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The axiomatic approach to the problem of sharing the revenue from bundled pricing*

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Abstract

We explore in this paper the axiomatic approach to the problem of sharing the revenue from bundled pricing. We formalize two models for this problem on the grounds of two different informational bases. In both models, we provide axiomatic rationale for natural rules to solve the problem. We, nonetheless, obtain drastic differences under each scenario, which highlights the importance of setting the appropriate informational basis of the problem.

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1 Introduction

It is well known that bundling products may increase revenue with respect to selling products independently (e.g., Adams and Yellen, 1976). Real-life instances in which bundling occurs abound. Think, for instance, of season tickets for art or sport shows at a given venue, unlimited streaming video or music downloads through periodic charges from digital video merchants or music sellers, or the transportation cards combining access to all the transportation means (e.g., bus, subway, tram) in a given city.

Our running example in this paper will refer to the case of museum passes, given their long tradition of use. In many cities (or, even, regions and countries) worldwide, museums join to give visitors unlimited access to their collections during a limited period of time. Passes are, obviously, more expensive than the individual ticket of each participating museum on the joint venture, but less expensive than the sum of all individual tickets. The problem we deal with in here is that of deciding how to share the net revenue from the sale of passes among the participating museums. Many issues arise to address this problem from a fairness viewpoint. For instance, is it fair to let more popular museums get a higher piece of the pie? Should less popular museums face a hurdle to allow them be part of the pass? Do we treat visits without the pass to the museums similarly to pass visits? We plan to tackle these (and related) issues by exploring the axiomatic approach to the problem, a somewhat unexplored approach in this case.¹

We consider two somewhat related, albeit fundamentally different, models to address this problem. Each of them reflects a different informational basis, depending on whether the identity of visitors using the pass is known or not. More precisely, in each case, we assume the existence of a group of heterogeneous museums. Heterogeneity is measured by their entrance prices, as well as by their number of visitors (with and without the pass). The price of the pass and the number of pass holders is also known in each case. Now, in the first model we assume that the actual subset of museums visited by each holder of a (museum) pass is observed, and that piece of information will play a role in the model; whereas in the second model the identity of the pass holders visiting each museum is not known.

Our first model will then rely on a richer informational basis that makes use of the identity of actual pass visitors to museums. This is more in line with the seminal contribution on museum

¹The reader is referred to Thomson (2001) for a hitchhiker's guide of the axiomatic method and some of its more popular applications to game theory and resource allocation.

pass problems by Ginsburgh and Zang (2003). Nevertheless, our model will be more general to capture additional aspects of the heterogeneity of museums (such that the number of visitors without the pass each museum has, which could also be interpreted as a sign of the individual status of that museum). We provide several characterization results for rules generalizing the canonical *Shapley* rule, proposed by Ginsburgh and Zang (2003), to our setting. Among other things, we provide axiomatic rationale for the rule that allocates the revenue from each pass (among the museums visited by the user of such pass) proportionally to the product of the entrance price and the number of visits without the pass of the museums. The results rely on standard axioms such as equal treatment of equals, dummy, additivity, or splitting-proofness.

Our second model relies on a less demanding informational basis in which the identity of visitors using the pass is not known. Our first result in this model highlights a fundamental difference with respect to the first model. More precisely, we shall see how an impossibility result arises when combining three natural axioms (equal treatment of equals, dummy and additivity), whereas many rules emerge in the first model. This raises the importance of setting the appropriate informational basis of the problem.

The related literature to the problem we analyze in this work is scarce.² As mentioned above, Ginsburgh and Zang (2003) could be considered the seminal contribution addressing this problem. They take a game-theoretical approach to the problem by proposing to associate it to a TU-game, which they claim should be solved by means of the Shapley value associated to it.³ Their model is similar to the first model we consider in this paper, as they include, for each pass holder, the subset of museums this pass holder actually visits. They, however, do not consider other individual characteristics of museums, in contrast to what we do, such as entrance prices or visitors without the pass. More recently, there have been some contributions approaching the problem in a more similar way to what we do with our second model (e.g., Estévez-Fernández et al., 2012; Casas-Mendez et al., 2011). Nevertheless, those contributions approach the museum-pass problem as a specific bankruptcy problem and endorse rules from the bankruptcy literature (e.g., O'Neill, 1982; Thomson, 2003) to solve museum-pass problems too, which contrasts to what we do in this paper.

²See, nevertheless, Casas-Mendez et al., (2013) for a recent survey.

³It turns out that the Shapley value can be easily computed in this context, thanks to the decomposition principle. Based on it, Ginsburgh and Zang (2003) convincingly make a case in favor of the Shapley value. They allege further virtues of that rule with respect to others in Ginsburgh and Zang (2004). See also Beal and Solal (2010) and Wang (2011).

