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The Rights Egalitarian Solution for *NTU* Sharing Problems

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Abstract

The purpose of this paper is to extend the Rights Egalitarian solution (Herrero, Maschler & Villar, 1999) to the context of non-transferable utility sharing problems. Such an extension is not unique. Depending on the kind of properties we want to preserve we obtain two different generalizations. One is the "proportional solution", that corresponds to the Kalai-Smorodinsky solution for surplus sharing problems and the solution in Herrero (1998) for rationing problems. The other is the "Nash solution", that corresponds to the standard Nash bargaining solution for surplus sharing problems and the Nash rationing solution (Mariotti & Villar (2005)) for the case of rationing problems.

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1 Introduction

Consider a group of agents that had formed a partnership in the past, each of them contributing with some amount of money. Now they are to dissolve the partnership and have to split its total value among the partners, taking into account their entitlements (initial contributions). The question is how to allocate the proceeds among the incumbents. This is a distributive problem with an extremely simple mathematical structure: there is a certain amount of money (a scalar that can be either positive, zero or negative), to be distributed among a group of agents characterized by their entitlements (a vector of real numbers of any sign). Note that all relevant data of the problem are expressed in the same units, a context that can be associated to the case of transferable utilities that are linear in the good under consideration. It is common to distinguishing between surplus sharing problems and rationing problems, depending on whether the amount to be divided exceeds or falls short of the aggregate entitlements.

Simple as it is, there is a good deal of possible solutions and a large stream of literature dealing with the properties of those solutions [e.g. Young (1987), (1994), Moulin (1988), (2001), Thomson (2003), for a review of the literature]. Herrero, Maschler & Villar (1999) analyze in this context a particular solution, called the *rights-egalitarian solution*. This solution corresponds to the equal losses solution for rationing problems and the equal-gains solution for surplus sharing problems, under the assumption of unlimited liability. The rights-egalitarian solution exhibits a number of interesting axiomatic properties and can be supported from a game-theoretic perspective.

The purpose of this paper is to extend the rights egalitarian solution to non-transferable utility sharing problems. We shall assume, in particular, the standard framework of NTU cooperative game theory. Namely, agents are characterized by cardinal non-comparable utility functions, and all the information of the problem refers to the joint utility space. We find in this context several solutions for surplus sharing (or bargaining) problems as well as solutions for rationing problems [e.g. Nash (1950), Kalai & Smorodinski (1975), Chun & Thomson (1992), Herrero (1998), Mariotti & Villar (2005)]. And also several solutions for the induced cooperative NTU games [in particular, Harsanyi (1963), Shapley (1969) and Maschler & Owen (1992)].[1]

The extension of the rights-egalitarian solution to the case of NTU sharing problems is not unique. Two different generalizations obtain, depending on the kind of properties we want to preserve. One is the "proportional solution", that corresponds to the Kalai-Smorodinsky solution for surplus

[1] See Hart (2004) for an interesting comparative discussion of those solutions.

sharing problems and the solution in Herrero (1998) for rationing problems. The other is the "Nash solution", that corresponds to the standard Nash bargaining solution for surplus sharing problems and the Nash rationing solution (Mariotti & Villar (2005)) for the case of rationing problems. The proportional solution preserves the self-duality nature of the rights-egalitarian solution in this more general context, whereas the Nash solution preserves the idea of egalitarian allocations.

The paper is structured as follows. Section 2 presents the rights-egalitarian solution in the conventional TU framework and provides a simple extension to the case of "hyperplane problems". Section 3 presents the general framework when agents' utilities are cardinal and non-comparable. It also analyzes the proportional solution and the Nash solution. Some final comments and remarks, in Section 4, close the paper.

2 The reference problem

2.1 Allocating a divisible good when agents have entitlements

A given amount of a divisible good is to be divided among a group of agents, each of them having an individual entitlement on it. Those agents form a partnership and the entitlements refer to their contributions to it. The amount of the good to be distributed corresponds to the liquidation value of the partnership. The problem is how to divide that value among the partners. A solution corresponds to a distribution that results from the application of some allocation rule. There is a number of sensible procedures to solve this problem, that can be associated to the nature of the property rights involved or the type of problem under consideration. Let us formalize these ideas.

