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Stochastic Frontier Models with Dependent Errors based on Normal and Exponential Margins

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ABSTRACT

Following the recent work of Gómez–Déniz and Pérez–Rodríguez (2014), this paper extends the results obtained there to the normal–exponential distribution with dependence. Accordingly, the main aim of the present paper is to enhance stochastic production frontier and stochastic cost frontier modelling by proposing a bivariate distribution for dependent errors which allows us to nest the classical models. Closed–form expressions for the error term and technical efficiency are provided. An illustration using real data from the econometric literature is provided to show the applicability of the model proposed.

Keywords: Technical and cost efficiencies; stochastic frontier; marginal distribution; dependence; Sarmanov model.
JEL classification: C01; C13; C21; C51.
MSC2010: 91B70; 62P20; 91G70.

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Modelos de frontera estocástica con errores dependientes basados en márgenes normal y exponencial

RESUMEN

Continuando el reciente trabajo de Gómez–Déniz y Pérez–Rodríguez (2014), el presente artículo extiende los resultados obtenidos a la distribución normal–exponencial con dependencia. En consecuencia, el principal propósito de este artículo es mejorar el modelado de la frontera estocástica tanto de producción como de coste proponiendo para ello una distribución bivariante para errores dependientes que nos permitan encajar los modelos clásicos. Se obtienen las expresiones en forma cerrada para el término de error y la eficiencia técnica. Se ilustra la aplicabilidad del modelo propouesto usando datos reales existentes en la literatura econométrica.

Palabras claves: eficiencias técnica y de coste; frontera estocástica; distribución marginal; dependencia; modelo de Sarmanov.
Clasificación JEL: C01; C13; C21; C51.
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1 Introduction

In general, the methods used for estimating technical and cost efficiency can be considered either parametric or non–parametric. The former involves the estimation of a stochastic production frontier (SPF) or a stochastic cost frontier (SCF) by imposing an explicit functional form and distribution assumption on the data (Aigner *et al.*, 1977; Meeusen and van den Broeck, 1977; Battese and Corra, 1977; Stevenson, 1980; Greene, 1980a, 1980b; Jondron *et al.*, 1982; Lee, 1983; Greene, 1990, 2003; Smith, 2008), where the output of a firm is a function of a set of inputs, plus inefficiency and random error. The second approach is the linear programming technique of data envelopment analysis (DEA), a non–parametric approach which does not impose any assumptions regarding functional form and which does not take into account random error (see Lovell and Schmidt, 1988, for an early survey). Both techniques have advantages and disadvantages; for example, SPF and SCF require the analyst to assume an underlying distribution about the error term, and independence between the inefficiency term and random error. On the other hand, DEA cannot take into account such statistical noise, and efficiency estimates may be biased if the production process is largely characterised by stochastic elements.

Between these two alternatives of modelling, our main interest is based on the the stochastic frontier model in a cross–section framework. The model in this scenario can be written as $y_i = f(x_i;\boldsymbol{\beta}) + \nu_i \pm u_i, \quad i = 1, 2, \ldots, n,\ u_i \geq 0$, where the sign of the last term depends on whether the frontier describes costs (positive) or production (negative). For example, if we assume that $f(x_i;\boldsymbol{\beta})$ takes the log–linear Cobb–Douglas form, then the stochastic production frontier (SPF) model can be written as: $\log y_i = \beta_0 + \sum_{j=1}^{k} \beta_j \log x_{ij} + \nu_i - u_i, \quad i = 1, 2, \ldots, n$, where $\log y_i$ is the natural logarithm of the production of the i-th firm; $\log x_i$ is a $k\times 1$ vector of (natural log transformations of the) input quantities of the i-th firm; $\boldsymbol{\beta}$ is a vector of unknown parameters, and the disturbance term $\varepsilon_i = \nu_i \pm u_i$ (which is asymmetric) is assumed to have two components: one with a strictly non–negative distribution, u_i (which is a non–negative component often referred to as the inefficiency term), and another with a symmetric distribution, ν_i (which is termed the idiosyncratic error). Although it is not an assumption of the model, independence of ν and u makes it easy to obtain the density of ε. The density of ε is then used to conduct maximum likelihood estimation of the model parameters. In addition, it is possible to obtain the conditional density of $u|\varepsilon$ and $E(u|\varepsilon)$. These serve as a basis to obtain estimates for firm–specific inefficiency.

The maximum likelihood method can be used to estimate β and u_i, the variances of the errors and the technical efficiency of each firm. Therefore, distributional assumptions are required for ν_i and u_i. In terms of v_i, and in general, these random variables are assumed to be independently and identically distributed (iid) $N\left(0, \sigma_\nu^2\right)$. On the other hand, in terms of u_i, various assumptions may be made; for example, Meeusen and van den Broeck (1977) assigned the exponential distribution to u_i, Battese and Corra (1977) assumed a half–normal distribution, while Aigner *et al.* (1977) considered both distributions. However, since the half–normal and exponential distributions are both single–parameter specifications with modes at zero, some scepticism has been expressed regarding their generality. Thus, Stevenson (1980) suggested the truncated normal and gamma distribution for u_i. Greene (1980a, 1980b) proposed the gamma distribution, Lee (1983) proposed a four–parameter Pearson family of distributions and Greene (1990, 2003) proposed the two–parameter gamma density as a more general alternative.

More recently, another way to model SPF and SCF are based on dependence of error terms such as Smith (2008) and Wiboonpongse *et al.* (2015) with copulas and El Mehdia and Hafner (2014) and Gómez–Déniz and Pérez–Rodríguez (2014) with closed–form solutions by using bivariate distributions. On the other hand, Tran and Tsionas (2015) and Amsler *et al.* (2016) study the correlation between the inputs and statistical noise or inefficiency. The former one proposes an approach which is based on copula function to directly model the correlation between the endogenous regressors and the composed errors assumed to be independent and identically distributed.

Accordingly, the main aim of the present paper is to enhance SPF and SCF modelling by proposing a closed form of a bivariate distribution for dependent errors which allows us to nest the classical models. In particular, we follow Gómez–Déniz and Pérez–Rodríguez (2014) and extend their results by using Sarmanov's family of distributions (Sarmanov, 1966; Lee, 1996; Gómez–Déniz and Pérez–Rodríguez, 2014; among others) to obtain closed–form expressions for the error term and technical efficiency. More specifically, we built a bivariate dependent SPF and SCF models by using normal and exponential distributions (NE), and thus we construct a general extension of the classical stochastic frontier model with these distributions.

The remainder of this paper is structured as follows. Section 2 introduces a brief note on the Sarmanov family of distributions which is used to estimate the technical (cost) efficiency in a cross–section framework. We analysed one parametric form, deriving in closed–form expression the log likelihood functions and technical (cost) efficiencies, based on the classical pdf distributions, by including the dependence structure. An application of the new model is discussed in Section 3. Finally, the main conclusions drawn are presented in Section 4.

