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Monotonicity in sharing the revenues from broadcasting sports leagues*

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Abstract

We explore the implications of the principle of monotonicity in the problem of sharing the revenues from broadcasting sports leagues. We formalize different forms of this principle as several axioms for sharing rules in this setting. We show that, combined with two other basic axioms (equal treatment of equals and additivity), they provide axiomatic characterizations of focal rules for this problem, as well as families of rules compromising among them. These results highlight the normative appeal of the (focal) equal-split rule.

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1 Introduction

The sports industry has enjoyed prolific growth in the last decades, outpacing the GDP growth of most countries. KPMG notes that the entire global sports market (including infrastructure, events, training and sports goods) is estimated to be worth between \$600 and \$700 billion per year.[1] This has prompted an increasing interest within the operations research community to study several aspects related with the sports industry. Just to mention a few recent cases, Song and Shi (2020) and Li et al. (2021) analyze the performance of teams in the National Basket Association; Elitzur (2020) explores the use of data analytics in the Major League Baseball, whereas Peeters et al. (2020) study the impact of managers therein. Goller and Krumer (2020) and Yi et al. (2020) analyze the impact of game scheduling in European football leagues. Arlegi and Dimitrov (2020) and Van Bulck et al. (2020) deal with the design of competitions. The reader is referred to Wright (2009, 2014) or Kendall and Lenten (2017) for surveys of the fast-expanding literature, and Palacios-Huerta (2014) or Csató (2021) for recent books.

In this paper, we shall be concerned with a major aspect of the sports industry: broadcasting. It is estimated that the 2016 Olympic Summer Games had global audience of approximately 3.2 billion, and the final game of the 2018 FIFA World Cup a combined 3.572 billion viewers (more than half of the global population aged four and over). For the 2019 regular season, (US) National Football League games averaged 16.4 million viewers, whereas the Super Bowl broadcast that season attracted an average TV audience of 99.9 million people. The sale of broadcasting and media rights is currently the biggest source of revenue for sports organizations, overcoming more traditional sources such as ticket sales, merchandising or sponsorship. According to Statista, the total value of the NFL's national TV deal with ESPN was worth a total of 15.2 billion US dollars from 2014 to 2021.

The allocation of these (huge) amounts collected from selling broadcasting rights is crucial for the management of sports organizations. In Bergantiños and Moreno-Ternero (2020a) we introduced a formal model in which the allocation process is based on the (broadcasting) audiences that games throughout the season generate. Therein, we studied the problem theoretically and empirically (applying our theoretical results to the Spanish football league). We have also explored further aspects of the problem theoretically (e.g., Bergantiños and Moreno-Ternero,

[1]https://assets.kpmg/content/dam/kpmg/in/pdf/2016/10/The-business-sports.pdf. Last accessed on April 25, 2021.

2020b, 2020c, 2021) and empirically (e.g., Bergantiños and Moreno-Ternero, 2020d, 2021). In this paper, we take the axiomatic approach for such a model to derive appropriate (allocation) rules.[2] The main contribution of this paper is, actually, to explore in that setting the implications of axioms formalizing the principle of *monotonicity*, which had not been explored in our previous work.

Monotonicity is a general principle of fair division with a long tradition within the economics and operations research literature. Early instances are Megiddo (1974), Kalai and Smorodinsky (1975), Kalai (1977), Thomson and Myerson (1980), Young (1985, 1987, 1988), Roemer (1986), Chun and Thomson (1988), Moulin and Thomson (1988), or Thomson (1999). It states that when the underlying data of a problem change in a specific way, the solution should change accordingly. A typical formulation is as follows. Let P and P' be two problems such that the situation of agent i at P is "better" than at P'. Then, the allocation for agent i at P should not be worse than at P'. Depending on the specific meaning of "better", various *monotonicity* axioms could be defined. We shall consider several meanings in our setting, giving rise to the following specific axioms:

Aggregate monotonicity: when the total audience of the tournament increases.

Overall monotonicity: when the audiences of all games in the tournament increase.

Pairwise monotonicity: when the aggregate audience of the game(s) played by any pair of teams increases.

Team monotonicity: when the audiences of the games played by a certain team increase.

Weak team monotonicity: when the audiences of the games played by a certain team increase and the rest of audiences remain the same.

Reciprocal monotonicity: when the audiences of the games not played by a certain team decrease and the audiences of such a team remain the same.

We shall explore the implications of each of the above axioms, in combination with two other basic axioms: *equal treatment of equals* (teams with the same audiences should receive the same), and *additivity* (the rule should be additive on the audiences).

Three focal rules exist for this model. The *uniform* rule (U) divides the total audience of the tournament equally among all teams. The *equal-split* rule (ES) is defined in two steps. First,

[2]Some recent papers also take the axiomatic approach to the related problem of prize/revenue allocation in sports competitions (e.g., Dietzenbacher and Kondratev, 2020; Petróczy and Csató, 2020).

the audience of each game is divided equally among the two teams playing such a game. Second, each team receives the sum over the games played by this particular team. *Concede-and-divide* (CD) is defined through a three-step procedure. First, the number of fans of each team is estimated.[3] Second, the audience of each game is divided by assigning to each team its number of fans and dividing the rest of the audience equally among both teams. Third, each team receives the sum over the games played by this particular team. Convex combinations of the three mentioned rules give rise to several natural families of rules compromising among them. The family of EC rules comprises the convex combinations of rules EC and CD (namely, $\lambda EC + (1-\lambda)\, CD$ with $\lambda \in [0,1]$). Similarly, the family of UC rules is made of the convex combinations of rules U and CD, whereas the family of UE rules is made of the convex combinations of rules U and ES.

Our results, summarized next, provide characterizations for some of the previous rules and families, as well as several extensions of them, when combining the *monotonicity* axioms with the basic axioms described above. More precisely, we show that the *uniform* rule is the unique rule satisfying *aggregate monotonicity*, whereas the *equal-split* rule is the unique rule satisfying *team monotonicity* (as a matter of fact, *additivity* is not needed for these results). And a rule satisfies *overall monotonicity* or *pairwise monotonicity* if and only if it is a certain linear (but not necessarily convex) combination of both rules. A rule satisfies *weak team monotonicity* if and only if it is a certain linear (but not necessarily convex) combination of the *uniform* rule and *concede-and-divide*. Finally, a rule satisfies *reciprocal monotonicity* if and only if it is a certain linear (but not necessarily convex) combination of the *equal-split* rule and *concede-and-divide*.

We can infer from the summary of results just presented that *monotonicity* axioms become a powerful tool to uncover the structure of the problem of sharing the revenues from broadcasting sports leagues. This is similar to what happens in some other related problems. Beyond the classical references mentioned above for the use of *monotonicity*, there have been recent instances in which these axioms have characterized rules (or families of rules) in related problems, such as the ones just mentioned, as well as bargaining problems, or TU games, among others (e.g., Tijs et al. 2006; Casajus and Huettner, 2013, 2014; Calleja and Llerena, 2017; Bergantiños et al. 2020; Calleja et al. 2021; Gaertner and Xu, 2020; Csató and Petróczy, 2021; Moreno-Ternero and Vidal-Puga, 2021).