A *problem* is a triple $[N, E, c]$, where $N = \{1, 2, ..., n\}$ represents the set of agents (a finite subset of $\mathcal{N}$, the set of potential agents), $E \in \mathbb{R}$ is the liquidation value, and $c \in \mathbb{R}^n$ is the vector of entitlements. Let Ω be the family of all problems. For any $\omega = [N, E, c] \in \Omega$, call $C(\omega) = \sum_{i \in N} c_i$, and $H(\omega) = \{z \in \mathbb{R}^n \mid \sum_{i \in N} z_i = E\}$. If $C(\omega) > E$, we are facing a problem of sharing losses from the aggregate entitlements, whereas if $C(\omega) < E$, our problem is one of surplus-sharing.

Definition 2 *A solution is a function* $F : \Omega \to \bigcup_{N \in \mathcal{N}} \mathbb{R}^{|N|}$, *such that, for any* $\omega \in \Omega$, $F(\omega) \in H(\omega)$ *and:* $F(\omega) \geq c$ *if* $E \geq C(\omega)$, $F(\omega) \leq c$ *if* $E \leq C(\omega)$.

A problem is thus defined by means of an hyperplane $H(\omega)$ with normal $(1,1,...,1)$ and a point c, both in $\mathbb{R}^n$. A solution is a point that satisfies two requirements: (i) it lies on $H(\omega)$, i.e., the sum of the shares equals the liquidation value; and (ii) it does not exceed the entitlement on any agent for rationing problems nor gives anybody less than her entitlement for surplus-sharing problems. Note that this implies $F(\omega) = c$ for those problems ω with $c \in H(\omega)$. In general, nonetheless, $c \notin H(\omega)$, namely, c lies in one of the semispaces in which the hyperplane $H(\omega)$ divides $\mathbb{R}^n$.

Consider now the following solution, introduced in Herrero, Maschler & Villar (1999):

Definition 3 *The* ***rights-egalitarian solution****,* F^{RE}*, is given by:*

$$F_i^{RE}(\omega) = c_i + \frac{1}{n}(E - C(\omega)) \qquad [1]$$

The rights-egalitarian solution assigns to each agent her entitlement plus an equal share of the difference between the estate and total entitlements. When $E > C(\omega)$ (resp. $E < C(\omega)$) this corresponds to a surplus sharing (resp. a rationing) problem that is solved by distributing equally the net proceeds among the partners. Note that agents with positive entitlements may end up with a negative allotment in the liquidation of the partnership. That is, this solution assumes that the agents are the owners of the liquidation value, if positive, but they are also fully responsible for the total losses, if negative. The rights-egalitarian solution can thus be viewed as a combination of the equal-awards and equal-loss principles (hence its name).

For a problem $\omega \in \Omega$, define a point $r(\omega)$ as follows: for each $i \in N$, $r_i(\omega)$ is given by:

$$r_i(\omega) = E - \sum_{j \neq i} c_j \qquad [2]$$

This value $r_i(\omega)$ tells us what agent i would obtain once all other agents receive their full entitlements. It is obvious that $\sum_{i \in N} r_i(\omega) < E$ when $C(\omega) > E$, and $\sum_{i \in N} r_i(\omega) > E$ if $C(\omega) < E$, that is, c and $r(\omega)$ always lie on opposite sides of $H(\omega)$. Furthermore, c and $r(\omega)$ are symmetric points from $H(\omega)$, i.e., they are mirror images of each other. Indeed, the problems $\omega = [N, E, c]$ and $\omega' = [N, E, r(\omega)]$ can be regarded as as *dual problems*, as $r(\omega') = c$. Let us call $r(\omega)$ the *reference point* of problem ω, and call c the *entitlements* or *claims point*.

Using the reference point $r(\omega)$ we can re-write the Rights Egalitarian solution [1] as follows:

$$F_i^{RE}(\omega) = c_i + \frac{1}{n}(r_i(\omega) - c_i) \qquad [3]$$

This expression provides still another interpretation of the Rights Egalitarian solution. It appears as the feasible point that assigns to each agent her entitlement plus the expected value of the lottery that gives equal probability to get c_i and equal probability to get $r_i(\omega)$. This is a well-known method of fair division with linear utilities (random priority).

2.2 An elementary extension: Hyperplane problems

The very definition of the rights-egalitarian solution implies that all the data of the problem are formulated in the same units, so that we can aggregate them. In order to extend this concept to more general environments we need first to extract the principle behind this allocation rule to make it independent on that common units feature.