2 Modelling the dependence

In addition to the distributional assumptions on the error terms, ν_i and u_i, in stochastic parametric frontier models, another important characteristic of the above cited models is the independence between them to construct the density and marginal distributions.

The classical stochastic frontier model with normal and exponential assumptions is described by the following stochastic representation: (i) $v_i \sim$ iid $N(0, \sigma_\nu^2)$; (ii) $u_i \sim$ iid exponential with parameter $\sigma_u > 0$; and (iii) u_i and v_i are distributed independently of each other and of the regressors. The probability density functions of v_i and u_i are as follows

$$f_{\sigma_\nu}(\nu) = \frac{1}{\sigma_\nu\sqrt{2\pi}}\, e^{-\frac{\nu^2}{2\sigma_\nu^2}}, \quad f_{\sigma_u}(u) = \frac{1}{\sigma_u}\, e^{-\frac{u}{\sigma_u}},$$

where $-\infty < \nu < \infty$, $\sigma_\nu > 0$, $u > 0$ and $\sigma_u > 0$.

In this case, we have

$$f_{\sigma_u,\sigma_\nu}(\varepsilon) = \frac{1}{\sigma_u}\,\Phi\left(-\frac{\varepsilon}{\sigma_\nu} - \frac{\sigma_\nu}{\sigma_u}\right)\exp\left\{\frac{\varepsilon}{\sigma_u} + \frac{\sigma_\nu^2}{2\sigma_u^2}\right\}, \tag{1}$$

$$f(u|\varepsilon) = \frac{1}{\sqrt{2\pi}\,\sigma_\nu\,\Phi(\widetilde{\mu}/\sigma_\nu)}\,\exp\left\{-\frac{1}{2\sigma_\nu^2}(u-\widetilde{\mu})^2\right\}, \tag{2}$$

where $\widetilde{\mu} = -\varepsilon - \sigma_\nu^2/\sigma_u$.

The marginal $f(\varepsilon)$ is asymmetrically distributed with given by $E(\varepsilon) = -\sigma_u$ and the variance by $var(\varepsilon) = \sigma_u^2 + \sigma_\nu^2$. On the other hand, $u|\varepsilon$ follows a half–normal distribution, $N^+(\widetilde{\mu}, \sigma_\nu^2)$, with mean

$$E(u|\varepsilon) = \widetilde{\mu} + \sigma_\nu\frac{\phi(-\widetilde{\mu}/\sigma_\nu)}{\Phi(-\widetilde{\mu}/\sigma_\nu)} = \sigma_\nu\left(\frac{\phi(A)}{\Phi(-A)} - A\right),$$

and where $A = -\widetilde{\mu}/\sigma_\nu$.

Following Gómez–Déniz and Pérez–Rodríguez (2014), we obtain closed–form expression for the likelihood function and technical efficiency for SPF and SCF likelihoods based on the classical mixture of normal and exponential distributions. We propose a broader, more general and flexible range of dependence, which is also easy to handle, for testing the independence between the inefficiency term and random error (the idiosyncratic component).

This family of distributions is implemented by assuming that $f_1(x_1)$ and $f_2(x_2)$ are univariate probability density functions, with supports defined on $A_1 \subseteq \mathbb{R}$ and $A_2 \subseteq \mathbb{R}$, respectively. Let $\varphi_s(t)$, $s = 1, 2$, be bounded nonconstant functions (the mixing functions) such that

$$\int_{-\infty}^{\infty} \varphi_s(t) f_s(t)\; dt = 0,$$

then the function defined by

$$f(x_1, x_2) = f_1(x_1) f_2(x_2) \left[1 + \omega\, \varphi_1(x_1)\, \varphi_2(x_2)\right] \tag{3}$$

is a bivariate joint density with margins $f_1(x_1)$ and $f_2(x_2)$, provided ω is a real number which satisfies the condition $1 + \omega\, \varphi_1(x_1)\, \varphi_2(x_2) \geq 0$, for all x_1 and x_2. Some methods to obtain the mixing function φ when $f_s(x_s)$, $s = 1, 2$, are members of the natural exponential family of distributions are described in Lee (1996). The Farlie–Gumbel–Morgernstern (FGM) family of copulas can be viewed as a special case of the above–mentioned construction, by setting $\varphi(x_s) = 1 - 2\, F(x_s)$, $s = 1, 2$ and therefore one of the models proposed in Smith (2008). Here $F(\cdot)$ represents the cumulative distribution function of the random variables with pdf $f(\cdot)$.

As we will see this family provide analytical expressions for the marginal pdf of the random variable ε, the conditional pdf of $u|\varepsilon$, $cov(u, \nu)$ and technical efficiency are provided in the SPF model, for the NE model. The classical independent models are derived as a particular case when $\omega = 0$ while $\omega \neq 0$ measures the dependence structure.

As in the classical SPF model, let $\nu = u + \varepsilon$. Using (3) we get

$$f_{\sigma_u,\sigma_\nu,\omega}(u, \varepsilon) = f_{\sigma_u}(u) f_{\sigma_\nu}(u + \varepsilon) \left[1 + \omega\, \varphi_{\sigma_u}(u)\, \varphi_{\sigma_\nu}(u + \varepsilon)\right], \tag{4}$$

by taking $\varphi_{\sigma_u}(u) = e^{-u} - \delta_u(\sigma_u)$, $\varphi_{\sigma_\nu}(\nu) = e^{-\nu^2 - 2\nu} - \delta_\nu(\sigma_\nu)$, with

$$\delta_u(\sigma_u) = \frac{1}{1 + \sigma_u}, \tag{5}$$

$$\delta_\nu(\sigma_\nu) = \frac{1}{\sqrt{1 + 2\sigma_\nu^2}} \exp\left\{\frac{2\sigma_\nu^2}{1 + 2\sigma_\nu^2}\right\}, \tag{6}$$

defines a bivariate distribution of (u, ν) with marginal distributions $f_{\sigma_\nu}(\nu)$ and $f_{\sigma_u}(u)$ as in the classical model and where $\omega_1 \leq \omega \leq \omega_2$, being

$$\omega_1 = \max\left\{\frac{-1}{\delta_u(\sigma_u)\delta_\nu(\sigma_\nu)}, \frac{-1}{(1-\delta_u(\sigma_u))(e-\delta_\nu(\sigma_\nu))}\right\}, \quad (7)$$

$$\omega_2 = \min\left\{\frac{1}{\delta_u(\sigma_u)(e-\delta_\nu(\sigma_\nu))}, \frac{1}{(1-\delta_u(\sigma_u))\delta_\nu(\sigma_\nu)}\right\}. \quad (8)$$