[3]See Bergantiños and Moreno-Ternero (2020a) for a more detailed explanation of this step.

Our results also highlight the normative appeal of the *equal-split* rule, as it exhibits a pivotal behavior with respect to the monotonicity axioms: it satisfies all of the axioms considered in this paper (except for one) and it separates the rules satisfying somewhat complementary axioms.

Finally, we stress that our broadcasting problem studied here is a specific resource allocation problem, akin to well-known problems already analyzed in the game-theory literature. Instances are airport problems (e.g., Littlechild and Owen, 1973), bankruptcy problems (e.g., O'Neill, 1982; Thomson, 2019), telecommunications problems (e.g., van den Nouweland et al. 1996), museum pass problems (e.g., Ginsburgh and Zang, 2003; Bergantiños and Moreno-Ternero, 2015), cost sharing in minimum cost spanning tree problems (e.g., Bergantiños and Vidal-Puga, 2007), cake-cutting (e.g., Segal-Halevi and Sziklai, 2018, 2019), or labelled network games (e.g., Algaba et al. 2019).

The rest of the paper is organized as follows. In Section 2, we introduce the model, rules, and axioms. In Section 3, we present the characterization results we obtain. We conclude in Section 4. For a smooth passage, we defer to the Appendix the proofs that our results are tight.

2 The model

We consider the model introduced by Bergantiños and Moreno-Ternero (2020a). Let N describe a finite set of teams. Its cardinality is denoted by n. We assume $n \geq 3$. For each pair of teams $i, j \in N$, we denote by a_{ij} the broadcasting audience (number of viewers) for the game played by i and j at i's stadium. We use the notational convention that $a_{ii} = 0$ for each $i \in N$. Let $A \in \mathcal{A}_{n\times n}$ denote the resulting matrix of broadcasting audiences generated in the whole tournament involving the teams within N. Each matrix $A \in \mathcal{A}_{n\times n}$ with zero entries in the diagonal will thus represent a *broadcasting problem* and we shall refer to the set of broadcasting problems as $\mathcal{P}$.[4]

Let $\alpha_i(A)$ denote the total audience achieved by team i, i.e.,

$$\alpha_i(A) = \sum_{j\in N} (a_{ij} + a_{ji}).$$

When no confusion arises, we write α_i instead of $\alpha_i(A)$.

[4]As the set N will be fixed throughout our analysis, we shall not explicitly consider it in the description of each broadcasting problem .

For each $A \in \mathcal{A}_{n\times n}$, let $||A||$ denote the total audience of the tournament. Namely,

$$||A|| = \sum_{i,j\in N} a_{ij} = \frac{1}{2}\sum_{i\in N} \alpha_i.$$

Without loss of generality, we normalize the revenue generated from each viewer to 1 (to be interpreted as the "pay per view" fee). Thus, we sometimes refer to $\alpha_i(A)$ by the *claim* of team i and to $||A||$ as the total revenue.

2.1 Rules

A (sharing) **rule** is a mapping that associates with each broadcasting problem the list of the amounts the teams get from the total revenue. Formally, $R : \mathcal{P} \to \mathbb{R}^n$ is such that, for each $A \in \mathcal{P}$,

$$\sum_{i\in N} R_i(A) = ||A||.$$

The following three rules have been highlighted as focal for the broadcasting problem (e.g., Bergantiños and Moreno-Ternero, 2020a; 2020b).

The *uniform rule* divides equally among all teams the overall audience of the whole tournament. Formally,

Uniform rule, U: for each $A \in \mathcal{P}$, and each $i \in N$,

$$U_i(A) = \frac{||A||}{n}.$$

The *equal-split rule* divides the audience of each game equally, among the two participating teams. Formally,

Equal-split rule, ES: for each $A \in \mathcal{P}$, and each $i \in N$,

$$ES_i(A) = \frac{\alpha_i}{2}.$$

Concede-and-divide compares the performance of a team with the average performance of the other teams. Formally,

Concede-and-divide, CD: for each $A \in \mathcal{P}$, and each $i \in N$,

$$CD_i(A) = \alpha_i - \frac{\sum\limits_{j,k\in N\setminus\{i\}} (a_{jk} + a_{kj})}{n-2} = \frac{(n-1)\,\alpha_i - ||A||}{n-2}.$$

The following family of rules (e.g., Bergantiños and Moreno-Ternero, 2020c) encompasses the above three rules.

UC-family of rules $\{UC^{\lambda}\}_{\lambda\in[0,1]}$: for each $\lambda \in [0,1]$, each $A \in \mathcal{P}$, and each $i \in N$,

$$UC_i^{\lambda}(A) = (1-\lambda)U_i(A) + \lambda CD_i(A).$$

Equivalently,

$$UC_i^{\lambda}(A) = (1-\lambda)\frac{||A||}{n} + \lambda\frac{(n-1)\,\alpha_i - ||A||}{n-2}.$$

At the risk of stressing the obvious, note that, when $\lambda = 0$, UC^{λ} coincides with the *uniform* rule, whereas, when $\lambda = 1$, UC^{λ} coincides with *concede-and-divide*. That is, $UC^0 \equiv U$ and $UC^1 \equiv CD$. Bergantiños and Moreno-Ternero (2020a) prove that for each $A \in \mathcal{P}$,

$$ES(A) = \frac{n}{2\,(n-1)}U(A) + \frac{n-2}{2\,(n-1)}CD(A).$$

That is, $UC^{\lambda} \equiv ES$, where $\lambda = \frac{n-2}{2(n-1)}$.[5]

Consequently, the *UC-family of rules* can be split in two.

On the one hand, the family of rules compromising between the uniform rule and the *equal-split* rule (e.g., Bergantiños and Moreno-Ternero, 2020c). Formally,

UE-family of rules $\{UE^{\beta}\}_{\beta\in[0,1]}$: for each $\beta \in [0,1]$, each $A \in \mathcal{P}$, and each $i \in N$,

$$UE_i^{\beta}(A) = (1-\beta)U_i(A) + \beta ES_i(A).$$

On the other hand, the family of rules compromising between the equal-split rule and *concede-and-divide* (e.g., Bergantiños and Moreno-Ternero, 2020c, 2021). Formally,

EC-family of rules $\{EC^{\gamma}\}_{\gamma\in[0,1]}$: for each $\gamma \in [0,1]$, each $A \in \mathcal{P}$, and each $i \in N$,

$$EC_i^{\gamma}(A) = (1-\gamma)ES_i(A) + \gamma CD_i(A).$$

As Figure 1 illustrates, the family of UC rules is indeed the union of the family of UE rules and EC rules. Note that $UE^0 \equiv UC^0 \equiv U$, $EC^1 \equiv UC^1 \equiv CD$, whereas $ES \equiv UE^1 \equiv EC^0 \equiv UC^{\frac{n-2}{2(n-1)}}$ is the unique rule belonging to both families.