We now consider a simple extension of the division problem discussed above. It refers to the case of allocating cardinal non-comparable utilities, rather than amounts of a given good, when the utility possibility set is defined by a hyperplane. This is a particular sub-family of the standard NTU sharing problem, to be analyzed below, that will play an auxiliary role in the ensuing discussion. Formally:

Definition 4 *The family* $\sum_H$ *of* ***hyperplane problems*** *consists of all those problems* $(N, H(p, E), c)$ *such that*

$$H(p, E) = \{s \in \mathbb{R}^n \quad / \quad \sum_{i \in N} p_i s_i \leq E\}$$

for $p_i > 0$, $i \in N$, $E \in \mathbb{R}$.

Hyperplane sharing problems are a special class of NTU problems introduced in Maschler & Owen (1989) in order to provide a first extension of the Shapley value to NTU cooperative games (see Hart (1994) for a discussion of the class of situations that may generate this kind of problems).[2] Hyperplane problems differ from the standard division problem considered above in that the slope of the hyperplane is not $-\mathbf{1}$ anymore and we are in a non-side payment scenario.

The extension of the rights egalitarian solution to this context is given by the following:

[2]For an extension of the consistent Shapley value to general NTU cooperative games see see also Maschler & Owen (1992), Hart & Mas-Colell (1996), Hart (2005).

Definition 5 *The* ***extended rights-egalitarian*** *solution,*F^{ERE}*, is the mapping* $F : \sum_H \to \bigcup_{N \in \mathcal{N}} \mathbb{R}^{|N|}$*, defined by the following condition, for all* $i \in N$ *:*

$$p_i F_i^{ERE}(\omega) = p_i c_i + \frac{1}{n}\left(E - \sum_{j=1}^{n} p_j c_j\right) \qquad [1']$$

The extended rights-egalitarian solution corresponds to the application of the same principle that defines the rights egalitarian solution, when units are different for different agents. Similarly, we define the reference point r_i in this context as follows:

$$p_i r_i(N, H(p, E), c) = E - \sum_{j \neq i} p_j c_j \qquad [2']$$

Consider now the following concepts:

Definition 6 *A* ***proportional allocation*** *for a hyperplane problem* $(N, H(p, E), c)$ *in* $\sum_H$ *is a point* s^* *such that* $\sum_{i \in N} p_i s_i^* = E$ *and, for all* $i, j \in N$*,*

$$\frac{c_i - s_i^*}{c_i - r_i(N, E, c)} = \frac{c_j - s_j^*}{c_j - r_j(N, E, c)}$$

A proportional allocation is a point in which the amounts obtained by the agents equalize their relative gains or losses and the liquidation worth of the partnership, E, is fully distributed.

Definition 7 *An* ***egalitarian allocation*** *for a hyperplane problem* $(N, H(p, E), c)$ *in* $\sum_H$*, is a point* s^* *such that* $\sum_{i \in N} p_i s_i^* = E$ *and, for all* $i, j \in N$*,*

$$p_i(s_i^* - c_i) = p_j(s_j^* - c_j)$$

Besides requiring Pareto efficiency this definition establishes that the weighted utility gains or losses of all agents should be equal, where the weights are given by the normal of the hyperplane that defines de hyperplane problem. Note that this notion can be interpreted as the outcome of maximizing a weighted utilitarian welfare function, in which agents enter with weights p_1, p_2, ..., p_n, in such a way that the allocation of utilities compensates the differences in those weights.

The following result is obtained:

Proposition 8 *For each problem* $(N, H(p, E), c) \in \sum_H$*, it follows that:*
(i) $s^* = F^{ERE}(\omega)$ *if and only if* s^* *is a proportional allocation.*
(ii) $s^* = F^{ERE}(\omega)$ *if and only if* s^* *is an egalitarian allocation.*

Proof

From $[1']$ and $[2']$ we immediately deduce:

$$s_i^* = c_i + \frac{1}{n}(r_i(N, H(p, E), c) - c_i)$$

(which is, precisely, equation [3] above). Then, it follows that:

$$\frac{s_i^* - c_i}{r_i(N, H(p, E), c) - c_i} = \frac{1}{n}$$

an expression that corresponds precisely to the notion of proportional allocation.

Moreover, the definition of extended rights-egalitarian allocation can be rewritten as:

$$p_i(s_i^* - c_i) = p_j(s_j^* - c_j) \ , \quad \forall \ i, j \in N$$

that is the definition of the egalitarian allocation.