To see this, observe that because $\frac{d}{du}\varphi_{\sigma_u}(u) < 0$ we have that $\varphi_{\sigma_u}(u)$ is a decreasing function on u and the range of variation of $\varphi_{\sigma_u}(u)$ is $(-\delta_u(\sigma_u), 1-\delta_u(\sigma_u))$. In the same way, it is simple to see that $\varphi_{\sigma_\nu}(\nu)$ has a maximum in $\nu = -1$ and therefore the range of variation of $\varphi_{\sigma_\nu}(\nu)$ results $(-\delta_\nu(\sigma_\nu), e-\delta_\nu(\sigma_\nu))$. Now, we have that $f_{\sigma_u,\sigma_\nu,\omega}(u,\nu)$ represents a probability density function if $1+\omega\,\varphi_{\sigma_u}(u)\,\varphi_{\sigma_\nu}(\nu) \geq 0$, and this occurs if

$$\omega \geq \frac{-1}{\varphi_{\sigma_u}(u)\,\varphi_{\sigma_\nu}(\nu)}, \quad \text{for} \quad \varphi_{\sigma_u}(u)\,\varphi_{\sigma_\nu}(\nu) > 0,$$
$$\omega \leq \frac{-1}{\varphi_{\sigma_u}(u)\,\varphi_{\sigma_\nu}(\nu)}, \quad \text{for} \quad \varphi_{\sigma_u}(u)\,\varphi_{\sigma_\nu}(\nu) < 0,$$

from which it is a simple exercise to see that range of ω is given by (ω_1, ω_2).

Although any other mixing functions satisfying that $\int_0^\infty f_{\sigma_u}(u)\varphi_{\sigma_u}(u)\,du = 0$ and that $\int_{-\infty}^\infty f_{\sigma_\nu}(\nu)\varphi_{\sigma_\nu}(\nu)\,d\nu = 0$ can be considered we have chosen the mixing functions above since: (i) The presence of the exponential term which is also present in the probability density function of the normal and exponential distributions facilitates the computations in order to obtain closed–form expressions for the marginal of ε as we will see in the next section; (ii) The square term in the exponential part of $\varphi_{\sigma_\nu}(\nu)$ is important to ensure appropriate bounds for the ω parameter. Dependence assumption is now depending on ω and the Sarmanov family with normal and half normal marginals studied here can also be considered as an extension of the classical Sarmanov family of distributions dealt in Lee (1996) for the normal case.

Some algebra provides the correlation coefficient, which is given by

$$\rho = \frac{2\,\omega\,\sigma_u\,\sigma_\nu}{(1+\sigma_u)^2(1+2\sigma_\nu^2)^{3/2}} \exp\left\{\frac{2\sigma_\nu^2}{1+2\sigma_\nu^2}\right\},$$

This correlation coefficient is bounded by (see Lee, 1996)

$$|\rho| \leq |\omega| \left[E\left(\varphi_{\sigma_u}^2(u)\right) E\left(\varphi_{\sigma_\nu}^2(\nu)\right)\right]^{1/2}.$$

Now, we have the following result which can be used to build the likelihood function for SPF (see Appendix 4 for the SCF model).

Theorem 1 *In the SPF model for NE distributions and assuming dependence, the marginal pdf of ε is given by*

$$
\begin{aligned}
f_{\sigma_u,\sigma_\nu,\omega}(\varepsilon) \;=\; & \frac{\Upsilon^1_{\sigma_u,\sigma_\nu}(\varepsilon)}{\sigma_u\sqrt{1+2\sigma_\nu^2}}\left\{\Phi\left(\frac{\widetilde{\mu}}{\sigma_\nu}\right)\left[\sqrt{1+2\sigma_\nu^2}+\frac{\omega\, e^{\frac{2\sigma_\nu^2}{1+2\sigma_\nu^2}}}{1+\sigma_u}\right]\right. \\
& -\omega\;\Upsilon^2_{\sigma_u,\sigma_\nu}(\varepsilon)\,\Phi\left(\frac{\widetilde{\mu}}{\sigma_\nu}-\sigma_\nu\right) \\
& -\frac{\omega}{1+\sigma_u}\,\Upsilon^3_{\sigma_u,\sigma_\nu}\;\Phi\left(-\frac{\sigma_\nu}{\sigma_u}\frac{1+2\sigma_u}{\sqrt{1+2\sigma_\nu^2}}-\frac{\varepsilon}{\sigma_\nu}\sqrt{1+2\sigma_\nu^2}\right) \\
& \left.+\,\omega\,\Upsilon^4_{\sigma_u,\sigma_\nu}(\varepsilon)\,\Phi\left(-\frac{\sigma_\nu}{\sigma_u}\frac{1+3\sigma_u}{\sqrt{1+2\sigma_\nu^2}}-\frac{\varepsilon}{\sigma_\nu}\sqrt{1+2\sigma_\nu^2}\right)\right\}, \qquad (9)
\end{aligned}
$$

where

$$
\begin{aligned}
\Upsilon^1_{\sigma_u,\sigma_\nu}(\varepsilon) &= \exp\left\{\frac{\varepsilon}{\sigma_u}+\frac{\sigma_\nu^2}{2\sigma_u^2}\right\}, \\
\Upsilon^2_{\sigma_u,\sigma_\nu}(\varepsilon) &= \exp\left\{\varepsilon+\frac{\sigma_\nu^2}{\sigma_u}+\frac{\sigma_\nu^2}{2}+\frac{2\sigma_\nu^2}{1+2\sigma_\nu^2}\right\}, \\
\Upsilon^3_{\sigma_u,\sigma_\nu} &= \exp\left\{\frac{\sigma_\nu^2}{2\sigma_u^2}\frac{4\sigma_u-2\sigma_\nu^2}{1+2\sigma_\nu^2}+\frac{2\sigma_\nu^2}{1+2\sigma_\nu^2}\right\}, \\
\Upsilon^4_{\sigma_u,\sigma_\nu}(\varepsilon) &= \exp\left\{\varepsilon+\frac{\sigma_\nu^2}{2\sigma_u^2}\frac{5\sigma_u^2+6\sigma_u-2\sigma_\nu^2}{1+2\sigma_\nu^2}+\frac{2\sigma_\nu^2}{1+2\sigma_\nu^2}\right\}.
\end{aligned}
$$

Proof: See Appendix 1. ■

Observe that when $\omega = 0$, i.e. the independence case, pdf (9) reduces to (1). Simple computations provide that the mean for the marginal pdf given in (9) is equal to $E(\varepsilon) = -\sigma_u$ while the variance results

$$var(\varepsilon) = \sigma_u^2 + \sigma_\nu^2\left[1-\frac{4\,\omega\,\sigma_u^2}{(1+\sigma_u^2)(1+2\sigma_\nu^2)^{3/2}}e^{\frac{2\sigma_\nu^2}{1+2\sigma_\nu^2}}\right].$$

Figure 1 shows examples of the marginal pdf (9) for different values of the model parameters.

Having obtained the main result of the likelihood function, we now show the conditional distribution of u given ε.