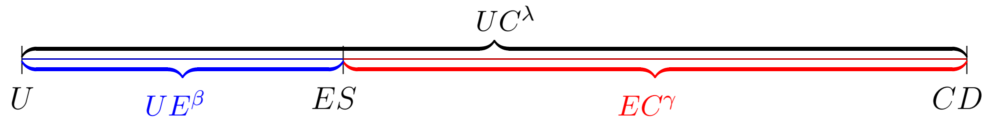

Figure 1. Illustration of the three families of rules.

[5]Note that λ approaches 0.5 (from below) for an arbitrarily large n.

We now present a generalization of the UC rules obtained by considering any linear (but not necessarily convex) combination between U and CD. Formally,

GUC-family of rules $\left\{GUC^{\lambda}\right\}_{\lambda\in\mathbb{R}}$: for each $\lambda \in \mathbb{R}$, each $A \in \mathcal{P}$, and each $i \in N$,

$$GUC_i^{\lambda}(A) = (1-\lambda)U_i(A) + \lambda CD_i(A).$$

Note that we could similarly obtain generalizations of the EC and UE rules, giving rise to the same generalized family. Formally,

$$\left\{GUC^{\lambda}\right\}_{\lambda\in\mathbb{R}} \equiv \left\{GEC^{\lambda}\right\}_{\lambda\in\mathbb{R}} \equiv \left\{GUE^{\lambda}\right\}_{\lambda\in\mathbb{R}}.$$

2.2 Basic axioms

We first present two basic axioms that will be used throughout the paper, together with each of the *monotonicity* axioms.

The first axiom says that if two teams generate the same audiences, then they should receive the same amount. Formally,

Equal treatment of equals (ETE). For each $A \in \mathcal{P}$, and each pair $i, j \in N$ such that $a_{ik} = a_{jk}$, and $a_{ki} = a_{kj}$, for each $k \in N \setminus \{i, j\}$,

$$R_i(A) = R_j(A).$$

The second axiom says that revenues should be additive on A. Formally,

Additivity (ADD). For each pair A and $A' \in \mathcal{P}$,

$$R(A + A') = R(A) + R(A').$$

The axiom of *additivity* has an interesting implication, which will be used in most of the ensuing results. More precisely, for each pair $i, j \in N$, with $i \neq j$, let $\mathbf{1}^{ij}$ denote the matrix with the following entries:

$$\mathbf{1}_{kl}^{ij} = \begin{cases} 1 & \text{if } (k,l) = (i,j) \\ 0 & \text{otherwise.} \end{cases}$$

Then, if R satisfies *additivity*,

$$R_i(A) = \sum_{j,k\in N: j\neq k} a_{jk} R_i\left(\mathbf{1}^{jk}\right), \qquad (1)$$

for each $A \in \mathcal{P}$ and each $i \in N$.

2.3 Monotonicity axioms

Our first *monotonicity* axiom says that if the overall audience in a tournament is higher than in another, then no team can lose from it. Formally,

Aggregate monotonicity (AM). For each pair A and $A' \in \mathcal{P}$ and each $i \in N$,

$$||A|| \leq ||A'|| \quad \Rightarrow \quad R_i(A) \leq R_i(A').$$

The next axiom says that the rule should be monotonic on (the entries of) A. Formally,

Overall monotonicity (OM). For each pair A and $A' \in \mathcal{P}$ and each $i \in N$,

$$a_{jk} \leq a'_{jk} \text{ for each } j,k \in N \quad \Rightarrow \quad R_i(A) \leq R_i(A').$$

Pairwise monotonicity says that if the aggregate audience of the game(s) played by any pair of teams increases, then no team can be worse off. Formally,

Pairwise monotonicity (PM). For each pair A and $A' \in \mathcal{P}$ and each $i \in N$,

$$a_{kj} + a_{jk} \leq a'_{kj} + a'_{jk} \text{ for each } j,k \in N \quad \Rightarrow \quad R_i(A) \leq R_i(A').$$

Another notion is the one requiring that a team does not suffer if it increases its audience. Formally,

Team monotonicity (TM). For each pair A and $A' \in \mathcal{P}$ and each $i \in N$,

$$\left.\begin{array}{l} a_{ij} \leq a'_{ij} \text{ for all } j \in N \setminus \{i\} \text{ and} \\ a_{ji} \leq a'_{ji} \text{ for all } j \in N \setminus \{i\} \end{array}\right\} \quad \Rightarrow \quad R_i(A) \leq R_i(A').$$

The previous axiom can be naturally weakened adding the proviso that the rest of the audiences do not change. Formally,

Weak team monotonicity (WTM). For each pair A and $A' \in \mathcal{P}$ and each $i \in N$,

$$\left.\begin{array}{l} a_{ij} \leq a'_{ij} \text{ for all } j \in N \setminus \{i\} \text{ and} \\ a_{ji} \leq a'_{ji} \text{ for all } j \in N \setminus \{i\} \\ a_{jk} = a'_{jk} \text{ when } i \notin \{j,k\} \end{array}\right\} \quad \Rightarrow \quad R_i(A) \leq R_i(A').$$

The last axiom says that if the audiences of all games not involving team i increase, whereas the rest remain the same, then team i cannot be better off.

Reciprocal monotonicity (RM). For each pair A and $A' \in \mathcal{P}$ and each $i \in N$,

$$\left.\begin{array}{l} a_{ij} = a'_{ij} \text{ for all } j \in N \setminus \{i\} \text{ and} \\ a_{ji} = a'_{ji} \text{ for all } j \in N \setminus \{i\} \\ a_{jk} \leq a'_{jk} \text{ when } i \notin \{j,k\} \end{array}\right\} \Rightarrow \quad R_i(A) \geq R_i(A').$$

The next proposition, whose straightforward proof we omit, summarizes the relations between the axioms introduced above.[6]

Proposition 1 *The following implications among monotonicity axioms hold:*

$$AM \Rightarrow PM \Rightarrow OM \Rightarrow WTM \Leftarrow TM$$

3 Characterization results

In this section we present several characterizations using the axioms introduced above. We combine each of the *monotonicity* axioms with the pair of basic axioms (*equal treatment of equals* and *additivity*) and we characterize the set of rules satisfying the three axioms (in some cases, *additivity* will be redundant as the combination of *equal treatment of equals* and the *monotonicity* axiom will suffice to characterize a rule).

3.1 Aggregate Monotonicity

Our first result states that *aggregate monotonicity* and *equal treatment of equals* characterize the uniform rule (without needing *additivity*).

Theorem 1 *A rule satisfies equal treatment of equals and aggregate monotonicity if and only if it is the uniform rule.*

Proof. It is straightforward to show that the *uniform* rule satisfies the two axioms. Conversely, let R be a rule satisfying the two axioms. Let $A \in \mathcal{P}$. Let A^e denote the resulting matrix from A after splitting all its entries equally. More precisely,

$$A^e_{ij} = \begin{cases} \frac{||A||}{(n-1)n} & \text{if } i \neq j \\ 0 & \text{otherwise.} \end{cases}$$

[6] $A \Rightarrow B$ means that if a rule satisfies property A, then it also satisfies property B.