Note that in both cases $\sum_{j=1}^{n} p_j s_j^* = E$, which ensures the uniqueness part and the fulfillment of the efficiency requirement. **Q.e.d.**

Note that, for hyperplane problems, proportional and egalitarian allocations are uniquely defined and do coincide. This will not be the case for general NTU sharing problems (indeed the very notion of egalitarian allocation should be redefined for such a context). Also observe that, for surplus sharing problems, proportional allocations correspond to the Kalai-Smorodinsky bargaining solution whereas egalitarian allocations correspond to the Nash bargaining solution. For the case of rationing problems, proportional allocations and egalitarian allocations correspond, respectively, to the Kalai-Smorodinsky and the Nash solution of the dual problem $(N, H(p, E), r(N, H(p, E))$.

Consider now the following properties, that adapt those with the same name in Herrero, Maschler & Villar (1999):

Symmetry: *For all* $\omega = [N, E, c]$, *if* $p_i c_i = p_j c_j$ *for all* $i, j \in N$, *then* $p_i F_i(\omega) = p_j F_j(\omega)$, *for all* $i, j \in N$.

Symmetry establishes that two agents whose weighted claims are equal will get equal amounts that also coincide when weighted in the same way.

Composition: *For any* $\omega = [N, H(p, E), c]$, and any $E_1, E_2 \in \mathbb{R}$ *such that* $E_1 + E_2 = E$, *it follows that* $F(\omega) = F[N, H(p, E_1), c] + F[N, H(p, E_2), c - F[N, E_1, c]]$.

Composition says that we can solve any problem in a sequential manner. The solution to the original problem coincides with the sum of the solutions of two sub-problems, one in which we first allocate a fraction of the liquidation value and the other that in which we allocate the rest, reducing the original claims according to what agents already obtained.

We obtain the following characterization result:

Proposition 9 *The extended rights-egalitarian solution, F^{ERE}, is the only solution on $\sum_H$ that satisfies symmetry and composition.*

Proof.-

For a problem $\omega = (N, H(p, E), c)$, define the following related problems, all in $\sum$:

$$\omega_1 = (N, H(p, \sum_{j=1}^{n} p_j c_j), c), \qquad \omega_2 = (N, H(p,\ E - \sum_{j=1}^{n} p_j c_j), 0)$$

By definition and symmetry, respectively, we have:

$$F_i(\omega_1) = c_i, \qquad F_i(\omega_2) = \frac{1}{p_i}\frac{1}{n}\left(E - \sum_{j=1}^{n} p_j c_j\right)$$

Composition implies:

$$F_i(\omega) = F_i(\omega_1) + F_i(\omega_2)$$

That is:

$$p_i F_i(\omega) = p_i c_i + \frac{1}{n}\left(E - \sum_{j=1}^{n} p_j c_j\right) = p_i F_i^{ERE}(\omega)$$

Q.e.d.

This result shows that the properties that characterize the rights egalitarian solution also characterize the extended version, once suitably translated to the new context. Needless to say, when $p_i = 1$ for all i, we are back to the standard set up.

3 The general model: NTU sharing problems

We now consider a more general social choice problem consisting of the allocation of utility gains and losses among a group of agents with non-transferable utilities and some utility value to be taken as the "entitlements utility". To be precise, let $N = \{1, 2, ..., n\}$ stand for a collection of agents, each of which is endowed with a cardinal non-comparable utility function u_i and a utility point c_i, to be interpreted as her *status quo* or her claims point. Again, we can think of this situation as a case in which agents in N have to share the proceeds of some collective project, be them gains or losses. The entitlements vector can be interpreted as an expression of rights, aspirations or secured outcomes, depending of the type of problem at hand. This type of problem can thus be summarized in a set of agents N, a utility possibility set $S \subset \mathbb{R}^n$, and a point $c \in \mathbb{R}^n$. A choice must be made out of the feasible set of utility allocations S depending on the distinguished utility vector c.

3.1 Preliminary definitions

Each agent $i \in N = \{1, 2, ..., n\}$ is characterized by a pair $(u_i,\ c_i)$, where u_i is a von Neumann-Morgenstern utility function, defined on some suitable (commodity) space, and c_i is a distinguished utility value. The set $S \subset \mathbb{R}^n$ describes the collection of utility allocations which are feasible, while the vector $c \in \mathbb{R}^n$ denotes the entitlements or claims vector. A *NTU* **sharing problem** (or *a problem*, for short) is a triple (N, S, c). There are two types of problems. One corresponds to *NTU rationing problems*, in which $c \notin S$ and the agents are to share the losses of some joint venture. The other refers to *NTU surplus sharing problems,* in which $c \in S$ and the question is how to allocate the gains of some cooperative enterprise.

