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Bilateral Redistribution

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Abstract

We explore the implications of three basic and intuitive axioms for income redistribution problems: *continuity*, *no transfer paradox* and *stability*. The combination of the three axioms characterizes in the two-agent case a large family of rules, which we call *threshold* rules. For each level of total income in society, a threshold is considered for each agent. It is impossible for both agents to be below their respective thresholds. If an agent's income is below the threshold, the difference is redistributed from the other agent; otherwise, the rule imposes laissez-faire.

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1 Introduction

Economics is essentially about scarcity. The problem of dividing scarce resources is faced by many on a daily basis, and the media is plagued with references to it, especially now in the aftermath of the COVID-19 pandemic. Albeit to a lower extent, attention to this problem is also being given within the scientific literature (e.g., Chambers and Moreno-Ternero, 2019). Income taxation and redistribution are basic tools to alleviate the negative consequences of scarcity. In this paper, we focus on the specific problem of income redistribution, which we approach with the axiomatic method. We consider a stylized model in which an income profile reflects the taxable and observable income of two agents. The issue is to construct rules that transform the given income profile into another income profile, with the requirement that nothing is wasted in the redistribution process, and that no agent ends up with a negative post-tax income. It is thus reminiscent of the seminal model introduced by O'Neill (1982) to analyze claims problems, surveyed by Thomson (2019). Young (1988) reinterpreted the same model to analyze taxation problems.[1] Here, we consider taxation problems with an important proviso: a zero tax revenue to be raised, and thus, negative taxes are possible. As such, our problems could be considered a case of the generalized claims problems as studied by Ju et al., (2007).[2]

We consider three basic and intuitive axioms for this model: *continuity* (small changes in the data of a problem should not lead to large changes in the solution), *no transfer paradox* (transferring some income before redistribution takes place does not increase income after redistribution) and *stability* (no further redistribution takes place once a solution is obtained). We show that the combination of the three axioms characterizes a large family, which we call *threshold* rules, that guarantee partial redistribution for unequal incomes. Our rules therefore convey a long-standing concern in the political philosophy literature that can be traced back to Rawls (1971) or, more recently, to Van Parijs and Vanderborght (2017). Lower bounds also have a long tradition within the literature on fair allocation (e.g., Thomson, 2011) and we have also explored them recently for taxation problems (e.g, Chambers and Moreno-Ternero, 2017).

Each rule within the family we characterize is described by two functions (over aggregate income) which determine the region where partial redistribution is imposed. The family is wide enough to encompass (while considering specific functions) rules ranging from *laissez-faire* to its polar *fully egalitarian* rule.

[1] See also Ju and Moreno-Ternero (2011) or Moreno-Ternero (2011), among others.

[2] See also Cowell (1985).

2 The model

Two *agents* have non-negative (taxable) *incomes* y_1 and y_2. We refer to $y = (y_1, y_2)$ as the income profile and $Y = y_1 + y_2$ as its overall income. Given an income profile $y \in \mathbb{R}^2_+$, a *tax profile* is a vector $t = (t_1, t_2) \in \mathbb{R}^2$ satisfying the following two conditions: (i) for each $i = 1, 2$, $t_i \leq y_i$ and (ii) $t_1 + t_2 = 0$. A *taxation scheme*, $T\colon \mathbb{R}^2_+ \to \mathbb{R}^2$, associates with each income profile $y \in \mathbb{R}^2_+$ a tax profile $T(y)$. For each taxation scheme, there is a corresponding *income redistribution rule* (or, simply, *rule*), $R\colon \mathbb{R}^2_+ \to \mathbb{R}^2_+$, which associates with each income profile $y \in \mathbb{R}^2_+$ a post-tax income profile $R(y) = y - T(y)$. Note that, in particular, $R_1(y) + R_2(y) = Y$.

An instance of rule is the so-called *egalitarian rule* R^E, which ensures both agents end up with the same (post-tax) income: for each $y \in \mathbb{R}^2_+$,

$$R^E(y) \equiv \left(\frac{Y}{2}, \frac{Y}{2}\right).$$

The polar case is the *identity rule* R^I, or *laissez-faire rule*, which taxes no agent, i.e., $R^I(y) = y$ for each $y \in \mathbb{R}^2_+$.

Between these two polar cases, there is room for implementing the idea of *partial redistribution*. For instance, consider the family of rules imposing a flat tax rate and personalized lump-sum transfers (e.g., Ju et al., 2007). Formally,

$$R_i^F(y) = (1 - F(Y))\, y_i + G_i(Y),$$

where $G_1(Y) + G_2(Y) = F(Y)Y$. In these rules, F determines the flat tax rate $F(Y)$ as a function of the overall income, Y, while G determines the reallocation scheme $(G_1(Y), G_2(Y))$ as a function of agents' identities subject to the budget balance.

We now present another family of rules that guarantee partial redistribution based on an endogenous portion of the aggregate income. We call them *threshold rules*. They behave as follows. For each level of total income in society, a threshold is considered for each agent. It is impossible for both agents to be below their respective thresholds. If an agent's income is below the threshold, the difference is redistributed from the other agent; otherwise, the rule imposes laissez-faire.

Formally, let $\alpha : \mathbb{R}_+ \to \mathbb{R}^2_+$ and $\beta : \mathbb{R}_+ \to \mathbb{R}^2_+$ be two continuous functions such that, for each $x \in \mathbb{R}_+$,

$$\alpha_1(x) + \alpha_2(x) = x = \beta_1(x) + \beta_2(x),$$

and

$$\alpha_2(x) \leq \beta_2(x).$$

We shall refer to the locus of the vectors chosen by the functions, i.e., $(\alpha_1(x), \alpha_2(x))$ and $(\beta_1(x), \beta_2(x))$, as x varies from 0 to $+\infty$, as the *paths* associated to the α and β functions. We shall impose to these paths a minimal monotonicity requirement indicating that they cannot intersect the lines of slope -1 more than once. Essentially, this implies an almost everywhere bound on their slope.