Notice that $||A^e|| = ||A||$. By *aggregate monotonicity*, $R(A) = R(A^e)$. Now, by *equal treatment of equals*,

$$R_k(A^e) = \frac{||A||}{n} = U_k(A),$$

for each $k \in N$, which concludes the proof. ■

3.2 Team Monotonicity

The next result states that replacing *aggregate monotonicity* by *team monotonicity* at Theorem 1, the *equal-split* rule is characterized (instead of the uniform rule).

Theorem 2 *A rule R satisfies equal treatment of equals and team monotonicity if and only if it is the equal-split rule.*

Proof. It is straightforward to show that the *equal-split* rule satisfies both axioms. Conversely, let R be a rule satisfying the two axioms. We make use of the following claim, whose straightforward proof we omit.

Claim. Let A, A', and $i \in N$ be such that for all $j \in N \setminus \{i\}$, $a_{ij} = a'_{ij}$ and $a_{ji} = a'_{ji}$. By *team monotonicity*, $R_i(A) = R_i(A')$.

We now prove that R coincides with ES by induction on

$$m(A) = |\{\{i, j\} \subset N : a_{ij} + a_{ji} > 0\}|.$$

If $m(A) = 0$, then $A = 0$. By *equal treatment of equals*, for all $i \in N$, $R_i(0) = 0 = ES_i(0)$.

If $m(A) = 1$, then there exist $i, j \in N$ such that $a_{kl} = 0$ when $\{k, l\} \neq \{i, j\}$. By the claim, for each $l \in N \setminus \{i, j\}$, $R_l(A) = R_l(0) = 0 = ES_l(A)$.

By *equal treatment of equals* $R_i(A) = R_j(A)$. Thus, for each $k \in \{i, j\}$, $R_k(A) = \frac{a_{ij}+a_{ji}}{2} = ES_k(A)$.

Assume that $R(A) = ES(A)$ when $m(A) \leq m$ with $m \geq 1$ and we prove that $R(A) = ES(A)$ when $m(A) = m + 1$. As $m \geq 1$ we can find $\{i^1, j^1\}$ and $\{i^2, j^2\}$ such that $\{i^1, j^1\} \neq \{i^2, j^2\}$, $a_{i^1 j^1} + a_{j^1 i^1} > 0$ and $a_{i^2 j^2} + a_{j^2 i^2} > 0$.

Let A^1 be obtained from A by making 0 the audiences of the games played between teams i^1 and j^1. Namely, $a^1_{i^1 j^1} = 0$, $a^1_{j^1 i^1} = 0$ and $a^1_{ij} = a_{ij}$ otherwise. Let $k \in N \setminus \{i^1, j^1\}$. By the claim, $R_k(A) = R_k(A^1)$. As $m(A^1) = m$, $R_k(A^1) = ES_k(A^1)$. Obviously, $ES_k(A^1) = ES_k(A)$. Thus, $R_k(A) = ES_k(A)$.

Let A^2 defined in a similar way to A^1. If we proceed with A^2 as with A^1 we obtain that for each $k \in N \setminus \{i^2, j^2\}$, $R_k(A) = ES_k(A)$.

As $\{i^1, j^1\} \neq \{i^2, j^2\}$ and $R_k(A) = ES_k(A)$ for all $k \in (N \setminus \{i^1, j^1\}) \cup (N \setminus \{i^2, j^2\})$ we deduce that $R_k(A) = ES_k(A)$ for all $k \in N$. $\blacksquare$

3.3 Weak Team Monotonicity

The previous two results do not make use of *additivity*, although the characterized rules satisfy this axiom. This implies that adding the axiom would not change the characterization. In particular, the *equal-split* rule is the only rule that satisfies *equal treatment of equals*, *additivity* and *team monotonicity*. The next result states the effect of weakening *team monotonicity* therein. It turns out that a wide range of generalized UC-rules (including the whole UC-family) are characterized by those axioms.

Theorem 3 *A rule R satisfies equal treatment of equals, additivity and weak team monotonicity if and only if $R \in \left\{GUC^{\lambda} : \lambda \geq \frac{-1}{n-1}\right\}$.*

Proof. It is not difficult to show that both the *uniform rule* and *concede-and-divide* satisfy all the axioms in the statement. It follows from there that all the members of the generalized *UC-family of rules* satisfy *additivity* and *equal treatment of equals*. As for weak *team monotonicity*, let A, A' and $i \in N$ be as in its definition. By (1),

$$\begin{aligned} UC_i^{\lambda}(A) &= \sum_{j,k \in N : i \in \{j,k\}} a_{jk} UC_i^{\lambda}\left(\mathbf{1}^{jk}\right) + \sum_{j,k \in N \setminus \{i\}} a_{jk} UC_i^{\lambda}\left(\mathbf{1}^{jk}\right) \text{ and} \\ UC_i^{\lambda}(A') &= \sum_{j,k \in N : i \in \{j,k\}} a'_{ij} UC_i^{\lambda}\left(\mathbf{1}^{jk}\right) + \sum_{j,k \in N \setminus \{i\}} a_{jk} UC_i^{\lambda}\left(\mathbf{1}^{jk}\right). \end{aligned}$$

Thus, $UC_i^{\lambda}(A) \leq UC_i^{\lambda}(A')$ provided $0 \leq UC_i^{\lambda}\left(\mathbf{1}^{jk}\right) = (1-\lambda)\frac{1}{n} + \lambda$ for each $j, k \in N$ with $i \in \{j, k\}$, which is precisely equivalent to $\lambda \geq \frac{-1}{n-1}$.

Conversely, let R be a rule satisfying the three axioms.

Let $k \in N$. By *additivity*,

$$R_k(A) = \sum_{i,j \in N : i \neq j} a_{ij} R_k\left(\mathbf{1}^{ij}\right).$$

By *equal treatment of equals*, for each pair $k, l \in N \setminus \{i, j\}$, $R_i\left(\mathbf{1}^{ij}\right) = R_j\left(\mathbf{1}^{ij}\right) = x^{ij}$, and $R_k\left(\mathbf{1}^{ij}\right) = R_l\left(\mathbf{1}^{ij}\right) = z^{ij}$. As $\sum_{k \in N} R_j\left(\mathbf{1}^{ij}\right) = \left|\left|\mathbf{1}^{ij}\right|\right| = 1$, we deduce that

$$z^{ij} = \frac{1 - 2x^{ij}}{n-2}.$$

Let $k \in N \setminus \{i,j\}$. By *additivity*, $R_j\left(\mathbf{1}^{ij}+\mathbf{1}^{ik}\right)=x^{ij}+z^{ik}$, and $R_k\left(\mathbf{1}^{ij}+\mathbf{1}^{ik}\right)=z^{ij}+x^{ik}$. By *equal treatment of equals*, $R_j\left(\mathbf{1}^{ij}+\mathbf{1}^{ik}\right)=R_k\left(\mathbf{1}^{ij}+\mathbf{1}^{ik}\right)$. Thus,

$$\begin{aligned} x^{ij}+\frac{1-2x^{ik}}{n-2} &= x^{ik}+\frac{1-2x^{ij}}{n-2} \Leftrightarrow \\ (n-2)\,x^{ij}+1-2x^{ik} &= (n-2)\,x^{ik}+1-2x^{ij} \Leftrightarrow \\ x^{ij} &= x^{ik}. \end{aligned}$$

