that if E increases, no creditor receives a smaller amount. The fact that each T^θ rule satisfies this axiom is shown at Proposition 4 in Moreno-Ternero and Villar (2006).

where λ and μ are chosen so that $\sum_{i\in N} RT_i^\theta(c,E) = E$.

Alternatively, the reverse TAL-family can be expressed as follows. For each $N \in \mathcal{N}$ and each $(c,E) \in \mathcal{B}^N$,

$$RT^\theta(c,E) = \begin{cases} CEL(\theta c, E) & \text{if } E \leq \sum_{i\in N}\theta c_i; \\ \theta c + CEA((1-\theta)c, E - \sum_{i\in N}\theta c_i) & \text{if } E \geq \sum_{i\in N}\theta c_i, \end{cases}$$

or, equivalently,

$$RT^\theta(c,E) = CEL(\theta c, \min\{E, \sum_{i\in N}\theta c_i\}) + CEA((1-\theta)c, \max\{E - \sum_{i\in N}\theta c_i, 0\}).$$

It would be worthwhile to offer strategic justifications of this family of rules. A characterization of the reverse Talmud rule is established by van den Brink et al. (2013). We conjecture that such a result would be a natural first step towards constructing non-cooperative games that strategically justify the *reverse TAL-family* of rules.

Finally, we should mention that the TAL-family is included within a more general family, known as the ICI-family of rules introduced by Thomson (2008). All ICI rules require that the evolution of each claimant's award, as a function of the endowment, is increasing first, constant next and finally increasing again. It turns out that when one imposes *bilateral consistency*, which is central to our analysis, as well as the mild notion of *scale invariance*, the ICI-family of rules shrinks precisely to the TAL-family of rules. In this sense, one might interpret our results as a partial strategic justification of the ICI-family of rules too.

References

Aumann, R., Maschler, M., 1985. Game theoretic analysis of a bankruptcy problem from the Talmud. Journal of Economic Theory 36, 195-213.

van den Brink, R., Funaki Y., van der Laan, G., 2013. Characterization of the reverse Talmud bankruptcy rule by exemption and exclusion properties. European Journal of Operational Research 228, 413-417.

van den Brink, R., Moreno-Ternero, J.D., 2017. The reverse TAL-family of rules for bankruptcy problems. Annals of Operations Research 254, 449-465.

Chang, C., Hu, C.-C., 2017. A non-cooperative interpretation of the kernel. International Journal of Game Theory 46, 185-204.

Chun, Y., 1999. Equivalence of axioms for bankruptcy problems. International Journal of Game Theory 28, 511-520.

Chun, Y., Schummer, J., Thomson, W., 2001. Constrained egalitarianism: a new solution for claims problems. Seoul Journal of Economics 14, 269-297.

Dagan, N., 1996. New characterization of old bankruptcy rules. Social Choice and Welfare 13, 51-59.

Dagan, N., Serrano, R., Volij, O., 1997. A noncooperative view of consistent bankruptcy rules. Games and Economic Behavior 18, 55-72.

Dominguez, D., Thomson, W., 2006. A new solution to the problem of adjudicating conflicting claims. Economic Theory 28, 283-307.

Hu, C.-C., Tsay, M.-H., Yeh, C.-H., 2012. Axiomatic and strategic justifications for the constrained equal benefits rule in airport problems. Games and Economic Behavior 75, 185-197.

Hu, C.-C., Tsay, M.-H., Yeh, C.-H., 2018. A study of the nucleolus in the nested cost-sharing problem: axiomatic and strategic perspectives. Games and Economic Behavior 109, 82-98.

Krishna, V., Serrano, R., 1996. Multilateral bargaining. Review of Economic Studies 63, 61-80.

Maschler, M., Solan, E., Zamir, S., 2013. Game Theory. Cambridge University Press. Cambridge, MA.

Moreno-Ternero, J.D., 2011a. Voting over piece-wise linear tax methods. Journal of Mathematical Economics 47, 29-36.

Moreno-Ternero, J.D., 2011b. A coalitional procedure leading to a family of bankruptcy rules. Operations Research Letters 39, 1-3.

Moreno-Ternero, J.D., Tsay, M.-H., Yeh, C.-H., 2020. A strategic justification of the Talmud rule based on lower and upper bounds. International Journal of Game Theory 49, 1045-1057.

Moreno-Ternero J., Villar A., 2004. The Talmud rule and the securement of agents' awards. Mathematical Social Sciences 47, 245-257.

Moreno-Ternero, J.D., Villar, A., 2006. The TAL-family of rules for bankruptcy problems. Social Choice and Welfare 27, 231-249.

Nash, J.F., 1953. Two person cooperative games. Econometrica 21, 128-140.

O'Neill, B., 1982. A problem of rights arbitration from the Talmud. Mathematical Social Sciences 2, 345-371.

Serrano, R., 1995. Strategic bargaining, surplus sharing problems and the nucleolus. Journal of Mathematical Economics 24, 319-329.

Serrano, R., 2005. Fifty years of the Nash program. 1953-2003. Investigaciones Economicas 29, 219-258.

Serrano, R., 2020. Sixty-seven years of the Nash program: time for retirement? SERIEs. Forthcoming. https://doi.org/10.1007/s13209-020-00221-5

Serrano, R., Vohra, R., 2002. Bargaining and bargaining sets. Games and Economic Behavior 39, 292-308.

Thomson, W., 2005. Divide-and-permute. Games and Economic Behavior 52, 186-200.

Thomson, W., 2008. Two families of rules for the adjudication of conflicting claims. Social Choice and Welfare 31, 667-692.

Thomson, W., 2011. Fair allocation rules, Chapter 21 in the Handbook of Social Choice and Welfare Vol. 2, Arrow, K., Sen, A., Suzumura, K., eds. North Holland, 393-506

Thomson W., 2019. How to divide when there isn't enough: from Aristotle, the Talmud, and Maimonides to the axiomatics of endowment allocation, Econometric Society Monograph. Cambridge University Press.

Tsay, M.-H., Yeh, C.-H., 2019. Relations among the central rules in bankruptcy problems: a strategic justification perspective. Games and Economic Behavior 113, 515-532.