Note that β reflects a threshold for agent 1, whereas α reflects a threshold for agent 2, as illustrated in Figure 1 below.

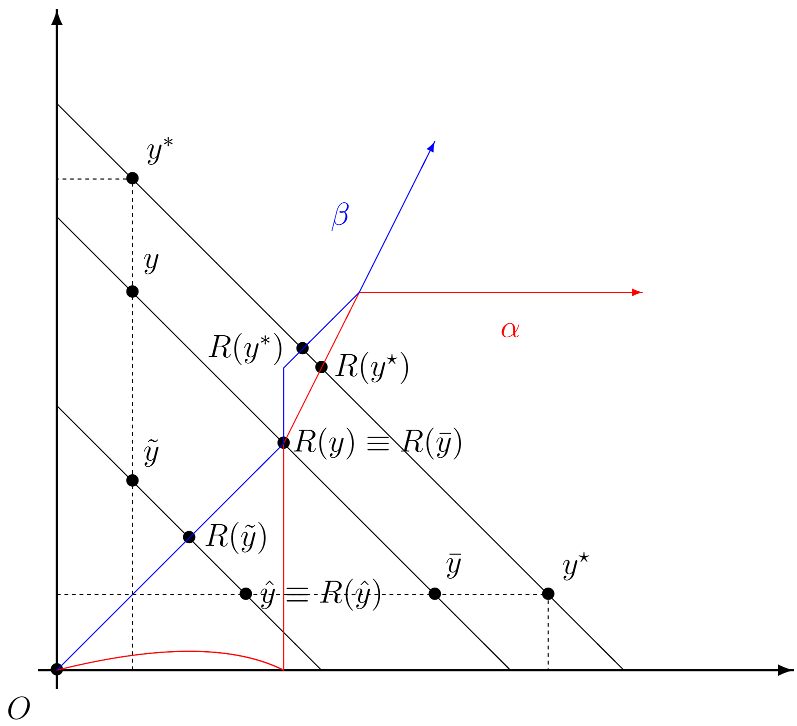

Figure 1: Threshold rules. This figure illustrates the threshold rule $R^{\alpha\beta}$. We first consider the *paths* associated to α and β. That is, the locus of the vectors chosen by the functions: $(\alpha_1(x), \alpha_2(x))$ and $(\beta_1(x), \beta_2(x))$, as x varies from 0 to $+\infty$. For each $x \in [0, +\infty)$, we consider the segment determined by $(\alpha_1(x), \alpha_2(x))$ and $(\beta_1(x), \beta_2(x))$. If the starting income profile is $\hat{y}$, then it is located within the corresponding segment determined by $(\alpha_1(\hat{Y}), \alpha_2(\hat{Y}))$ and $(\beta_1(\hat{Y}), \beta_2(\hat{Y}))$, and thus the rule suggests no redistribution. If the starting profile is $\tilde{y}$, which is not located within that segment, although it conveys the same aggregate income as $\hat{y}$, then the rule suggests to redistribute from agent 2 to agent 1 until the former stays with $\beta_2(\hat{Y})$ (and the latter reaches her threshold $\beta_1(\hat{Y}) = \hat{Y} - \beta_2(\hat{Y})$). For both y and $\bar{y}$ the rule suggests the same solution; namely, $(\alpha_1(Y), \alpha_2(Y)) = (\beta_1(Y), \beta_2(Y))$, where $Y = \bar{y}_1 + \bar{y}_2$. Finally, if the starting income profile is y^*, then the rule suggests to redistribute from agent 2 to agent 1 until the former stays at $\beta_2(y_1^* + y_2^*)$ (and the latter reaches her threshold $\beta_1(y_1^* + y_2^*)$), whereas if the starting income profile is $y^\star$, then the rule suggests to redistribute from agent 1 to agent 2 until the former stays with $\alpha_1(y_1^\star + y_2^\star)$ (and the latter reaches her threshold $\alpha_2(y_1^\star + y_2^\star)$).

Then, the corresponding rule $R^{\alpha\beta}$ is defined as follows. For each $y \in \mathbb{R}^2_+$,

$$R^{\alpha\beta}(y) = \begin{cases} (\alpha_1(Y), \alpha_2(Y)), & \text{if } y_2 \leq \alpha_2(Y). \\ y, & \text{if } \alpha_2(Y) \leq y_2 \leq \beta_2(Y). \\ (\beta_1(Y), \beta_2(Y)), & \text{if } y_2 \geq \beta_2(Y). \end{cases} \tag{1}$$

Equivalently,

$$R^{\alpha\beta}(y) = \begin{cases} (\alpha_1(Y), \alpha_2(Y)), & \text{if } y_1 \geq \alpha_1(Y). \\ y, & \text{if } \alpha_1(Y) \geq y_1 \geq \beta_1(Y). \\ (\beta_1(Y), \beta_2(Y)), & \text{if } y_1 \leq \beta_1(Y). \end{cases}$$

In other words, the rule $R^{\alpha\beta}$ behaves as laissez-faire in a somewhat middle range of income distributions (determined by the α and β functions). For very unequal incomes, in which one agent lies below the threshold (again, determined by the α and β functions), then the rule imposes partial redistribution to guarantee the threshold for that agent. Note that the threshold is endogenous, as it is determined by the overall income in the income profile. Figure 1 illustrates one of the rules within this family.

Note that, as also illustrated in Figure 1, the *paths* associated to the α and β functions do not need to be strictly increasing.