Therefore, there exists $x \in \mathbb{R}$ such that for each $\{i,j\} \subset N$,

$$\begin{aligned} R_i\left(\mathbf{1}^{ij}\right) &= R_j\left(\mathbf{1}^{ij}\right)=x, \text{ and} \\ R_l\left(\mathbf{1}^{ij}\right) &= \frac{1-2x}{n-2} \text{ for each } l \in N \setminus \{i,j\}. \end{aligned}$$

Let $\lambda=\frac{nx-1}{n-1}$. Then,

$$GUC_k^{\lambda}\left(\mathbf{1}^{ij}\right)=(1-\lambda)U_k\left(\mathbf{1}^{ij}\right)+\lambda CD_k\left(\mathbf{1}^{ij}\right)=\begin{cases} (1-\lambda)\frac{1}{n}+\lambda=x & \text{if } k=i,j \\ (1-\lambda)\frac{1}{n}-\lambda\frac{1}{n-2}=\frac{1-2x}{n-2} & \text{otherwise.} \end{cases}$$

Thus, $GUC^{\lambda}\left(\mathbf{1}^{ij}\right)=R\left(\mathbf{1}^{ij}\right)$. As GUC^{λ} and R satisfy *additivity*, we deduce from here that $GUC^{\lambda}(A)=R(A)$, for each $A \in \mathcal{P}$.

Finally, by *weak team monotonicity*, $x=R_i\left(\mathbf{1}^{ij}\right) \geq R_i(0)=0$, where the last equality follows by *additivity*.

Thus, $\lambda \in \left[-\frac{1}{n-1},\infty\right)$, which concludes the proof. ■

3.4 Overall Monotonicity and Pairwise Monotonicity

As stated in the next result, if we replace *weak team monotonicity* in the previous result by either *overall monotonicity* or *pairwise monotonicity* we have the same effect. Namely, within the rules in the family characterized in Theorem 3, only those "to the left" of the *equal-split* rule (see Figure 1) survive. This implies that the whole UE-family of rules is included, whereas the whole EC-family of rules is excluded.

Theorem 4 *A rule R satisfies equal treatment of equals, additivity, and overall monotonicity or pairwise monotonicity, if and only if* $R \in \left\{GUC^{\lambda}: -\frac{1}{n-1} \leq \lambda \leq \frac{n-2}{2(n-1)}\right\}$.

Proof. As mentioned above, all the members of the generalized *UC-family of rules* satisfy *additivity* and *equal treatment of equals*. As for *pairwise monotonicity* (which implies *overall* *monotonicity*), let A, A' and $i \in N$ be as in its definition. By (1),

$$\begin{aligned} UC_i^{\lambda}(A) &= \sum_{j \in N \setminus \{i\}} UC_i^{\lambda}\left(a_{ij}\mathbf{1}^{ij} + a_{ji}\mathbf{1}^{ji}\right) + \sum_{j,k \in N \setminus \{i\}} UC_i^{\lambda}\left(a_{jk}\mathbf{1}^{jk} + a_{kj}\mathbf{1}^{kj}\right) \text{ and} \\ UC_i^{\lambda}(A') &= \sum_{j \in N \setminus \{i\}} UC_i^{\lambda}\left(a'_{ij}\mathbf{1}^{ij} + a'_{ji}\mathbf{1}^{ji}\right) + \sum_{j,k \in N \setminus \{i\}} UC_i^{\lambda}\left(a'_{jk}\mathbf{1}^{jk} + a'_{kj}\mathbf{1}^{kj}\right). \end{aligned}$$

Then, it is enough to prove that

$$UC_i^{\lambda}\left(a_{ij}\mathbf{1}^{ij} + a_{ji}\mathbf{1}^{ji}\right) \leq UC_i^{\lambda}\left(a'_{ij}\mathbf{1}^{ij} + a'_{ji}\mathbf{1}^{ji}\right),$$

for each pair $j \in N \setminus \{i\}$, and

$$UC_i^{\lambda}\left(a_{jk}\mathbf{1}^{jk} + a_{kj}\mathbf{1}^{kj}\right) \leq UC_i^{\lambda}\left(a'_{jk}\mathbf{1}^{jk} + a'_{kj}\mathbf{1}^{kj}\right),$$

for each pair $j, k \in \setminus \{i\}$, with $j \neq k$.

Let $j \in N \setminus \{i\}$. Then,

$$UC_i^{\lambda}\left(a_{ij}\mathbf{1}^{ij} + a_{ji}\mathbf{1}^{ji}\right) = (a_{ij} + a_{ji})\left((1-\lambda)\frac{1}{n} + \lambda\right).$$

As $-\frac{1}{n-1} \leq \lambda$, we have $(1-\lambda)\frac{1}{n} + \lambda \geq 0$. Then,

$$\begin{aligned} UC_i^{\lambda}\left(a_{ij}\mathbf{1}^{ij} + a_{ji}\mathbf{1}^{ji}\right) &= (a_{ij} + a_{ji})\left((1-\lambda)\frac{1}{n} + \lambda\right) \\ &\leq (a'_{ij} + a'_{ji})\left((1-\lambda)\frac{1}{n} + \lambda\right) \\ &= UC_i^{\lambda}\left(a'_{ij}\mathbf{1}^{ij} + a'_{ji}\mathbf{1}^{ji}\right). \end{aligned}$$

Let $j, k \in N \setminus \{i\}$. Then,

$$UC_i^{\lambda}\left(a_{jk}\mathbf{1}^{jk} + a_{kj}\mathbf{1}^{kj}\right) = (a_{jk} + a_{kj})\left((1-\lambda)\frac{1}{n} + \lambda\frac{-1}{n-2}\right).$$

As $\lambda \leq \frac{n-2}{2(n-1)}$, we have $(1-\lambda)\frac{1}{n} + \lambda\frac{-1}{n-2} \geq 0$. then,

$$\begin{aligned} UC_i^{\lambda}\left(a_{jk}\mathbf{1}^{jk} + a_{kj}\mathbf{1}^{kj}\right) &= (a_{jk} + a_{kj})\left((1-\lambda)\frac{1}{n} + \lambda\frac{-1}{n-2}\right) \\ &\leq (a'_{ij} + a'_{ji})\left((1-\lambda)\frac{1}{n} + \lambda\frac{-1}{n-2}\right) \\ &= UC_i^{\lambda}\left(a'_{jk}\mathbf{1}^{jk} + a'_{kj}\mathbf{1}^{kj}\right). \end{aligned}$$

Conversely, let R be a rule satisfying *equal treatment of equals*, *additivity*, and *overall monotonicity* (which is weaker than *pairwise monotonicity*). By an analogous argument to that in the proof of Theorem 3, it follows that, for each $\{i, j\} \subset N$,

$$R_k\left(\mathbf{1}^{ij}\right) = GUC_k^{\lambda}\left(\mathbf{1}^{ij}\right) = (1-\lambda)U_k\left(\mathbf{1}^{ij}\right) + \lambda CD_k\left(\mathbf{1}^{ij}\right) = \begin{cases} (1-\lambda)\frac{1}{n} + \lambda = x & \text{if } k = i, j \\ (1-\lambda)\frac{1}{n} - \lambda\frac{1}{n-2} = \frac{1-2x}{n-2} & \text{otherwise} \end{cases}$$

where $\lambda = \frac{nx-1}{n-1}$.