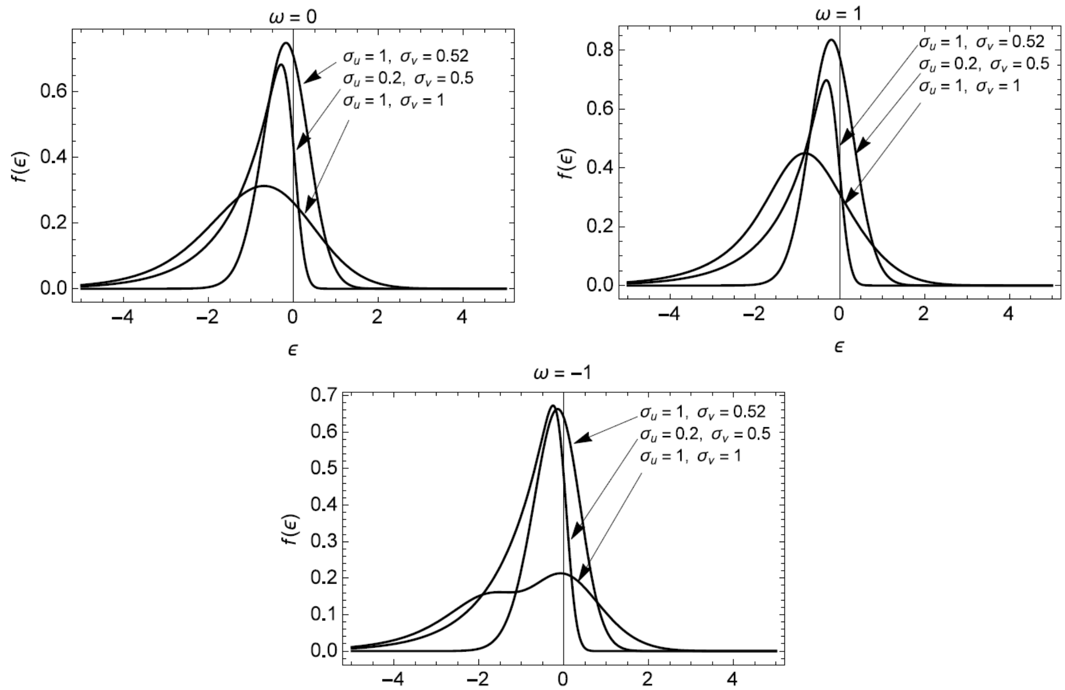

Figure 1: Marginal distribution in the SPF–NE model for selected parameter values.

Proposition 1 *In the SPF model under NE distributions, assuming dependence, the conditional pdf of u given ε is given by*

$$f_{\sigma_u,\sigma_\nu,\omega}(u|\varepsilon) = \frac{f_{\sigma_u,\sigma_\nu}(u|\varepsilon) + \omega\,\Psi^0_{\sigma_u,\sigma_\nu}(u|\varepsilon)}{1 + \omega\,\Psi^1_{\sigma_u,\sigma_\nu}(\varepsilon)}, \tag{10}$$

for $u > 0$, where $f_{\sigma_u,\sigma_\nu}(u|\varepsilon)$ is given in (2) and

$$\Psi^0_{\sigma_u,\sigma_\nu}(u|\varepsilon) = \varphi_{\sigma_u}(u)\,\varphi_{\sigma_\nu}(u+\varepsilon)\,f_{\sigma_u,\sigma_\nu}(u|\varepsilon), \tag{11}$$

$$\begin{aligned}\Psi^1_{\sigma_u,\sigma_\nu}(\varepsilon) = {} & \frac{1}{\sqrt{1+2\sigma_\nu^2}}\,\frac{1}{\Phi\left(\frac{\widetilde{\mu}}{\sigma_\nu}\right)}\left[\Phi\left(\frac{\widetilde{\mu}}{\sigma_\nu}\right)\frac{e^{\frac{2\sigma_\nu^2}{1+2\sigma_\nu^2}}}{1+\sigma_u} - \Upsilon^2_{\sigma_u,\sigma_\nu}(\varepsilon)\,\Phi\left(\frac{\widetilde{\mu}}{\sigma_\nu} - \sigma_\nu\right)\right. \\ & - \frac{1}{1+\sigma_u}\,\Upsilon^3_{\sigma_u,\sigma_\nu}\,\Phi\left(-\frac{\sigma_\nu}{\sigma_u}\frac{1+2\sigma_u}{\sqrt{1+2\sigma_\nu^2}} - \frac{\varepsilon}{\sigma_\nu}\sqrt{1+2\sigma_\nu^2}\right) \\ & \left. + \Upsilon^4_{\sigma_u,\sigma_\nu}(\varepsilon)\,\Phi\left(-\frac{\sigma_\nu}{\sigma_u}\frac{1+3\sigma_u}{\sqrt{1+2\sigma_\nu^2}} - \frac{\varepsilon}{\sigma_\nu}\sqrt{1+2\sigma_\nu^2}\right)\right].\end{aligned} \tag{12}$$

Proof: See Appendix 2. ■

The point estimation for technical efficiency used in this study is the expression given by Battese and Coelli (1988), which is related to the conditional expectation of e^{-u_i}, that is, $E(e^{-u_i}|\varepsilon = \hat{\varepsilon})$ which is given in the next result.[1]

Proposition 2 *The point estimation for the technical efficiency of the i–th producer in the SPF model under NE distributions, assuming dependence, is given by*

$$TE_i = \frac{\Psi^{01}_{\sigma_u,\sigma_\nu,\omega}(\varepsilon_i) + \omega\,\Psi^{02}_{\sigma_u,\sigma_\nu}(\varepsilon_i)}{1 + \omega\,\Psi^{1}_{\sigma_u,\sigma_\nu}(\varepsilon_i)}, \quad i = 1, 2, \ldots, n,$$

where

$$\Psi^{01}_{\sigma_u,\sigma_\nu,\omega}(\varepsilon_i) = [1 + \omega\,\delta_u(\sigma_u)\,\delta_\nu(\sigma_\nu)]\,\frac{\Phi(-A-\sigma_\nu)}{\Phi(-A)}\,e^{\sigma_\nu(A+\sigma_\nu/2)},$$

and

$$\begin{aligned}\Psi^{02}_{\sigma_u,\sigma_\nu}(\varepsilon_i) \quad = \quad & \frac{1}{\Phi(-A)}\left[\frac{1}{\sqrt{1+2\sigma_\nu^2}}\,\Upsilon^{11}_{\sigma_u,\sigma_\nu}(\varepsilon_i)\,\Phi\left(-\frac{A + 2(2+\varepsilon_i)\sigma_\nu}{\sqrt{1+2\sigma_\nu^2}}\right)\right.\\ & -\,\delta_\nu(\sigma_\nu)\,\Upsilon^{22}_{\sigma_u,\sigma_\nu}(\varepsilon)\,\Phi(-A-2\sigma_\nu)\\ & \left.-\,\frac{1}{\sqrt{1+2\sigma_\nu^2}}\,\delta_u(\sigma_u)\,\Upsilon^{33}_{\sigma_u,\sigma_\nu}(\varepsilon_i)\,\Phi\left(-\frac{A+\sigma_\nu(3+2\varepsilon_i)}{\sqrt{1+2\sigma_\nu^2}}\right)\right],\end{aligned}$$