Interesting members within this family are the so-called $\delta-$threshold rules, obtained in the above definition for each $\delta \in (0, +\infty)$, and

$$\alpha(x) = \begin{cases} (x, 0), \text{ if } 0 \leq x \leq \delta. \\ (\delta, x-\delta), \text{ if } \delta \leq x \leq 2\delta. \\ (x-\delta, \delta), \text{ if } x \geq 2\delta. \end{cases}$$

and

$$\beta(x) = \begin{cases} (0, x), \text{ if } 0 \leq x \leq \delta. \\ (x-\delta, \delta), \text{ if } \delta \leq x \leq 2\delta. \\ (\delta, x-\delta), \text{ if } x \geq 2\delta. \end{cases}$$

respectively.

A $\delta-$threshold rule imposes partial redistribution for unequal income profiles in which one agent lies below δ, whereas the other lies above. For those cases, the rule forces the latter agent transfer to the former until she reaches δ. If both agents are above δ, or below it, the rule says nothing.

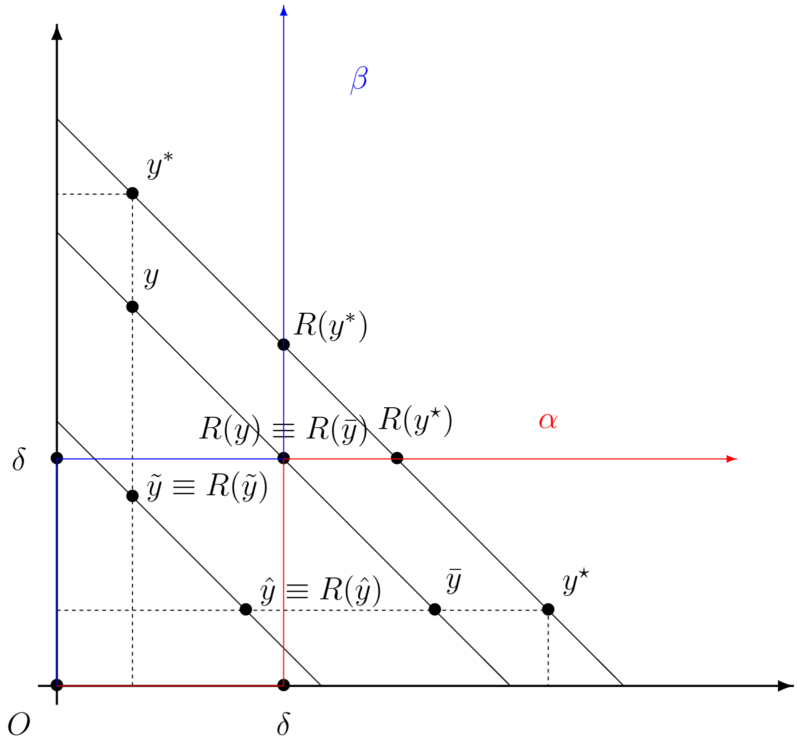

Figure 2: δ-threshold rules. This figure illustrates the δ-threshold rule R^δ. We first note that the *paths* associated to α and β are made of three linear pieces each: a first one over one axis a second one parallel to the other axis and a third one parallel to the first piece. That is, the locus of the vectors $(\alpha_1(x), \alpha_2(x))$ follows the horizontal axis as x varies from 0 to δ, a vertical line as x varies from δ to 2δ, from where it follows again a horizontal line. Similarly, the locus of the vectors $(\beta_1(x), \beta_2(x))$, follows the vertical axis as x varies from 0 to δ, a horizontal line as x varies from δ to 2δ, from where it follows again a vertical line. If the starting income profile is $\hat{y}$ or $\tilde{y}$, which both have the same aggregate income and none of the incomes is above δ, then the rule suggests no redistribution. For both y and $\bar{y}$, which both have the same aggregate income (2δ), the rule suggests the same solution; namely, (δ, δ). Finally, if the starting income profile is y^*, then the rule suggests to redistribute from agent 2 to agent 1 until agent 1 reaches δ, whereas if the starting income profile is $y^\star$, then the rule suggests to redistribute from agent 1 to agent 2 until agent 2 reaches δ.

Other somewhat focal members of the family are, precisely, the *egalitarian* rule and the *identity* rule introduced above. Formally, if $\alpha(x) = \beta(x) = (\frac{x}{2}, \frac{x}{2})$, for each $x \in \mathbb{R}_+$, then $R^{\alpha\beta} \equiv R^E$. Also, if $\alpha(x) = (x, 0)$ for each $x \in \mathbb{R}_+$ and $\beta(x) = (0, x)$ for each $x \in \mathbb{R}_+$, then $R^{\alpha\beta} \equiv R^I$. In other words, both the egalitarian rule and the identity rule are (admittedly, extreme) threshold rules.

We now introduce our fundamental axioms in this context.

The first axiom says that small changes in the income profile should only produce small changes in the solution. Besides being a technical axiom, it is also ethically attractive as it models *non-arbitrariness* of the rule, or robustness to measurement errors.

Continuity. For each sequence $y^n \in \mathbb{R}^2_+$, and each $y \in \mathbb{R}^2_+$, if y^n converges to y, then $R(y^n)$ converges to $R(y)$.

The next axiom precludes the existence of a transfer paradox, i.e., a situation in which an agent gets more income after redistribution, provided she has transferred some amount of income to the other agent, before redistribution takes place.[3]

No Transfer Paradox. For each $y \in \mathbb{R}^2_+$, each $i = 1, 2$ and each $\varepsilon \in [0, y_i]$, then

$$R_i(y) \geq R_i(y_i - \varepsilon, y_j + \varepsilon).$$

The third axiom is a requirement of stability. It says that after a rule redistributes income for a given income profile, the rule proposes no further redistribution for the resulting income profile. In other words, after redistribution takes place, no more redistribution is optimal.

Stability. For each $y \in \mathbb{R}^2_+$, $R(R(y)) = R(y)$.

3 The characterization result

Our main result states that the three previous axioms characterize the family of threshold rules.