The set of utility allocations that are **admissible,** denoted by $\mathcal{A}(N, S, c)$, is defined as follows:

$$\mathcal{A}(N,S,c) = \begin{cases} \{s \in S \ / \ s \leq c\} & \text{if } c \notin S \\ \{s \in S \ / \ s \geq c\} & \text{if } c \in S \end{cases}$$

This set is made out of those utility allocations in which agents obtain utilities which are bounded by the reference vector c, above or below depending on whether (N, S, c) is a rationing or a surplus sharing problem. Moreover, we define the (weak) Pareto frontier of the set of admissible allocations, as follows:

$$\mathcal{PA}(N,S,c) = \{s \in \mathcal{A}(N,S,c) \ / \ s' >> s \Longrightarrow s' \notin \mathcal{A}(N,S,c)\}$$

We concentrate on a family $\sum$ of problems that satisfies some elementary restrictions.

Definition 10 *The family* $\sum$ *of* ***standard*** *NTU* ***sharing problems*** *consists of all those problems* (N, S, c) *such that: (i)* $S \subset \mathbb{R}^n$ *is closed, convex, and comprehensive; and (ii) For all* $i \in N$, $\mathcal{PA}(N,S,c) \cap \{s \in \mathbb{R}^n | s_{-i} > c_{-i}\} \neq \varnothing$.

The set S is closed and convex when utility functions are continuous and concave. Comprehensiveness means that if $s \in S$ and $s' \in \mathbb{R}^n$ is such that

$s' \leq s,$ then $s' \in S$. It is related to the monotonicity of the utility functions and implies that the relevant boundary of the utility possibility set is downward sloping and coincides with the set of weakly efficient utility allocations. Part (ii) of the definition says that agents' admissible gains and losses are bounded. From a geometrical viewpoint it implies that $\mathcal{PA}(N,S,c)$ intersects all axes of $c+\mathbb{R}^n$. Note that these properties ensure that $\mathcal{PA}(N,S,c)$ is a non-empty compact subset of $\mathbb{R}^n$ (more specifically of $c-\mathbb{R}^n_+$ for rationing problems and of $c+\mathbb{R}^n_+$ for surplus sharing problems).

Definition 11 *A **solution** to a NTU sharing problem is a mapping* $\phi : \sum \to \bigcup\limits_{N\in\mathcal{N}} \mathbb{R}^{|N|}$ *that for all* $(N,S,c) \in \sum$ *selects a subset* $\phi(N,S,c) \neq \emptyset$ *in* $\mathcal{PA}(N,S,c)$.

Points in $\phi(N,S,c)$ represent sensible compromises in the allocation of utility gains or losses, depending on the nature of the problem, that is chosen in the Pareto frontier of the set of admissible allocations. Note that the way in which this notion is defined implies that $s_i \leq c_i$ for all $i \in N$, whenever $s \in \phi(N,S,c)$ and (N,S,c) is a rationing problem (resp. $s_i \geq c_i$ for all i, whenever $s \in \phi(N,S,c)$ and (N,S,c) is a surplus sharing problem).

One more element is to be defined. For a given problem $(N,S,c) \in \sum$ the point $r_i(N,S,c)$ describes the maximum value of agent i's utility when $u_j = c_j$ for all $j \neq i$.

$$r_i(\omega) = \begin{cases} \sup\{s_i|(c_{-i},s_i) \in \mathcal{PA}(\omega) \cap c-\mathbb{R}^N_+\} & \text{if } c \notin S \\ \sup\{s_i|(c_{-i},s_i) \in \mathcal{PA}(\omega) \cap c+\mathbb{R}^N_+\} & \text{if } c \in S \end{cases}$$

Notice that $r_i(N,S,c)$ represents the highest value of s_i that is compatible with all other agents getting their entitlements (c in this case) in full. When (N,S,c) is a rationing problem this scalar represents agent i's *worst admissible outcome.* On the contrary, in a surplus problem $r_i(N,S,c)$ tells us agent i's *best possible outcome.*