By *additivity*, $GUC^{\lambda}(A) = R(A)$, for each $A \in \mathcal{P}$.

Now, by *overall monotonicity* (and *additivity*), $x = R_i\left(\mathbf{1}^{ij}\right) \geq R_i(0) \geq 0$ and $\frac{1-2x}{n-2} = R_l\left(\mathbf{1}^{ij}\right) \geq R_l(0) \geq 0$. Thus, $x \geq 0$ and $\frac{1-2x}{n-2} \geq 0$, which implies that $x \in \left[0, \frac{1}{2}\right]$, or, equivalently, $\lambda \in \left[-\frac{1}{n-1}, \frac{n-2}{2(n-1)}\right]$, which concludes the proof. ■

The following result is a direct consequence of Proposition 1 and Theorem 4.

Proposition 2 *Under ETE and ADD, the following implications hold*

$$AM \Rightarrow PM \iff OM \Rightarrow WTM \Leftarrow TM$$

3.5 Reciprocal Monotonicity

We conclude by providing a characterization of the family of rules satisfying *equal treatment of equals*, *additivity* and *reciprocal monotonicity.* It turns out that within the rules in the family characterized in Theorem 3, only those "to the right" of the *equal-split* rule (see Figure 1) survive. This implies that the whole *EC*-family of rules is included, whereas the whole *UE*-family of rules is excluded. In other words, the families characterized in Theorems 4 and 5 are complementary to obtain the whole family characterized in Theorem 3. The *equal-split* rule is the rule merging both families, which actually renders this rule pivotal (among those satisfying the basic axioms) with respect to the monotonicity properties. It is the only one satisfying all the *monotonicity* axioms we considered in this paper (except for *aggregate monotonicity*). Furthermore, it separates those satisfying *reciprocal monotonicity* from those satisfying *pairwise monotonicity* or *overall monotonicity.*

Theorem 5 *A rule R satisfies equal treatment of equals, additivity, and reciprocal monotonicity if and only if* $R \in \left\{GUC^{\lambda} : \lambda \geq \frac{n-2}{2(n-1)}\right\}$.

Proof. As mentioned above, all the members of the generalized *UC-family of rules* satisfy *additivity* and *equal treatment of equals.* As for *reciprocal monotonicity*, let A, A' and $i \in N$ be as in its definition. By (1),

$$\begin{aligned} UC_i^{\lambda}(A) &= \sum_{j \in N \setminus \{i\}} UC_i^{\lambda}\left(a_{ij}\mathbf{1}^{ij} + a_{ji}\mathbf{1}^{ji}\right) + \sum_{j,k \in N \setminus \{i\}} a_{jk} UC_i^{\lambda}\left(\mathbf{1}^{jk}\right) \text{ and} \\ UC_i^{\lambda}(A') &= \sum_{j \in N \setminus \{i\}} UC_i^{\lambda}\left(a_{ij}\mathbf{1}^{ij} + a_{ji}\mathbf{1}^{ji}\right) + \sum_{j,k \in N \setminus \{i\}} a'_{jk} UC_i^{\lambda}\left(\mathbf{1}^{jk}\right) \end{aligned}$$

Thus, it suffices to show that $UC_i^\lambda\left(\mathbf{1}^{jk}\right) = (1-\lambda)\frac{1}{n} - \lambda\frac{1}{n-2} \leq 0$ for all $j,k \in N\setminus\{i\}$. But this happens precisely when $\lambda \geq \frac{n-2}{2(n-1)}$.

Conversely, let R be a rule satisfying the three axioms. By an analogous argument to that in the proof of Theorem 3, it follows that, for each $\{i,j\} \subset N$,

$$R_k\left(\mathbf{1}^{ij}\right) = GUC_k^\lambda\left(\mathbf{1}^{ij}\right) = (1-\lambda)U_k\left(\mathbf{1}^{ij}\right) + \lambda CD_k\left(\mathbf{1}^{ij}\right) = \begin{cases} (1-\lambda)\frac{1}{n} + \lambda = x & \text{if } k = i,j \\ (1-\lambda)\frac{1}{n} - \lambda\frac{1}{n-2} = \frac{1-2x}{n-2} & \text{otherwise.} \end{cases}$$

where $\lambda = \frac{nx-1}{n-1}$.

By *additivity*, $GUC^\lambda(A) = R(A)$, for each $A \in \mathcal{P}$.

Now, by *reciprocal monotonicity* (and *additivity*), $\frac{1-2x}{n-2} = R_l\left(\mathbf{1}^{ij}\right) \leq R_l(0) = 0$. Thus, $x \geq \frac{1}{2}$, or, equivalently, $\lambda \geq \frac{n-2}{2(n-1)}$, which concludes the proof. ■

We now state characterizations of the *equal-split* rule, which are straightforward corollaries from Theorems 3, 4, and 5, and complement the discussion preceding Theorem 5.

Corollary 1 *The following statements hold:*

1. *A rule satisfies equal treatment of equals, additivity, reciprocal monotonicity and overall monotonicity if and only if it is the equal-split rule.*

2. *A rule satisfies equal treatment of equals, additivity, reciprocal monotonicity and pairwise monotonicity if and only if it is the equal-split rule.*

3.6 Summary

We now summarize all the results obtained in this section. Table 1 yields the parameter ranges for which the corresponding rules within the GUC-family satisfy the axioms in the same row (and are actually characterized by them). Figure 2 illustrates the contents of Table 1. Finally, Table 2 yields the performance of the rules and families mentioned above with respect to the *monotonicity* axioms. For the case of families, we only state YES when the whole family satisfies the corresponding axiom (appearing in the same row).