Theorem 1. *A rule satisfies continuity, no transfer paradox and stability if and only if it is a threshold rule.*

Proof. It is straightforward to show that the threshold rules satisfy the three axioms in the statement. Conversely, let R be a rule satisfying continuity, no transfer paradox and stability. For each $x \geq 0$, let

$$E(x) = \left\{(y_1, y_2) \in \mathbb{R}^2_+ \text{ such that } Y = x\right\}.$$

That is, $E(x)$ is the set of profiles whose overall income is x. By the definition of rules, $R(E(x)) \subseteq E(x)$. By *continuity*, as $E(x)$ is a compact and connected set, so is $R(E(x))$. By

[3] In two-agent exchange economies, the transfer paradox takes place when a transfer from one agent to the other leads to a welfare gain for the donor and a welfare loss for the recipient. The concept has long been studied within mathematical economics (e.g., Gale, 1974; Balasko, 1975; Demuynck et al., 2016).

stability, $R(E(x))$ coincides with the set of fixed points of R (restricted to $E(x)$). Thus, there exist $a(x) \in \mathbb{R}_+$ and $b(x) \in \mathbb{R}_+$ such that the projection of $R(E(x))$ over the horizontal axis is given by the interval $[a(x), b(x)]$. In other words, for each $y \in E(x)$, we have the following:

$$R_1(y) = y_1 \text{ for all } y_1 \in [a(x), b(x)],$$

and

$$R_1(y) \in [a(x), b(x)] \text{ for all } y_1 \notin [a(x), b(x)].$$

Now, by *no transfer paradox*,

$$R_1(y) = a(x) \text{ for all } y \text{ such that } y_1 < a(x),$$

and

$$R_1(y) = b(x) \text{ for all } y \text{ such that } y_1 > b(x).$$

Let $\alpha : \mathbb{R}_+ \to \mathbb{R}^2_+$ and $\beta : \mathbb{R}_+ \to \mathbb{R}^2_+$ be the functions defined such that, for each $x \in \mathbb{R}_+$, $\alpha(x) = (b(x), x - b(x))$, and $\beta(x) = (a(x), x - a(x))$. By *continuity*, it follows that both α and β are continuous functions. Furthermore, for each $x \in \mathbb{R}_+$, $\alpha_1(x) + \alpha_2(x) = \beta_1(x) + \beta_2(x) = x$, and $\alpha_2(x) \leq \beta_2(x)$. It then follows that R is the threshold rule $R^{\alpha\beta}$. $\square$

Theorem 1 says that the properties of continuity, no transfer paradox and stability force us to redistribute (at least partially) on the quite unequal cases, while preserving the status quo (i.e., laissez-faire) for the somewhat equal cases. In doing so, a certain income level is guaranteed for each agent (albeit this income level may be zero in an extreme case). Nevertheless, this level is endogenous and can be arbitrary small (as mentioned before, even 0, for the extreme case of the identity rule) or as large as possible (even the mean income of the population, for the case of the egalitarian rule, the other extreme threshold rule).

4 Further insights

4.1 Refining the family

We explore further the threshold rules, obtaining necessary and sufficient conditions on (α, β) under which the corresponding rule $R^{\alpha\beta}$ satisfies some additional (secondary) axioms introduced below.

First, two axioms of impartiality. On the one hand, the requirement that agents with equal (pre-tax) income end up having the same (post-tax) income.

Equal treatment of equals. For each $y \in \mathbb{R}^2_+$ such that $y_1 = y_2$, $R_1(y) = R_2(y)$.

On the other hand, the stronger requirement that the order of agents has no influence on the outcome of the rule.

Anonymity. For each $y \in \mathbb{R}^2_+$, $(R_2(y), R_1(y)) = (R_1(y_2, y_1), R_2(y_2, y_1))$.

Then, an axiom indicating that if the income of one agent increases, ceteris paribus, the agent cannot be hurt by it.

Agent monotonicity. For each $y \in \mathbb{R}^2_+$, and $\varepsilon > 0$, $R_i(y_i + \varepsilon, y_j) \geq R_i(y)$.

Alternatively, we could consider the axiom indicating that if the income of both agents increase, none of the post-tax incomes can decrease.

Pair monotonicity. For each pair $y, y' \in \mathbb{R}^2_+$, such that $y \geq y'$, $R(y) \geq R(y')$.

Finally, a standard technical axiom.

Homogeneity. For each $y \in \mathbb{R}^2_+$ and each $\lambda \in \mathbb{R}_+$, $R(\lambda y) = \lambda R(y)$.

The performance of the threshold rules, with respect to the above axioms, is stated in the following result, whose proof we relegate to the appendix.

Proposition 1. *Let R satisfy continuity, stability, and no transfer paradox. Then,*

1. *R satisfies equal treatment of equals if and only if it is a threshold rule for which $\alpha_2(x) \leq \frac{x}{2} \leq \beta_2(x)$ for each $x \in \mathbb{R}_+$, i.e., the 45^0-line is comprised within the region determined by the paths of α and β.*

2. *R satisfies anonymity if and only if it is a threshold rule for which $\alpha_2(x) + \beta_2(x) = x$ for each $x \in \mathbb{R}_+$, i.e., the paths of α and β are symmetric with respect to the 45^0-line.*

3. *R satisfies agent monotonicity, or pair monotonicity, if and only if it is a threshold rule for which $\alpha(x') \geq \alpha(x)$ and $\beta(x') \geq \beta(x)$ for each pair $x, x' \in \mathbb{R}_+$ such that $x' \geq x$, i.e., the paths of α and β are non-decreasing.*

4. *R satisfies homogeneity if and only if it is a (proportional) threshold rule for which $\alpha_2(x) = \lambda_\alpha x$ and $\beta_2(x) = \lambda_\beta x$ for some $\lambda_\alpha, \lambda_\beta \in \mathbb{R}_+$ such that $\lambda_\alpha \leq \lambda_\beta$, i.e., the paths of α and β are straight lines.*

4.2 The general case

The results above (and the model itself) are set for the case of two agents. To extend our analysis to the case of $n > 2$, it is natural to consider a variable-population model. In such a setting, the axiom formalizing a principle of *solidarity* stating that the arrival of new agents should affect all original agents in the same direction (all gain, all lose, or all remain the same as before) has strong normative appeal.[4] It can be shown, for instance, that in this variable-population setting, a rule satisfies *continuity*, *stability*, *no transfer paradox*, *solidarity*, and *homogeneity* if and only if it is either the egalitarian rule or the identity rule. That is, among the two-agent *proportional* threshold rules, characterized in Proposition 1.4, only the *egalitarian rule* and the *identity rule* pass the *solidarity* test and admit extensions. If *homogeneity* is dropped from that result, the analysis becomes more cumbersome.