If we consider the problems (N,S,c) and $(N,S,r(N,S,c))$, it happens that $r(N,S,r(N,Sc)) = c$, that is those are *dual problems.*

3.2 The Proportional solution to sharing problems

Let $(N,S,c) \in \sum$ be a sharing problem and let $[c,r(N,S,c)]$ denote the line segment that joins points c and $r(N,S,c)$. We now extend the notion of proportionality involved in the rights-egalitarian solution to this context in a natural way:

Definition 12 *The* ***proportional solution*** *is the mapping* $P : \sum \to \bigcup_{N \in \mathcal{N}} \mathbb{R}^{|N|}$ *such that, for all* $(N, S, c) \in \sum$*, selects the (unique) point in the intersection of* $\mathcal{PA}(N, S, c)$ *with* $[c, r(N, S, c)]$.

Trivially $s^* = P(N, S, c)$ if and only if s^* is a proportional allocation, as defined above.

In order to characterize the proportional solution, let us consider the following axioms:

Affine invariance: *Let* $\tau(S) = \{y \in \mathbb{R}^n \ / \ y = \tau(s), \textit{for some } s \in S, \textit{with } \tau_i(s) = \alpha_i s_i + \beta_i, \ \alpha_i > 0\}$. *Then,* $\phi(N, \tau(S), \tau(c)) = \tau(\phi(N, S, c))$.

This axiom postulates that solutions must be independent of positive affine transformations. It simply translates the underlying assumption of cardinal non-comparable utility functions.

Symmetry: *For all* $(N, S, c) \in \sum$*, if* S *is symmetric with respect the* 45^o *line, and* $c_i = c_j$ *for all* $i, j \in N$*, then* $\{\lambda' \mathbf{1}\} \in \phi(N, S, 0)$ *for some scalar* λ'.

Symmetry is an equity restriction. It establishes that if agents cannot be distinguished in a problem, they cannot be distinguished in a solution.

Monotonicity: *For all* $(N, S, c), (N, S', c) \in \Sigma$*, if* $S \subseteq S'$*, and* $r(N, S, c) = r(N, S', c)$*, then* $\phi(N, S, c) \leq \phi(N, S', c)$.

Monotonicity says that an expansion in the set of opportunities without changes in the claims and reference points, does not hurt any agent. Monotonicity is borrowed from Kalai & Smorodinski (1975) characterization of the KS bargaining solution.

The following result is easily obtained:

Proposition 13 *The proportional solution* P *is the only solution in* $\sum$ *satisfying affine invariance, symmetry, and monotonicity.*

Proof. Obviously, P satisfies all the requirements. Let now ϕ be a solution fulfilling them all. Let $(N, S, c) \in \sum$ be a problem. By affine invariance, we may apply a transformation τ so that $\tau(c) = 0$, and $\tau[r(N, S, c)] = \mathbf{1}$ (if (N, S, c) is a surplus-sharing problem) or $\tau[r(N, S, c)] = -\mathbf{1}$ (otherwise). In either case, $P[\tau(N, S, c)] = \lambda \mathbf{1}$ (with $0 < \lambda < 1$ for surplus-sharing problems, and $-1 < \lambda < 0$ for rationing problems). Let $S' = CoCom\{P[\tau(N, S, c)], (0_{-i}, \lambda)_{i \in N}\}$. Since $(N, S', 0)$ is a symmetric problem, and $\lambda \mathbf{1} \in \mathcal{PA}(N, S', 0)$, symmetry implies that $F(N, S', 0) = \lambda \mathbf{1}$. Since $S' \subset \tau(S)$, and $r(N, S', 0) = \mathbf{1} = r(N, \tau(S), 0)$, monotonicity says that $\phi(N, \tau(S), 0) = \lambda \mathbf{1} = \phi(N, S', 0) = \lambda \mathbf{1}$. By affine invariance, $\phi(N, S, c) = P(N, S, c)$. **Q.e.d.** ■

3.3 Egalitarian allocations and the Nash solution to sharing problems.

Let now $(N, S, c) \in \sum$ denote a general convex sharing problem. We now extend the notion of egalitarian allocations to this context in a natural way:

Definition 14 *Let (N, S, c) be a sharing problem in $\sum$. An **egalitarian allocation** is a point $s^* \in \mathcal{PA}(N, S, c)$ for which there exists a vector of weights $p^* \in \mathbb{R}^n_{++}$, with $\sum_{i=1}^n p_i^* = 1$, such that:*
(i) $p^* s^* \geq p^* s$ for all $s \in S$.
(ii) $p_i^*(s_i^* - c_i) = p_j^*(s_j^* - c_j)$ for all $i, j \in N$.