with

$$\begin{aligned}\Upsilon^{11}_{\sigma_u,\sigma_\nu}(\varepsilon_i) \quad &= \quad \exp\left\{-\frac{\varepsilon_i^2 - A\sigma_\nu(4-A\sigma_\nu) - 8\sigma_\nu^2 + 2\varepsilon_i(1-\sigma_\nu(A+2\sigma_\nu))}{1+2\sigma_\nu^2}\right\},\\ \Upsilon^{22}_{\sigma_u,\sigma_\nu}(\varepsilon_i) \quad &= \quad \exp\left\{2\sigma_\nu(A+\sigma_\nu)\right\},\\ \Upsilon^{33}_{\sigma_u,\sigma_\nu}(\varepsilon_i) \quad &= \quad \exp\left\{-\frac{\varepsilon_i^2 - A\sigma_\nu(3-A\sigma_\nu) - 9/2\sigma_\nu^2 + 2\varepsilon_i(1-\sigma_\nu(A+\sigma_\nu))}{1+2\sigma_\nu^2}\right\},\end{aligned}$$

while $\delta_u(\sigma_u)$, $\delta_\nu(\sigma_\nu)$ *and* $\Psi^1_{\sigma_u,\sigma_\nu}(\varepsilon_i)$ *are given in (5), (6) and (12), respectively.*

Proof: See Appendix 3. ■

[1]This estimator is particularly useful when u_i is not close to zero. However, the estimate of technical efficiency is inconsistent because the variation associated with the distribution of $u_i|\varepsilon$ is independent of i (Kumbhakar and Lovell, 2000, p.78)

3 Numerical application

In this section, we use the theoretical results obtained in Section 3 to estimate and test independence between the inefficiency term and the idiosyncratic error, using one empirical framework.

In the example we estimate the proposed model normal–exponential by using data obtained from several bank branches of a large Spanish commercial bank during from January 2011 to December 2014 (monthly data). Specifically, data corresponds to gross operating annual cost as output and the inputs we use are labour and capital prices but also the annual revenues (total income) for each bank branch.

We estimate a log-linear Cobb-Douglas cost function for the 5009 pooled data and without imposing linear homogeneity in the input prices. The estimated model is written as follows

$$\log c_i = \beta_0 + \beta_1 \log l_i + \beta_2 \log k_i + \beta_3 \log y_i + \nu_i + u_i, \quad i = 1, 2, \ldots, 5009,$$

where the variable $\log c_i$ is the natural log–transformed annual operating cost, $\log l_i$ is the natural log-transformed labour price, $\log k_i$ is the natural log-transformed price of capital and $\log y_i$ is the natural log-transformed annual revenue.

The maximum likelihood estimates for the cost frontier NE model and for a sample of n bank branches can be obtained by maximizing the log–likelihood function derived from (17) restricted by (7) and (8), respectively. After some algebra, it is given by

$$\begin{aligned}
\log L \;=\; & \frac{n}{2}\left[\frac{\sigma_\nu^2}{2\sigma_u^2} - 2\log\sigma_u - \log(1+\sigma_\nu^2)\right] + \frac{1}{\sigma_u}\sum_{i=1}^{n}\varepsilon_i \\
& + \sum_{i=1}^{n}\log\left\{\Phi\left(\frac{\widetilde{\widetilde{\mu}}}{\sigma_\nu}\right)\left[\sqrt{1+2\sigma_\nu^2} + \frac{\omega}{1+\sigma_u}\exp\left(\frac{2\sigma_\nu^2}{1+2\sigma_\nu^2}\right)\right]\right. \\
& -\omega\left[\sqrt{1+2\sigma_\nu^2}\,\Xi^2_{\sigma_u,\sigma_\nu}(\varepsilon_i)\,\delta_\nu(\sigma_\nu)\,\Phi\left(\frac{\widetilde{\widetilde{\mu}}}{\sigma_\nu} - \sigma_\nu\right)\right. \\
& + \Xi^3_{\sigma_u,\sigma_\nu}(\varepsilon_i)\,\delta_u(\sigma_u)\,\Phi\left(\frac{\sigma_\nu(2\sigma_u-1)}{\sigma_u\sqrt{1+2\sigma_\nu^2}} + \frac{\varepsilon_i\sqrt{1+2\sigma_\nu^2}}{\sigma_\nu}\right) \\
& \left.\left. -\;\Xi^4_{\sigma_u,\sigma_\nu}(\varepsilon_i)\,\Phi\left(\frac{\sigma_\nu(\sigma_u-1)}{\sigma_u\sqrt{1+2\sigma_\nu^2}} + \frac{\varepsilon_i\sqrt{1+2\sigma_\nu^2}}{\sigma_\nu}\right)\right]\right\},
\end{aligned} \tag{13}$$

where $\Xi^1_{\sigma_u,\sigma_\nu}(\varepsilon_i)$, $\Xi^2_{\sigma_u,\sigma_\nu}(\varepsilon_i)$, $\Xi^3_{\sigma_u,\sigma_\nu}(\varepsilon_i)$ and $\Xi^4_{\sigma_u,\sigma_\nu}(\varepsilon_i)$, $i = 1, \ldots, n$, are given in (18), (19), (20) and (21), respectively.

Table 1 shows three estimation methods applied to the data: ordinary least squares (OLS), unrestricted maximum likelihood (UML) (by taking $\omega = 0$ in (13)) and restricted maximum likelihood (RML) (by using directly (13)).

Table 1: Stochastic cost frontier estimates

Variable	Coeff	t–Stat	Coeff	t–Stat	Coeff	t–Stat
	OLS		***UML–NE***		***RML–NED***	
β_0	0.1013	0.28	0.2744	21.43	0.2220	1.75
β_1	0.3238	9.16	0.2847	29.79	0.2835	21.33
β_2	0.2278	13.29	0.2549	14.14	0.2515	13.90
β_3	0.5072	36.07	0.4666	35.36	0.4709	33.76
σ_u			0.4336	31.95	0.4757	30.52
σ_v			0.2441	10.91	0.2573	11.42
ω					–1.3970	–119.36
ρ					–0.14648	
					(0.0141)	
ρ_S					–0.01246	
					(0.0093)	
$L_{\max}$		–5508.38		–3221.14		–3092.30

Note: Between parenthesis standard error is indicated.

The OLS estimates are compared with those obtained from stochastic frontier models, using the exponential distribution (UML–NE) and normal–exponential with dependence (RML–NED). Also shown is the maximum value of the log–likelihood function ($L_{\max}$), together with some correlation coefficients. It is known that if X and Y are two random variables with cdf $F(x)$ and $G(y)$, respectively, Spearman's coefficient, denoted by ρ_S , is given by $\rho_S = \mathrm{Corr}(F(X), G(Y))$, i.e. the ordinary (Pearson) correlation coefficient of the random variables $F(x)$ and $G(y)$ (see Fredricks and Nelsen, 2007). This coefficient was computed numerically for the analyzed data, and is also shown in the tables, together with ρ, the classical coefficient of correlation. The standard errors of both, Spearman's coefficient and correlation coefficient, appear between parenthesis. The first was calculated by using the expression $\sigma_{\rho_S} = \sqrt{0.437/(n-4)}$ (see Bonnet, 2000).