Axioms	GUC^{λ} where
$ETE + AM$	$\lambda = 0 \Leftrightarrow U$
$ETE + TM$	$\lambda = \frac{n-2}{2(n-1)} \Leftrightarrow ES$
$ETE + ADD + WTM$	$\lambda \geq -\frac{1}{n-1}$
$ETE + ADD + \{OM, PM\}$	$-\frac{1}{n-1} \leq \lambda \leq \frac{n-2}{2(n-1)}$
$ETE + ADD + RM$	$\lambda \geq \frac{n-2}{2(n-1)}$

Table 1: Characterization results

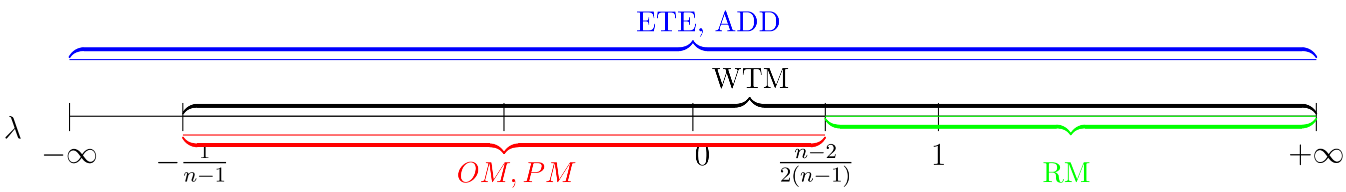

Figure 2. Performance of the rules with respect to the axioms.

	U	ES	CD	UC	EC	UE
AM	✓	✗	✗	✗	✗	✗
TM	✗	✓	✗	✗	✗	✗
WTM	✓	✓	✓	✓	✓	✓
OM	✓	✓	✗	✗	✗	✓
PM	✓	✓	✗	✗	✗	✓
RM	✗	✓	✓	✗	✓	✗

Table 2: Performance of the rules with respect to the axioms.

We conclude providing an example with four teams to illustrate how the different rules perform "in practice".

Let $A(a_{12})$ be the audience matrix given by

$$\begin{pmatrix} 0 & a_{12} & 750 & 700 \\ 850 & 0 & 300 & 250 \\ 770 & 330 & 0 & 200 \\ 680 & 280 & 190 & 0 \end{pmatrix}$$

We consider two cases: $a_{12} = 800$ and $a_{12} = 1200$. Thus, $A(1200)$ is obtained from $A(800)$ by increasing the audience of the game between teams 1 and 2 (at 1's stadium) by 400 (the rest of the audiences remain the same). Note that in both cases the audiences of team 1 are much larger than the audiences of the other three teams (which are somewhat similar, albeit decreasingly ordered). More precisely,

$$\begin{aligned}
||A(800)|| &= 6100, \\
||A(1200)|| &= 6500, \\
\alpha\left(A(800)\right) &= \left(4550, 2810, 2540, 2300\right), \text{ and} \\
\alpha\left(A(1200)\right) &= \left(4950, 3210, 2540, 2300\right).
\end{aligned}$$

We now compute the three focal rules (U, ES, and CD) and a rule in each family UC^{λ}, UE^{β}, and EC^{γ} (we take $\lambda = \beta = \gamma = 0.7$). The results appear in Table 3 below.

Matrix	$A(800)$				$A(1200)$			
Teams	1	1	3	4	1	1	3	4
U	1525	1525	1525	1525	1625	1625	1625	1625
ES	2275	1405	1270	1150	2475	1605	1270	1150
CD	3775	1165	760	400	4175	1565	560	200
$UC^{0.7}$	3100	1273	989.5	737.5	3410	1583	879.5	627.5
$UE^{0.7}$	2050	1441	1346.5	1262.5	2220	1611	1376.5	1292.5
$EC^{0.7}$	3325	1237	913	625	3665	1577	773	485

Table 3: A numerical example.

We briefly discuss the results from Table 3. CD is the most unequal allocation among teams, whereas U is obviously the most equal, and ES is, thus, in between. When we move from $A(800)$ to $A(1200)$, all teams increase by the same amount under U. Under ES, the amounts for teams 1 and 2 increase by the same amount, whereas the allocations for teams 3 and 4 remain the same. Under CD, the amounts for teams 1 and 2 increase, but not by the same amount, whereas the amounts for teams 3 and 4 decrease. The latter is natural because, under CD, the amount received by a team depends on how the audiences of the team are with respect to the audiences of the game not played by the team.

4 Discussion

We have considered several monotonicity axioms for the broadcasting problem of sharing the revenues from broadcasting sports leagues. Monotonicity axioms turned out to be very strong in other domains of problems, sometimes even being incompatible with very elementary requirements of efficiency and fairness (e.g., Thomson 2011). Along these lines, Csató (2019a,b,c) have recently argued that monotonicity generally implies impossibility in models based on pairwise data. In our setting, they are not extremely strong to carry out impossibilities, but they have far-reaching implications.[7] To wit, we have combined them with two basic axioms (*additivity* and *equal treatment of equals*) obtaining as a result several characterizations. In some cases, the characterizations are for single rules, such as the *uniform* rule and the *equal-split* rule. In other cases, the characterizations are for families of rules containing the UE rules, the EC rules, or the UC rules (which comprises the previous two).

We have also shown that the *equal-split* rule exhibits a pivotal behavior with respect to *monotonicity.* Except for one (*aggregate monotonicity*), it satisfies all of the axioms considered in this paper. It also separates the rules satisfying somewhat complementary *monotonicity* axioms. This reinforces the normative appeal of this rule, which had also been singled-out from a game-theoretical perspective (e.g., Bergantiños and Moreno-Ternero, 2020a).

We conclude by discussing how the results in this paper relate to other existing results in the literature.

We characterize the *uniform* rule here with *aggregate monotonicity* and *equal treatment of equals.* The two axioms had also been used separately in alternative characterizations of the same rule (e.g., Theorems 1 and 4 in Bergantiños and Moreno-Ternero, 2020b). The characterization result we present in this paper is reminiscent of other results in alternative contexts. For instance, the characterization of the egalitarian solution for bargaining problems in Kalai (1977) and the characterization of the equal division value for TU games in van den Brink (2007) and Casajus and Huettner (2014).[8]

We characterize the *equal-split* rule here with *team monotonicity* and *equal treatment of*

[7] An exception is, perhaps, obtained from combining our Theorems 1 and 2, which would imply that there exists no symmetric rule satisfying aggregate monotonicity and overall monotonicity together.

[8] Casajus and Huettner (2014) refer to the counterpart axiom of *aggregate monotonicity* in their setting as *grand coalition monotonicity.*

equals. The latter axiom has been used in earlier characterizations of the same rule (e.g., Theorem 1 in Bergantiños and Moreno-Ternero, 2020a; and Theorem 2 in Bergantiños and Moreno-Ternero, 2020b). Alternative characterizations with different axioms also exist (e.g., Theorem 5 in Bergantiños and Moreno-Ternero, 2020b; and Proposition 6 in Bergantiños and Moreno-Ternero, 2020c). The characterization result we present in this paper is also reminiscent of other results in alternative contexts. For instance, the characterization of the Shapley value for TU games in Young (1985).