Part (i) is an efficiency condition and establishes that the vector p^* of weights is *perpendicular* to S at the boundary point $s^* \in S$. In that way, p^* provides an endogenous weighting system for a weighted utilitarian social welfare function, and s^* is as a maximizer of such a function. Part (ii) says that those weights inversely proportional to their utilities (i.e. we give more weight in social welfare to those agents with smaller utilities).[3]

An egalitarian allocation on a general convex problem selects points $s^* \in \mathcal{PA}(N, S, 0)$ which admit supporting hyperplanes that define hyperplane problems whose solution is, precisely, s^*. Therefore, the notion of egalitarian allocations extends further the concept of rights-egalitarian allocations to situations where the feasible utility space does not have a linear frontier and, as a consequence, the lottery-equivalent method of division yields Pareto-dominated outcomes (see [3]).

The next result tells us that egalitarian allocations always exist (and also how they look like):[4]

Proposition 15 *Every NTU sharing problem (N, S, c) in $\sum$ has an egalitarian allocation.*

Proof.

We divide the proof in two parts, one for the case of rationing problems and the other for the case of surplus sharing problems. Without loss of generality we take the normalized version of the problem, that is, we let $c = 0$.

(i) Rationing problems. Define a mapping $\phi : \sum \to \bigcup_{N \in \mathcal{N}} \mathbb{R}^{|N|}$ as follows: for each problem $(N, S, 0) \in \sum$, $\phi(N, S, 0)$ is the set of maximizers of

[3] Taking $p_i^* > 0$ for all i excludes the trivial case in which $p_i^* s_i^* = 0$ for all $i \in N$, which would satisfy parts (i) and (ii) in the definition.

[4] The proof follows closely that in Mariotti & Villar (2005, prop. 2) and is included here for the sake of completeness.

$\Pi_{i=1}^{n}(-s_i)$ over the set $Q = \{s \in \mathbb{R}_{-}^{n} \quad / \quad s \notin intS\}$. The set $\phi(N,S,0)$ is trivially nonempty as the objective function is continuous and the feasible set is compact. To see that the points $s' \in \phi(N,S,0)$ satisfy the required conditions, first observe that s' must be a point on the relative interior of the boundary of the convex set S. Therefore there exists a hyperplane with normal $\mathbf{p}^* >> 0$ that supports S at s'. That is, $\alpha = \mathbf{p}^* s' \geq \mathbf{p}^* s$ for all $s \in S$. Define now $T = \{\mathbf{z} \in \mathbb{R}_{-}^{n} \quad / \quad \mathbf{p}^*\mathbf{z} \geq \alpha\}$ and consider the problem of maximizing $\Pi_{i=1}^{n}(-s_i)$ over T. As $T \subset Q$, s' must be a solution of this problem. Since the relevant boundary of T is smooth we can immediately deduce from the first order conditions that $p_i^* s_i' = p_j^* s_j'$ for all $i, j \in N$.

(ii) Surplus sharing problems. Define a mapping $\varphi : \sum \to \mathbb{R}^n$ as follows: for each $(N,S,0) \in \sum$, $\varphi(N,S,0)$ is the set of maximizers of $\Pi_{i=1}^{n}(s_i)$ over S. Reasoning as above we conclude that $\varphi(N,S,0)$ satisfies the requirements of an egalitarian allocation. **Q.e.d.**

Egalitarian allocations obtain from the extreme values of the product of the agents' utilities on different sets. For rationing problems they result from the minimization of the product of the unfeasible allocations. For surplus sharing problems they correspond to the maximization of the product on the set of possible gains. Note that, due to the convexity properties of the feasible sets and the function that is maximized, a problem $(N,S,0) \in \sum$ may have several egalitarian allocations when it is a rationing problem, but admits a unique egalitarian allocation when it corresponds to a surplus sharing problem. Indeed, egalitarian allocations correspond to the Nash bargaining solution (Nash (1950)) for the case of surplus sharing problems and to the Nash rationing solution (Mariotti & Villar (2005)) for the case of rationing problems.