For example, to move to the 3-agent setting, it would be natural to consider a 2-*agent coalition stability* notion (implied by the *solidarity* axiom outlined above).[5] If so, one might think of the rule $R^{(3)}(y) = R^{(3)}(y_1, y_2, y_3)$ that satisfies

$$R^{(2)}\left(R_i^{(3)}(y), R_j^{(3)}(y)\right) = \left(R_i^{(3)}(y), R_j^{(3)}(y)\right) \text{ for each pair } i, j \in \{1, 2, 3\}, \tag{2}$$

for some 2-agent redistribution rule $R^{(2)}$. We have shown that, under *continuity*, (2-agent) *stability*, and (2-agent) *no-transfer paradox*, $R^{(2)}$ must be a threshold rule with some $\alpha\left(R_i^{(3)}(y) + R_j^{(3)}(y)\right)$ and $\beta\left(R_i^{(3)}(y) + R_j^{(3)}(y)\right)$. If *anonymity* is further imposed (see Proposition 1.2 above), then it suffices to work with a single α such that α_1 defines a lower threshold with respect to the first agent in $R^{(2)}$.

Thus, any $R^{(3)}$ that satisfies (2) should also verify

$$R_i^{(3)}(y) \geq \max\left\{\alpha_1(Y - R_k^{(3)}(y)), \alpha_1(Y - R_j^{(3)}(y))\right\},$$

and

$$R_i^{(3)}(y) \leq \min\left\{Y - R_k^{(3)}(y) - \alpha_1(Y - R_k^{(3)}(y)), Y - R_j^{(3)}(y) - \alpha_1(Y - R_j^{(3)}(y))\right\},$$

where $Y = y_1 + y_2 + y_3$. From here it follows that, for each $i = 1, 2, 3$,

$$R_i^{(3)}(y) \geq \alpha_1\left(R_i^{(3)}(y) + \alpha_1(Y - R_i^{(3)}(y))\right),$$

[4]See Moreno-Ternero and Roemer (2006, 2012) for a formalization of this axiom in a related setting. The axiom is a strengthening of the so-called *consistency* axiom, which has played a central role in a large number of recent axiomatic studies (see, for instance, Thomson (2012) and the literature cited therein).

[5]We thank an anonymous referee for providing this specific suggestion, presented next.

which is a recursive inequality in α_1. As α_1 is continuous, it follows from the intermediate value theorem that there exists $\gamma(Y) \in [0, Y]$ such that

$$\gamma(Y) = \alpha_1 \left(\gamma(Y) + \alpha_1(Y - \gamma(Y))\right).$$

If we take the smallest of such $\gamma(Y)$ (if there are more than one), we have

$$\gamma(Y) \leq R_i^{(3)}(y) \leq Y - 2\gamma(Y),$$

thus defining the lower and upper thresholds for the 3-agent problem.

In words, in the three-agent case, the threshold rule would then behave as follows. Let $y = (y_1, y_2, y_3)$ be a three-agent vector for which we assume, without loss of generality, that $y_1 \leq y_2 \leq y_3$. If all incomes are above the threshold, then the rule does nothing (i.e., it behaves as the identity). If agent 1 is below the threshold then agent 3 will transfer income to agent 1 until either agent 1 achieves the threshold, or agent 3 ends up having the same income as agent 2; whichever comes first. In the latter case, the process would continue by transferring from both agents (2 and 3) equally to agent 1 until the threshold is achieved, either by 1, or by 2 and 3. Finally, if two agents are below the threshold, then agent 3 will transfer (equal amounts of) income to agents 1 and 2, until either agent 2 achieves the threshold, or agent 3 does so; whichever comes first. In the former case, the process would continue by transferring from agent 3 to agent 1 until the threshold is achieved either by 1 or by 3. Again, whichever comes first.

5 Discussion

We have studied a stylized model of income redistribution, from an axiomatic viewpoint. We have shown that, in the bilateral case, three basic and intuitive axioms (*continuity*, *no transfer paradox* and *stability*) characterize a large family, which we call *threshold* rules, that guarantee partial redistribution for unequal incomes. More precisely, each rule within the family we characterize is described by two functions (over aggregate income) which determine the region where partial redistribution is imposed. The family is wide enough to encompass (while considering specific functions) rules ranging from *laissez-faire* to its polar *fully egalitarian* rule. Adding extra axioms, formalizing basic principles of impartiality, monotonicity, or invariance, we shrink the family in various meaningful ways. We have also explored how to extend these results to a general context with more than two agents, which calls for a variable-population axiom formalizing the principle of *solidarity*.

Our work relates to recent, general interest, economics research. Income redistribution has declined in most OECD countries over the past two decades. The role of policy factors (e.g., changes in the size and design of tax and transfer systems) and non-policy factors (e.g., the impact of globalization, technological changes or population aging) in driving such a trend during this period has been scrutinized (e.g., Causa et al., 2019). In general, within that literature, income redistribution is measured as the relative reduction in the inequality of pre-tax and transfer income (so-called market income) that is achieved by personal income taxes, employees' social security contributions and cash transfers, based on household-level data. Our work is, thus, providing normative foundations for such a measurement process in a basic and stylized model.