The two basic axioms of *equal treatment of equals* and *additivity* characterize themselves the family of GUC rules (e.g., Theorem 5 in Bergantiños and Moreno-Ternero, 2020c). Our remaining results in this paper characterize subfamilies of the GUC rules combining the two basic axioms with a *monotonicity* axiom. Other subfamilies have been characterized before with at least one of the basic axioms, but without *monotonicity* axioms. For instance, the family of EC rules (e.g., Theorem 1 in Bergantiños and Moreno-Ternero, 2021; and Proposition 3 in Bergantiños and Moreno-Ternero, 2020c), the family of UC rules (e.g., Theorem 1 in Bergantiños and Moreno-Ternero, 2020c) and the family of UE rules (e.g., Corollary 3 in Bergantiños and Moreno-Ternero, 2020c). The family we characterize in Theorem 3 contains the family of UC rules (and, thus, the families of EC rules and UE rules). The families we characterize in Theorems 3 and 5 contain the family of EC rules. The families we characterize in Theorems 3 and 4 contain the family of UE rules. The family characterized in Theorem 4 has also been previously characterized upon replacing the monotonicity axioms by *non-negativity* (e.g., Theorem 3 in Bergantiños and Moreno-Ternero, 2020c). Those characterization results are also reminiscent of other results in alternative contexts. For instance, van den Brink et al. (2013) and Casajus and Huettner (2014) characterize a set of rules similar to our family of UE rules. In Bergantiños and Moreno-Ternero (2020c), we also characterize other subfamilies of the GUC rules, different from the ones characterized here.

Finally, our broadcasting problem is obviously related to the literature on sharing profits and costs. We treat the broadcasting problem axiomatically, but we acknowledge that one could follow a game-theoretical approach to solve our broadcasting problems. To wit, one could always define a transferable utility game associated to a broadcasting problem and import well-known solutions (values) from the former to suggest rules for the latter. One particular ("optimistic") option is to assume that each coalition of teams can generate the broadcasting revenues its members generated while being members of the league. If we would endorse this option, the Shapley value (e.g., Shapley, 1953) of the resulting game would yield the same allocations as the *equal-split* rule (e.g., Theorem 2 in Bergantiños and Moreno-Ternero, 2020a).[9]

The game-theoretical approach to allocation problems has been used often in related models such as some of those mentioned at the introduction. Nevertheless, although both natural and tractable, it is not without loss of generality. This is why we prefer the (direct) axiomatic approach, which we endorse in our paper, to the (indirect) game-theoretical approach to analyze our broadcasting problems.

Appendix

We show in this appendix that all of our results are tight.

Remark 1 *The axioms used in Theorem 1 are independent.*

(a) The equal-split rule satisfies equal treatment of equals but violates aggregate monotonicity.

(b) Let $\beta = (\beta_i)_{i \in N}$ be such that $\beta_i > 0$ for all $i \in N$ and $\beta_i \neq \beta_j$ when $i \neq j$. Let U^β be such that for each A and each i,

$$U_i^\beta (A) = \frac{\beta_i}{\sum\limits_{j \in N} \beta_j} ||A|| .$$

The rule U^β satisfies aggregate monotonicity but violates equal treatment of equals.

Remark 2 *The axioms used in Theorem 2 are independent.*

(a) The uniform rule satisfies equal treatment of equals but violates team monotonicity.

[9] Proposition 1 in Bergantiños and Moreno-Ternero (2020a) states that, in order to satisfy the core constraints, we should divide the revenue generated by the audience of a game between the two teams playing the game. The *equal-split rule* (or, equivalently, the Shapley value of this game) states that the revenue generated by the audience of a game be divided equally between the two teams playing the game. In that sense, it represents an even compromise (or somewhat middle point) within the bounds the core delimits. Instead of the Shapley value or the core, one might also be interested into another focal concept such as the Nash solution (e.g., Nash, 1950). We have not explored the possibility of deriving rules in our setting associated to it, as one might argue that the allocation process of revenues raised from selling broadcasting rights is centralized and, as such, a bargaining procedure among teams seems unrealistic to solve it. It is, nevertheless, a plausible avenue that we leave for further research.

(b) Let $\beta = (\beta_i)_{i \in N}$ be such that $\beta_i > 0$ for all $i \in N$ and $\beta_i \neq \beta_j$ when $i \neq j$. For each A and each i we define

$$ES_i^{\beta}(A) = \sum_{j \in N \backslash \{i\}} \frac{\beta_i}{\beta_i + \beta_j} (a_{ij} + a_{ji}).$$

The rule ES^{β} satisfies team monotonicity but violates equal treatment of equals.

Remark 3 *The axioms used in Theorem 3 are independent.*

(a) The rule GUC^{λ} with $\lambda < -\frac{1}{n-1}$ satisfies additivity and equal treatment of equals but violates weak team monotonicity.

(b) The rule U^{β}, defined as in Remark 1, satisfies additivity and weak team monotonicity but violates equal treatment of equals.

(c) Given $A \in \mathcal{P}$, let $H(A)$ denote the set of teams with the highest audience. Namely,

$$H(A) = \left\{ i \in N : \alpha_i = \arg\max_{j \in N} \{\alpha_j\} \right\}.$$

Let R^H denote the rule that divides the total audience equally among the agents with the highest audience. Namely,

$$R_i^H(A) = \begin{cases} \frac{||A||}{|H(A)|} & \text{if } i \in H(A) \\ 0 & \text{otherwise.} \end{cases}$$

The rule R^H satisfies equal treatment of equals and weak team monotonicity but violates additivity.

Remark 4 *The axioms used in Theorem 4 are independent.*

(a) The rule GUC^{λ} with $\lambda > \frac{n-2}{2(n-1)}$ satisfies additivity and equal treatment of equals but violates monotonicity.

(b) The rule ES^{β}, defined as in Remark 2, satisfies additivity and overall monotonicity but violates equal treatment of equals.

(c) Given A, we define A^x the matrix obtained from A by reducing the audience of the games played by each pair of teams by x units, with the condition that no game can have a negative audience. Let A^x be such that for each $i, j \in N$, $a_{ij}^x + a_{ji}^x = a_{ij} + a_{ji} - \min\{x, a_{ij} + a_{ji}\}$. We define $A^* = A - A^x$. We now define the rule R^x as $R^x(A) = U(A^*) + ES(A^x)$.[10] The rule R^x satisfies equal treatment of equals and overall monotonicity but violates additivity.

[10]Although, given A and x, several A^x could be defined. $R^x(A)$ does not depend on the choice of A^x.

Remark 5 *The axioms used in Theorem 5 are independent.*

(a) The rule GUC^{λ} *with* $\lambda < \frac{n-2}{2(n-1)}$ *satisfies additivity and equal treatment of equals but violates Reciprocal monotonicity.*

(b) The rule ES^{β}*, defined as in Remark 2, satisfies additivity and reciprocal monotonicity but violates equal treatment of equals.*

(c) The rule R^{H}*, defined as in Remark 3, satisfies equal treatment of equals and reciprocal monotonicity but violates additivity.*

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