The second was computed by using the expression $\sigma_\rho = 1/\sqrt{n-3}$, provided also in Bonnet (2000) which produces a similar result than the one obtained by using $\sigma_\rho = (1-\rho^2)/\sqrt{n^2-1}$ (see Dingman and Perry, 1956 for details). As pointed out by Shubina and Lee (2004), the Spearman correlation coefficient for a Sarmanov family of distribution is situated in the interval $[-3/4, 3/4]$.

The model was estimated by using restricted maximum likelihood in two–stages. The first stage is based on the simplex method, a search procedure that requires only function evaluations, not derivatives. To apply simplex, OLS initial values are used for β_0, β_1 and β_2 and then values for σ_u, σ_ν and ω are determined (these values are equal to 2.0, 2.0 and 2.0, respectively). The most important use of simplex is to refine initial estimates before applying one of the derivative–based methods, which are more sensitive to the choice of initial estimates. For all models, we used 5 iterations in this stage. In the second stage, the BFGS (Broyden, Fletcher, Goldfarb and Shanno) algorithm was applied to obtain the final estimates of the parameters and the asymptotic variance–covariance matrix estimated by the final iteration of the approximation of the inverse Hessian. Finally, we computed regression standard errors and the covariance matrix allowing for heteroscedasticity.[2]

It is noteworthy that the estimated parameters are very similar between the UML–NE and RML–NED estimates and, in general, they are statistically significant at any significant level and positive, indicating a positive relationship between the total cost and all input prices and total revenue. On the other hand, the estimated value of the dependence parameter, that is, ω, it is statistically significant at any significant level and negative (t–statistic is equal to –119.36, p-value = 0.0). Moreover, the correlation coefficient (ρ) and Spearman's ρ_S measure (the probabilistic concordance among errors) indicate that correlation is low and negative among errors but it is not zero. Finally, the maximum value of likelihood function is higher for RML–NED than UML–NE and OLS estimates.

In terms of models fitting, we compare both the RML–NED and UML–NE models by using a likelihood ratio test. Therefore, we can also show evidence of the dependence assumption

[2]All the computations were performed using RATS software.

being tested. Specifically, the likelihood ratio test for the null hypothesis where $\omega = 0$ is 257.68 (p-value $= 0.0$) which also indicates that independence between errors is rejected by our data.

As a final point, to illustrate the behavior of estimated cost efficiencies for RML–NED and UML–NE models by using results in Table 1, we show in Figure 2 their kernel densities (Epanechnikov kernel). As we can observe, there are clear differences among the estimated cost efficiencies. In general, we can see as UML–NE overestimates cost efficiencies regarding RML–NED estimates.

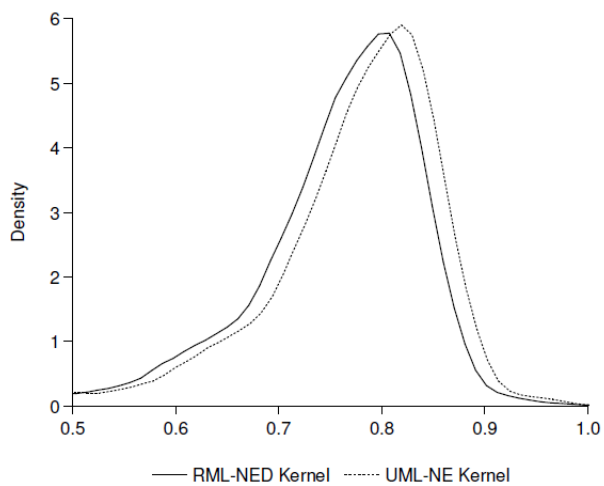

Figure 2: Kernel densities for estimated cost efficiencies.

Therefore, we can conclude that our data confirm a value added of our parametric specification and its practical relevance, that is, RML–NED is better fitted than UML–NE when the assumption of dependence is rejected.

4 Concluding remarks

In this paper, we have propose a new stochastic frontier model which introduces a flexible correlation structure between the probability of the normal error term, ν, and the inefficiency term, u. The formulae derived are closed form expressions for the marginal density of the estimated error, which allow us to introduce the classical assumptions applicable to the idiosyncratic error and the inefficiency term.

Our approach presents the following advantages: (*i*) Probabilistic interpretation, in the context of stochastic frontier models, is straightforward. (*ii*) It adds flexibility to the model by taking into account both the effects of the independent case and the correlation among variables. Furthermore, it allows a wider range of dependence. Any correlation sign is allowed, including the possibility of negative correlation among variables, thus reducing the possibility of misspecification. (*iii*) It is easily modelled in a maximum likelihood framework to test the dependence assumption, and it is computationally simpler than other models like, for example, Frank and Plackett copulas. (*iv*) It focuses on the classical case normal–exponential, but can also be extended to other distributions, including the normal–truncated normal and normal–gamma, among others.

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Appendix 1. Proof of Theorem 1

From (4) we have

$$
\begin{aligned}
f(\varepsilon) &= \int_0^\infty f_{\sigma_u,\sigma_\nu,\omega}(u,\varepsilon)\,du \\
&= \int_0^\infty f_{\sigma_u}(u)\,f_{\sigma_\nu}(u+\varepsilon)\,du + \omega \int_0^\infty f_{\sigma_u}(u)\,f_{\sigma_\nu}(u+\varepsilon)\,\varphi_{\sigma_u}(u)\,\varphi_{\sigma_\nu}(u+\varepsilon)\,du. \qquad (14)
\end{aligned}
$$

The first integral in (14) coincides with (1). Then, we have

$$
\begin{aligned}
f(\varepsilon) &= [1+\omega\,\delta_u(\sigma_u)\,\delta_\nu(\sigma_\nu)]\,\frac{1}{\sigma_u}\,\Phi\left(-\frac{\varepsilon}{\sigma_\nu}-\frac{\sigma_\nu}{\sigma_u}\right)\exp\left\{\frac{\varepsilon}{\sigma_u}+\frac{\sigma_\nu^2}{2\sigma_u^2}\right\} \\
&\quad -\omega\,(\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3)\,,
\end{aligned}
$$

where $\delta_u(\sigma_u)$, $\delta_\nu(\sigma_\nu)$ are given in (5), (6), respectively, while

$$
\begin{aligned}
\mathcal{J}_1 &= \int_0^\infty e^{-u-(u+\varepsilon)^2-2(u+\varepsilon)}\,f_{\sigma_u}(u)\,f_{\sigma_\nu}(u+\varepsilon)\,du, \\
\mathcal{J}_2 &= \delta_\nu(\sigma_\nu)\int_0^\infty e^{-u}\,f_{\sigma_u}(u)\,f_{\sigma_\nu}(u+\varepsilon)\,du, \\
\mathcal{J}_3 &= \delta_u(\sigma_u)\int_0^\infty e^{-(u+\varepsilon)^2-2(u+\varepsilon)}\,f_{\sigma_u}(u)\,f_{\sigma_\nu}(u+\varepsilon)\,du.
\end{aligned}
$$