That suggests the following:

Definition 16 *The mapping $\phi^N : \sum \to \mathbb{R}^n$ that associates to each problem $(N,S,0)$ the set of egalitarian allocations will be called the* ***Nash solution*** *to sharing problems.*

In order to characterize this solution we need to introduce the following familiar axiom:

Axiom 17 (Contraction Consistency) *For all $(N,S,c), (N,T,c) \in \Sigma$ with $S \subset T$ and $\phi(N,T,c) \cap S \neq \emptyset : \phi(N,S,c) \subset \phi(N,T,c) \cap S$.*

This axiom says the following: Take a given problem and suppose that the utility possibility set is reduced, without the reference vector c being altered. Suppose furthermore that the original solution is still part of the reduced set.

Then, the solution of the new problem must be part of the solution of the original one.

Our next result shows that egalitarian allocations correspond precisely to the outcome of the unique minimal solution that satisfies affine invariance, symmetry and contraction consistency. Formally:

Proposition 18 *The Nash Solution is the only minimal (in the order of set inclusion) solution ϕ satisfying affine invariance, symmetry and contraction consistency.*

Proof.

For rationing problems, see Mariotti & Villar (2005). For surplus sharing problems, see Nash (1950). ■

Note that the Nash solution may be multi-valued for rationing problems and it is single-valued for surplus-sharing problems.

4 Final comments

In this paper we have explored how the rights-egalitarian solution extends to NTU sharing problems. This solution recommends an equal split of the net worth of the partnership. The fact that the standard division problem is linear and symmetric, together with the unlimited liability assumption, implies that many solutions coincide yield the allocation prescribed by the rights-egalitarian one in the TU case. When we consider a richer domain of problems, the notion of equal split should be redefined. There is no unique way of making such an extension. In the special case of hyperplane problems we have provided two alternative definitions of the rights egalitarian solution, that stress two different aspects embedded in this notion. One is the idea of proportionality: all agents get equal relative gains or losses. The other is associated with the notion of egalitarian allocations, defined by equal weighted net gains or losses from the entitlements point. Both principles yield the same (unique) allocation for hyperplane problems. Yet, when we apply them to general NTU sharing problems, they give rise to two different solutions in that context: the proportional solution and the Nash solution.

The proportional solution coincides with the Kalai-Smorodinsky solution for NTU surplus-sharing problems, and coincides with the solution proposed by Herrero (1998) for the case of NTU rationing problems. The Nash solution coincides with the Nash bargaining solution for surplus-sharing problems and with the Nash-rationing solution (Mariotti & Villar, 2005) for rationing problems. Both ways of extending the rights egalitarian solution can be axiomatically characterized by means of properties very much in the spirit of their bargaining counterparts. Note that the proportional solution gives chooses the same allocation for a problem and its dual, whereas this is not the case for the Nash solution.

Finally, let us underline that those solutions can be supported from a game theoretic viewpoint and coincide with some well-known game theoretical solutions. Two TU games can be associated to any TU sharing problem:

$$v(S) = \begin{cases} \Sigma_{i \in S} c_i & S \neq N \\ E & S = N \end{cases}$$

$$z(S) = \begin{cases} E - \Sigma_{i \notin S} c_i & S \neq \varnothing \\ 0 & S = \varnothing \end{cases}$$

Some interesting properties of this games are the following (Herrero, Maschler & Villar, 1999):

(1) (N, v) and (N, z) are dual games;

(2) The rights-egalitarian solution coincides with both the Shapley value and the prenucleolus of both (N, v) and (N, z);

(3) If $C(\omega) \leq E$, then $F^{RE}(\omega)$ coincides with the Shapley value, the Tau value, the Prenucleolus, the nucleolus, the prekernel and the kernel of the game (N, v); and

(4) If $C(\omega) \geq E$, then $F^{RE}(\omega)$ coincides with the Shapley value, the Tau value, the Prenucleolus, the nucleolus, the prekernel and the kernel of the game (N, z).

Given a NTU sharing problem $(N, S, c) \in \sum$, we can associate a NTU game as follows

$$V(T) = \begin{cases} S & T = N \\ \{s \in \mathbb{R}^n : (s_T, c_{N \setminus T}) \in S\} & T \neq N \end{cases}$$

It is easy to check that:

(i) The proportional solution coincides with the *compromise value* (see Borm et al.)

(ii) Egalitarian allocations coincide with the Harsanyi value for weights $p^* \in \mathbb{R}^n_{++}$ [see Harsanyi (1963), Hart (1985), Hart & Mas-Colell (1989)].

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