The social choice literature has also brought the axiomatic approach to analyze redistribution/taxation problems. In a series of influential papers (e.g., Young, 1987, 1988, 1990), Peyton Young analyzed the problem of distributive justice in taxation, with a special emphasis on the connection between the principle of equal sacrifice, which can be traced back to John Stuart Mill, and progressive taxation. In Young (1988), he characterizes the family of equal-sacrifice (tax) rules in a similar model to ours. The only basic difference between our model and Young's (which is itself a reinterpretation of the original model introduced by O'Neill (1982) to analyze claims problems) is to consider a zero tax revenue, which thus allows for negative taxes (not allowed in the O'Neill-Young setting). This motivates the use of our critical axiom of *stability*, which does not have bite in the O'Neill-Young setting. The remaining axioms we consider (including those introduced in Section 4) are either identical or straightforward adaptations from existing axioms in that setting.

To the best our our knowledge, the only case where our same model was considered before is in Ju et al., (2007). Therein, they study a general class of claims problems, one of which happens to be O'Neill-Young setting with zero tax revenue (or, equivalently, an endowment equal to the aggregate claim). They focus on the notion of *reallocation-proofness* and show that this axiom, together with some other basic ones (including *no transfer paradox*) characterize the family of income-tax schedules with a flat tax rate and personalized lump-sum transfers, introduced in Section 2.

We therefore believe that our *threshold* rules are new in the literature. They are, nevertheless, reminiscent of some other rules considered in the literature. For instance, in the O'Neill-Young setting, generalized equal-sacrifice rules have been characterized (e.g., Chambers and Moreno-Ternero, 2017). Some of them impose constrained equal sacrifice with respect to some lower and upper bounds, allowing for a possible set of agents being exempted, to be interpreted as *agents below the poverty line*. Those bounds/thresholds are exogenously described by the brackets defining the rule, although the ones eventually being used are determined by the characteristics (set of agents, claims and endowment) of the problem at hand.

6 Appendix. Proof of Proposition 1

Let R satisfy *continuity*, *stability*, and *no transfer paradox*. Then, by Theorem 1, we know R is a threshold rule $R^{\alpha\beta}$.

1. Suppose $R^{\alpha\beta}$ satisfies *equal treatment of equals* too. Then, for each $y \in \mathbb{R}^2_+$ such that $y_1 = y_2$, $R^{\alpha\beta}(y) = y$. Assume, by contradiction, that there exists $x \in \mathbb{R}_+$ such that $\alpha_2(x) > \frac{x}{2}$. Then, $R^{\alpha\beta}(\frac{x}{2}, \frac{x}{2}) = (\alpha_1(x), \alpha_2(x)) \neq (\frac{x}{2}, \frac{x}{2})$, which is a contradiction. Similarly, assume, by contradiction, that there exists $x \in \mathbb{R}_+$ such that $\beta_2(x) < \frac{x}{2}$. Then, $R^{\alpha\beta}(\frac{x}{2}, \frac{x}{2}) = (\beta_1(x), \beta_2(x)) \neq (\frac{x}{2}, \frac{x}{2})$, which is a contradiction. Conversely, if $\alpha_2(x) \leq \frac{x}{2} \leq \beta_2(x)$, for each $\mathbb{R}_+$, it follows that $\alpha_2(Y) \leq y_2 \leq \beta_2(Y)$, for each $y \in \mathbb{R}^2_+$ such that $y_1 = y_2$. Thus, from (1), $R^{\alpha\beta}(y) = y$, which proves that $R^{\alpha\beta}$ satisfies *equal treatment of equals*.

2. Suppose $R^{\alpha\beta}$ satisfies *anonymity* instead. Then, for each $y \in \mathbb{R}^2_+$, $(R_2(y), R_1(y)) = (R_1(y_2, y_1), R_2(y_2, y_1))$. Assume, by contradiction, that there exists $x \in \mathbb{R}_+$ such that $\alpha_2(x) + \beta_2(x) < x$. Let $y \in \mathbb{R}^2_+$ be such that $y = (\alpha_1(x), \alpha_2(x))$. Then, $(R^{\alpha\beta}_2(y), R^{\alpha\beta}_1(y)) = (\alpha_2(x), \alpha_1(x)) = (\alpha_2(x), x - \alpha_2(x)) \neq (x - \beta_2(x), \beta_2(x)) = (\beta_1(x), \beta_2(x)) = (R^{\alpha\beta}_1(y_2, y_1), R^{\alpha\beta}_2(y_2, y_1))$, which represents a contradiction. Similarly, assume, by contradiction, that there exists $x \in \mathbb{R}_+$ such that $\alpha_2(x) + \beta_2(x) > x$. Let $y \in \mathbb{R}^2_+$ be such that $y = (\beta_1(x), \beta_2(x))$. Then, $(R^{\alpha\beta}_2(y), R^{\alpha\beta}_1(y)) = (\beta_2(x), \beta_1(x)) = (\beta_2(x), x - \beta_2(x)) \neq (x - \alpha_2(x), \alpha_2(x)) = (\alpha_1(x), \alpha_2(x)) = (R^{\alpha\beta}_1(y_2, y_1), R^{\alpha\beta}_2(y_2, y_1))$, which represents a contradiction.

   Conversely, if the condition $\alpha_2(x) + \beta_2(x) = x$ holds for each $x \in \mathbb{R}_+$ (and, therefore, $(\beta_2(x), \beta_1(x)) = (\alpha_1(x), \alpha_2(x))$, for each $x \in \mathbb{R}_+$), it follows that $R^{\alpha\beta}$ satisfies *anonymity*. To show that, let $y \in \mathbb{R}^2_+$. We distinguish several cases.