Again, simple but tedious computations lead to the following:

$$
\begin{aligned}
\mathcal{J}_1 &= \frac{1}{\sigma_u\sqrt{1+2\sigma_\nu^2}}\,\Upsilon^1_{\sigma_u,\sigma_\nu}(\varepsilon)\,\Upsilon^4_{\sigma_u,\sigma_\nu}(\varepsilon)\,\Phi\left(-\frac{\sigma_\nu}{\sigma_u}\frac{1+3\sigma_u}{\sqrt{1+2\sigma_\nu^2}}-\frac{\varepsilon}{\sigma_\nu}\sqrt{1+2\sigma_\nu^2}\right), \\
\mathcal{J}_2 &= \frac{1}{\sigma_u\sqrt{1+2\sigma_\nu^2}}\,\Upsilon^1_{\sigma_u,\sigma_\nu}(\varepsilon)\,\Upsilon^2_{\sigma_u,\sigma_\nu}(\varepsilon)\;\Phi\left(-\frac{\varepsilon}{\sigma_\nu}-\frac{\sigma_\nu}{\sigma_u}-\sigma_\nu\right), \\
\mathcal{J}_3 &= \frac{1}{\sigma_u\sqrt{1+2\sigma_\nu^2}}\,\Upsilon^1_{\sigma_u,\sigma_\nu}(\varepsilon)\,\Upsilon^{3p}_{\sigma_u,\sigma_\nu}\;\Phi\left(-\frac{\sigma_\nu}{\sigma_u}\frac{1+2\sigma_u}{\sqrt{1+2\sigma_\nu^2}}-\frac{\varepsilon}{\sigma_\nu}\sqrt{1+2\sigma_\nu^2}\right),
\end{aligned}
$$

from which (9) is obtained.■

Appendix 2. Proof of Proposition 1

We start with the wellknown relation

$$
f_{\sigma_u,\sigma_\nu,\omega}(u|\varepsilon) = \frac{f_{\sigma_u,\sigma_\nu,\omega}(u,\varepsilon)}{f_{\sigma_u,\sigma_\nu,\omega}(\varepsilon)}
$$

and replace in this expression the numerator and the denominator by (4) and (9), respectively. Now by taking the common factor for ω in both, numerator and denominator, we get the result after some computations.

Appendix 3. Proof of Proposition 2

From (9) it is easy to see

$$E\left(e^{-u}|\varepsilon\right) \propto \int_0^\infty e^{-u} f_{\sigma_u,\sigma_\nu}(u|\varepsilon)du + \omega \int_0^\infty e^{-u}\Psi^0_{\sigma_u,\sigma_\nu}(u|\varepsilon)du, \tag{15}$$

where the proportionality factor is given by $\left[1+\omega\Psi^1_{\sigma_u,\sigma_\nu}(\varepsilon)\right]^{-1}$.

The first integral in (15) is simple to solve by using the pdf of the half.normal provided in (2). For the second integral we use the expression given in (11) and with simple but tedious computations we get the result.

Appendix 4. The stochastic cost frontier model

The corresponding expressions for the stochastic cost frontier (SCF) are derived easily from the fact that now we assume $v = -u + \varepsilon$.

In this case and under the classical model we have that,

$$\begin{aligned} f_{\sigma_u,\sigma_\nu}(\varepsilon) &= \frac{1}{\sigma_u}\Phi\left(\frac{\varepsilon}{\sigma_\nu} - \frac{\sigma_\nu}{\sigma_u}\right)\exp\left\{-\frac{\varepsilon}{\sigma_u} + \frac{\sigma_\nu^2}{2\sigma_u^2}\right\}, \\ f_{\sigma_u,\sigma_\nu}(u|\varepsilon) &= \frac{1}{\sqrt{2\pi}\,\sigma_\nu\,\Phi(\widetilde{\widetilde{\mu}}/\sigma_\nu)}\exp\left\{-\frac{1}{2\sigma_\nu^2}(u-\widetilde{\widetilde{\mu}})^2\right\}, \end{aligned} \tag{16}$$

where $\widetilde{\widetilde{\mu}} = \varepsilon - \sigma_\nu^2/\sigma_u$.

Again, the marginal $f(\varepsilon)$ is asymmetrically distributed with mean $E(\varepsilon) = \sigma_u$ and variance $var(\varepsilon) = \sigma_u^2 + \sigma_\nu^2$.

The corresponding expressions under the dependence assumption are given bellow.

The marginal pdf of ε is given by

$$\begin{aligned} f_{\sigma_u,\sigma_\nu,\omega}(\varepsilon) &= \frac{\Xi^1_{\sigma_u,\sigma_\nu}(\varepsilon)}{\sigma_u\sqrt{1+2\sigma_\nu^2}}\left\{\Phi\left(\frac{\widetilde{\widetilde{\mu}}}{\sigma_\nu}\right)\left[\sqrt{1+2\sigma_\nu^2} + \frac{\omega}{1+\sigma_u}\exp\left(\frac{2\sigma_\nu^2}{1+2\sigma_\nu^2}\right)\right]\right. \\ & -\omega\left[\sqrt{1+2\sigma_\nu^2}\,\Xi^2_{\sigma_u,\sigma_\nu}(\varepsilon)\,\delta_\nu(\sigma_\nu)\,\Phi\left(\frac{\widetilde{\widetilde{\mu}}}{\sigma_\nu} - \sigma_\nu\right)\right. \\ & + \Xi^3_{\sigma_u,\sigma_\nu}(\varepsilon)\,\delta_u(\sigma_u)\,\Phi\left(\frac{\sigma_\nu(2\sigma_u-1)}{\sigma_u\sqrt{1+2\sigma_\nu^2}} + \frac{\varepsilon\sqrt{1+2\sigma_\nu^2}}{\sigma_\nu}\right) \\ & \left.\left. - \Xi^4_{\sigma_u,\sigma_\nu}(\varepsilon)\,\Phi\left(\frac{\sigma_\nu(\sigma_u-1)}{\sigma_u\sqrt{1+2\sigma_\nu^2}} + \frac{\varepsilon\sqrt{1+2\sigma_\nu^2}}{\sigma_\nu}\right)\right]\right\}, \end{aligned} \tag{17}$$

where

$$\Xi^1_{\sigma_u,\sigma_\nu}(\varepsilon) = \exp\left\{-\frac{\varepsilon}{\sigma_u} + \frac{\sigma_\nu^2}{2\sigma_u^2}\right\}, \tag{18}$$

$$\Xi^{2}_{\sigma_u,\sigma_\nu}(\varepsilon) = \exp\left\{-\varepsilon + \frac{\sigma_\nu^2}{\sigma_u} + \frac{\sigma_\nu^2}{2}\right\}, \tag{19}$$

$$\Xi^{3}_{\sigma_u,\sigma_\nu} = \exp\left\{\frac{\sigma_\nu^2}{2\sigma_u^2}\frac{4\sigma_u(\sigma_u-1)-2\sigma_\nu^2}{1+2\sigma_\nu^2}\right\}, \tag{20}$$

$$\Xi^{4}_{\sigma_u,\sigma_\nu}(\varepsilon) = \exp\left\{-\varepsilon + \frac{\sigma_\nu^2}{2\sigma_u^2}\frac{\sigma_u(\sigma_u-2)-2\sigma_\nu^2}{1+2\sigma_\nu^2}\right\}. \tag{21}$$