   Case 1. $y_2 \geq \beta_2(Y)$. In this case, y is above the path of β and, therefore, $R^{\alpha\beta}(y) = \beta(Y) = Y - \alpha(Y)$. Likewise, (y_2, y_1) is below the path of α, as $y_1 = Y - y_2 \leq Y - \beta_2(Y) = \alpha_2(Y)$ and, therefore, $R^{\alpha\beta}(y_2, y_1) = \alpha(Y) = Y - \beta(Y)$. Thus, $(R^{\alpha\beta}_2(y), R^{\alpha\beta}_1(y)) = (R^{\alpha\beta}_1(y_2, y_1), R^{\alpha\beta}_2(y_2, y_1))$, as desired.

   Case 2. $\alpha_2(Y) \leq y_2 \leq \beta_2(Y)$. In this case, both y and (y_2, y_1) are within the paths of α and β and, therefore, $R^{\alpha\beta}(y) = y$ and $R^{\alpha\beta}(y_2, y_1) = (y_2, y_1)$. Thus, $(R^{\alpha\beta}_2(y), R^{\alpha\beta}_1(y)) = (R^{\alpha\beta}_1(y_2, y_1), R^{\alpha\beta}_2(y_2, y_1))$, as desired.

   Case 3. $\alpha_2(Y) \geq y_2$. In this case, y is below the path of α and, therefore, $R^{\alpha\beta}(y) = \alpha(Y) = Y - \beta(Y)$. Likewise, (y_2, y_1) is above the path of β, as $y_1 = Y - y_2 \geq Y - \alpha_2(Y) = \beta_2(Y)$

and, therefore, $R^{\alpha\beta}(y_2, y_1) = \beta(Y)$. Thus, $(R_2^{\alpha\beta}(y), R_1^{\alpha\beta}(y)) = (R_1^{\alpha\beta}(y_2, y_1), R_2^{\alpha\beta}(y_2, y_1))$, as desired.

3. Suppose $R^{\alpha\beta}$ satisfies *agent monotonicity* instead. Then, for each $y \in \mathbb{R}_+^2$, and $\varepsilon > 0$, $R_i^{\alpha\beta}(y_i + \varepsilon, y_j) \geq R_i^{\alpha\beta}(y)$.[6]

   Assume, by contradiction, that there exist $x, x' \in \mathbb{R}_+$, such that $x' < x$ and $\alpha_1(x') > \alpha_1(x)$. Let $y = (x, 0)$ and $y' = (x', 0)$. Then, $y_1 = y_1' + \varepsilon$, where $\varepsilon = x - x' > 0$, and $R_1^{\alpha\beta}(y) = \alpha_1(x) < \alpha_1(x') = R_1^{\alpha\beta}(y')$, which is a contradiction.

   Similarly, assume, by contradiction, that there exist $x, x' \in \mathbb{R}_+$, such that $x' < x$ and $\beta_1(x') > \beta_1(x)$. By continuity, we can assume, without loss of generality, that $x - x' < \beta_1(x)$. Let $y = (x - x', x')$ and $y' = (0, x')$. Then, $y_2 = y_2'$, $y_1 = y_1' + \varepsilon$, where $\varepsilon = x - x' > 0$, and $R_1^{\alpha\beta}(y) = \beta_1(x) < \beta_1(x') = R_1^{\alpha\beta}(y')$, which is a contradiction.

   Similarly, assume, by contradiction, that there exist $x, x' \in \mathbb{R}_+$, such that $x' < x$ and $\alpha_2(x') > \alpha_2(x)$. By continuity, we can assume, without loss of generality, that $x - x' < \alpha_2(x)$. Let $y = (x', x - x')$ and $y' = (x', 0)$. Then, $y_1 = y_1'$, $y_2 = y_2' + \varepsilon$, where $\varepsilon = x - x' > 0$, and $R_2^{\alpha\beta}(y) = \alpha_2(x) < \alpha_2(x') = R_2^{\alpha\beta}(y')$, which is a contradiction.

   Finally, assume, by contradiction, that there exist $x, x' \in \mathbb{R}_+$, such that $x' < x$ and $\beta_2(x') > \beta_2(x)$. Let $y' = \beta(x')$ and $y = (\beta_1(x'), x - \beta_1(x'))$. Then, $y_1 = y_1'$, $y_2 = y_2' + \varepsilon$, where $\varepsilon = x - x' > 0$, and $R_2^{\alpha\beta}(y) = \beta_2(x) < \beta_2(x') = R_2^{\alpha\beta}(y')$, which is a contradiction.

   Conversely, if $R^{\alpha\beta}$ is such that $\alpha(x') \geq \alpha(x)$ and $\beta(x') \geq \beta(x)$ for each pair $x, x' \in \mathbb{R}_+$ such that $x' \geq x$, it follows that $R^{\alpha\beta}$ satisfies *pair monotonicity*.[7] To show that, let $y, y' \in \mathbb{R}_+^2$ be such that $y \geq y'$. We distinguish several cases.

   Case 1. $y_2' \geq \beta_2(Y')$. In this case, y' is above the path of β and, therefore, $R^{\alpha\beta}(y') = \beta(Y')$. As for y, it could be in any of the three regions defined by both paths. If above the path of β, i.e., $y_2 \geq \beta_2(Y)$, then, $R_1^{\alpha\beta}(y) = \beta_1(Y) \geq \beta_1(Y') = R_1^{\alpha\beta}(y')$. If below the path of α, i.e., $y_2 \leq \alpha_2(Y)$, then, $R^{\alpha\beta}(y) = \alpha(Y) \geq \beta(Y') = R^{\alpha\beta}(y')$. Finally, if within both paths, i.e., $\beta_2(Y) \geq y_2 \geq \alpha_2(Y)$, then $R^{\alpha\beta}(y) = y \geq \beta(Y') = R^{\alpha\beta}(y')$.

   Case 2. $\alpha_2(Y') \leq y_2' \leq \beta_2(Y')$. In this case, y' is between the paths of α and β and, therefore, $R^{\alpha\beta}(y') = y'$. As for y, it could be in any of the three regions defined by both paths. If above the path of β, i.e., $y_2 \geq \beta_2(Y)$, then, $R^{\alpha\beta}(y) = \beta(Y) \geq y' = R^{\alpha\beta}(y')$. If

[6] As *pair monotonicity* is a stronger axiom, it suffices to prove this implication.

[7] And, thus, *agent monotonicity* too, as it is a weaker axiom.

below the path of α, i.e., $y_2 \leq \alpha_2(Y)$, then, $R^{\alpha\beta}(y) = \alpha(Y) \geq y' = R^{\alpha\beta}(y)$. Finally, if within both paths, i.e., $\beta_2(Y) \geq y_2 \geq \alpha_2(Y)$, then $R^{\alpha\beta}(y) = y \geq y' = R^{\alpha\beta}(y')$.

Case 3. $\alpha_2(Y') \geq y_2'$. In this case, y' is below the path of α and, therefore, $R^{\alpha\beta}(y') = \alpha(Y')$. As for y, it could be in any of the three regions defined by both paths. If above the path of β, i.e., $y_2 \geq \beta_2(Y)$, then, $R^{\alpha\beta}(y) = \beta(Y) \geq \alpha(Y') = R^{\alpha\beta}(y')$. If below the path of α, i.e., $y_2 \leq \alpha_2(Y)$, then, $R^{\alpha\beta}(y) = \alpha(Y) \geq \alpha(Y') = R^{\alpha\beta}(y)$. Finally, if within both paths, i.e., $\beta_2(Y) \geq y_2 \geq \alpha_2(Y)$, then $R^{\alpha\beta}(y) = y \geq \alpha(Y') = R^{\alpha\beta}(y')$.

4. Suppose $R^{\alpha\beta}$ satisfies *homogeneity* instead. Then, for each $y \in \mathbb{R}^2_+$ and each $\lambda \in \mathbb{R}_+$, $R^{\alpha\beta}(\lambda y) = \lambda R^{\alpha\beta}(y)$. Assume, by contradiction, that there exist $x, x' \in \mathbb{R}_+$, such that $\frac{\alpha_2(x)}{x} \neq \frac{\alpha_2(x')}{x'}$. Let $\lambda = \frac{x'}{x}$. Then, by *homogeneity*,

$$(\alpha_1(x'), \alpha_2(x')) = R((x', 0)) = R((\lambda x, 0)) = \lambda R((x, 0)) = \lambda((\alpha_1(x), \alpha_2(x)),$$

which implies that $\frac{\alpha_2(x)}{x} = \frac{\alpha_2(x')}{x'}$, a contradiction. Similarly, assume, by contradiction, that there exist $x, x' \in \mathbb{R}_+$, such that $\frac{\beta_2(x)}{x} \neq \frac{\beta_2(x')}{x'}$. Let $\lambda = \frac{x'}{x}$. Then, by *homogeneity*,

$$(\beta_1(x'), \beta_2(x')) = R((0, x')) = R((0, \lambda x)) = \lambda R((0, x)) = \lambda((\beta_1(x), \beta_2(x)),$$

which implies that $\frac{\beta_2(x)}{x} = \frac{\beta_2(x')}{x'}$, a contradiction.

The above shows that, for each $x \in \mathbb{R}_+$, $\alpha_2(x) = \lambda_\alpha x$ and $\beta_2(x) = \lambda_\beta x$, for some $\lambda_\alpha, \lambda_\beta \in \mathbb{R}_+$. As $\alpha_2(x) \leq \beta_2(x)$, by the definition of $R^{\alpha\beta}$, it follows that $\lambda_\alpha \leq \lambda_\beta$.

Conversely, if $R^{\alpha\beta}$ is such that $\alpha_2(x) = \lambda_\alpha x$ and $\beta_2(x) = \lambda_\beta x$, for some $\lambda_\alpha, \lambda_\beta \in \mathbb{R}_+$ such that $\lambda_\alpha \leq \lambda_\beta$, it follows that $R^{\alpha\beta}$ satisfies *homogeneity*. To show that, let $y \in \mathbb{R}^2_+$ and $\lambda \in \mathbb{R}_+$. We distinguish several cases.

Case 1. $y_2 \geq \beta_2(Y)$. In this case, $R^{\alpha\beta}(y) = \beta(Y)$. As $\lambda y_2 \geq \lambda\beta_2(Y) = \beta_2(\lambda Y)$, it follows that $R^{\alpha\beta}(\lambda y) = (\beta_1(\lambda Y), \beta_2(\lambda Y)) = \lambda(\beta_1(Y), \beta_2(Y)) = \lambda R^{\alpha\beta}(y)$, as desired.

Case 2. $\alpha_2(Y) \leq y_2 \leq \beta_2(Y)$. In this case, $R^{\alpha\beta}(y) = y$. As $\alpha_2(\lambda Y) = \lambda\alpha_2(Y) \leq \lambda y_2 \leq \lambda\beta_2(Y) = \beta_2(\lambda Y)$, it follows that $R^{\alpha\beta}(\lambda y) = \lambda y = \lambda R^{\alpha\beta}(y)$, as desired.

Case 3. $y_2 \leq \alpha_2(Y)$. In this case, $R^{\alpha\beta}(y) = \alpha(Y)$. As $\lambda y_2 \leq \lambda\alpha_2(Y) = \alpha_2(\lambda Y)$, it follows that $R^{\alpha\beta}(\lambda y) = (\alpha_1(\lambda Y), \alpha_2(\lambda Y)) = \lambda(\alpha_1(Y), \alpha_2(Y)) = \lambda R^{\alpha\beta}(y)$, as desired.

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