The conditional pdf of u given ε is given by

$$f_{\sigma_u,\sigma_\nu,\omega}(u|\varepsilon) = \frac{f_{\sigma_u,\sigma_\nu}(u|\varepsilon) + \omega\,\Im^{0}_{\sigma_u,\sigma_\nu}(u|\varepsilon)}{1+\omega\,\Im^{1c}_{\sigma_u,\sigma_\nu}(\varepsilon)}, \tag{22}$$

for $u > 0$, where $f_{\sigma_u,\sigma_\nu}(u|\varepsilon)$ is the pdf given in (16) and

$$\begin{aligned}
\Im^{0}_{\sigma_u,\sigma_\nu}(u|\varepsilon) &= \varphi_{\sigma_u}(u)\,\varphi_{\sigma_\nu}(u+\varepsilon)\,f^{c}_{\sigma_u,\sigma_\nu}(u|\varepsilon),\\
\Im^{1}_{\sigma_u,\sigma_\nu}(\varepsilon) &= \delta_u(\sigma_u)\,\delta_\nu(\sigma_\nu) - \frac{1}{\sqrt{1+2\sigma_\nu^2}\,\Phi\left(\frac{\tilde{\tilde{\mu}}}{\sigma_\nu}\right)} \times \Bigg[\sqrt{1+2\sigma_\nu^2}\,\Xi^{2}_{\sigma_u,\sigma_\nu}(\varepsilon)\,\delta_\nu(\sigma_\nu)\,\Phi\left(\frac{\tilde{\tilde{\mu}}}{\sigma_\nu}-\sigma_\nu\right)\\
&\quad + \Xi^{3}_{\sigma_u,\sigma_\nu}(\varepsilon)\,\delta_u(\sigma_u)\,\Phi\left(\frac{\sigma_\nu(2\sigma_u-1)}{\sigma_u\sqrt{1+2\sigma_\nu^2}} + \frac{\varepsilon\sqrt{1+2\sigma_\nu^2}}{\sigma_\nu}\right)\\
&\quad - \Xi^{4}_{\sigma_u,\sigma_\nu}(\varepsilon)\,\Phi\left(\frac{\sigma_\nu(\sigma_u-1)}{\sigma_u\sqrt{1+2\sigma_\nu^2}} + \frac{\varepsilon\sqrt{1+2\sigma_\nu^2}}{\sigma_\nu}\right)\Bigg].
\end{aligned}$$

The point estimation for the efficiency in the SCF model under NE distributions, assuming dependence, is given by

$$\mathrm{CE}_i = \frac{\Im^{01}_{\sigma_u,\sigma_\nu,\omega}(\varepsilon_i) + \omega\,\Im^{02}_{\sigma_u,\sigma_\nu}(\varepsilon_i)}{1+\omega\,\Im^{1}_{\sigma_u,\sigma_\nu}(\varepsilon_i)}, \quad i = 1,2,\ldots,n,$$

where

$$\Im^{01}_{\sigma_u,\sigma_\nu,\omega}(\varepsilon_i) = [1+\omega\,\delta_u(\sigma_u)\,\delta_\nu(\sigma_\nu)]\,\frac{\Phi\left(\frac{\tilde{\tilde{\mu}}}{\sigma_\nu}-\sigma_\nu\right)}{\Phi\left(\frac{\tilde{\tilde{\mu}}}{\sigma_\nu}\right)}\,e^{-\varepsilon_i+\sigma_\nu^2/\sigma_u+\sigma_\nu^2/2}$$

and

$$\begin{aligned}
\Im^{02}_{\sigma_u,\sigma_\nu}(\varepsilon_i) &= \frac{1}{\Im\left(\frac{\tilde{\tilde{\mu}}}{\sigma_\nu}\right)}\Bigg[\frac{1}{\sqrt{1+2\sigma_\nu^2}}\,\Xi^{11}_{\sigma_u,\sigma_\nu}(\varepsilon_i)\,\Phi\left(\frac{\varepsilon_i}{\sigma_\nu}\sqrt{1+2\sigma_\nu^2} - \frac{\sigma_\nu}{\sigma_u\sqrt{1+2\sigma_\nu^2}}\right)\\
&\quad - \delta_\nu(\sigma_\nu)\,\Xi^{12}_{\sigma_u,\sigma_\nu}(\varepsilon_i)\,\Phi\left(\frac{\tilde{\tilde{\mu}}}{\sigma_\nu} - 2\sigma_\nu\right)\\
&\quad - \frac{\delta_u(\sigma_u)}{\sqrt{1+2\sigma_\nu^2}}\,\Xi^{13}_{\sigma_u,\sigma_\nu}(\varepsilon_i)\,\Phi\left(\frac{\varepsilon_i}{\sigma_\nu}\sqrt{1+2\sigma_\nu^2} + \frac{\sigma_u-1}{\sigma_u}\frac{\sigma_\nu}{\sqrt{1+2\sigma_\nu^2}}\right)\Bigg],
\end{aligned}$$

with

$$
\begin{aligned}
\Xi^{11}_{\sigma_u,\sigma_\nu}(\varepsilon_i) &= \exp\left\{-2\varepsilon_i - \frac{\sigma_\nu^4}{(1+2\sigma_\nu^2)\sigma_u^2}\right\}, \\
\Xi^{12}_{\sigma_u,\sigma_\nu}(\varepsilon_i) &= \exp\left\{-2\varepsilon_i + \frac{2\sigma_\nu^2}{\sigma_u}(1+\sigma_u)\right\}, \\
\Xi^{13}_{\sigma_u,\sigma_\nu}(\varepsilon_i) &= \exp\left\{-\varepsilon_i + \frac{\sigma_\nu^2(\sigma_u(\sigma_u-2)-2\sigma_\nu^2)}{2\sigma_u^2(1+2\sigma_\nu^2)}\right\},
\end{aligned}
$$

while $\delta_u(\sigma_u)$ and $\delta_\nu(\sigma_\nu)$ are given in (5), (6), respectively.