The rest of the paper is organized as follows. In Section 2, we introduce the first model we study, the axioms and rules we consider therein, as well as our (axiomatic and game-theoretical) results for that model. In Section 3, we turn to the analysis of our second model. Section 4 concludes.

2 The first model

We consider a generalization of the museum pass problem introduced by Ginsburgh and Zang (2003). As these authors, we assume that the actual subset of museums visited by each holder of a (museum) pass is observed. Besides, we also include in our model the entrance prices (without the pass) to each museum, and the number of visits without the pass.

2.1 Preliminaries

The main mathematical conventions and notations, used here, are as follows. The set of non-negative (positive) real numbers is \mathbb{R}_+ (\mathbb{R}_{++}). The set of non-negative (positive) integer numbers is \mathbb{Z}_+ (\mathbb{Z}_{++}). Vector inequalities are denoted by $>$ and \geq . More precisely, $x > y$ means that each coordinate of x is greater than the corresponding coordinate of y , whereas $x \geq y$ allows some of them to be equal. Finally, given a set S and a subset T , we denote the projection of the vector $x \in \mathbb{R}_+^{|S|}$ over T as x_T , i.e., $x_T = (x_i)_{i \in T}$.

Let \mathbb{M} represent the set of all potential museums, which may be finite or infinite. Let \mathcal{M} be the family of all finite (non-empty) subsets of \mathbb{M} . An element $M \in \mathcal{M}$ describes a finite set of museums. Its cardinality is denoted by m . Now, let \mathbb{N} represent the set of all potential costumers, i.e., individuals who might be interested in acquiring a museum pass (which would grant access to all museums) and let \mathcal{N} be the family of all finite (non-empty) subsets of \mathbb{N} . An element $N \in \mathcal{N}$ describes a finite set of costumers. Its cardinality is denoted by n .⁴

A (museum) **problem** is a 6-tuple (M, N, π, K, p, v) where:

- $M \in \mathcal{M}$ is the set of museums.
- $N \in \mathcal{N}$ is the set of costumers acquiring a (museum) pass.
- $\pi \in \mathbb{R}_+$ is the price of a pass.

⁴Note that this is similar although not identical to the general model described by Casas-Mendez et al., (2013) to survey all the recent contributions in the related literature.

- $K \in 2^{nM}$ is the profile of sets of museums visited by each customer (thanks to the pass), i.e., $K = (K_l)_{l \in N}$, and, for each $l \in N$, $K_l \subset M$ denotes the set of museums visited by customer l (thanks to the pass).
- $p \in \mathbb{R}_+^m$ is the profile of entrance prices.
- $v \in \mathbb{Z}_+^m$ is the profile of visits without the pass.

For ease of notation, we refer to the revenue generated from selling passes by E , i.e., $E = n\pi$.

For each $i \in M$, let $U_i(K)$ denote the set of customers visiting museum i with the pass. Namely, $U_i(K) = \{j \in N : i \in K_j\}$. As for cardinalities, we consider the following notational conventions: $\nu_i = |U_i(K)|$, for each $i \in M$, and $k_l = |K_l|$, for each $l \in N$.

The family of all the problems described as above is denoted by \mathcal{P}^{pv} .

We now formalize some focal subclasses of problems that will play a role in our analysis.

Let \mathcal{P} be the set of problems defined by the 4-tuple (M, N, π, K) , in which each component is described as above. In other words, \mathcal{P} corresponds to situations where we neither take into account the entrance price of a museum, nor the number of visits without pass. This is the class of problems studied in Ginsburgh and Zang (2003, 2004), as well as in Beal and Solal (2009) and Wang (2011).

Let \mathcal{P}^p be the set of problems defined by the 5-tuple (M, N, π, K, p) , in which each component is described as above. In other words, \mathcal{P}^p corresponds to situations where we do not take into account the number of visits without pass.

2.2 Rules

A **rule** is a mapping that associates with each problem an allocation indicating the amount each museum gets from the revenue generated by passes sold. Formally, $R : \mathcal{P}^{pv} \rightarrow \mathbb{R}_+^m$ is such that, for each $(M, N, \pi, K, p, v) \in \mathcal{P}^{pv}$

$$\sum_{i \in M} R_i(M, N, \pi, K, p, v) = E.$$

The vector $R(M, N, \pi, K, p, v)$ represents a desirable way of dividing the overall revenue from the sales of passes among the museums in M . Note that the entire amount is allocated, which imposes a sort of efficiency condition.

We now give some examples of rules.

The *Shapley rule* (e.g., Ginsburgh and Zang, 2003) allocates the price of each pass equally among the museums visited by the user of such pass.⁵ Namely, for each $(M, N, \pi, K, p, v) \in \mathcal{P}^{pv}$, and $i \in M$,

$$S_i(M, N, \pi, K, p, v) = \sum_{l \in N, i \in K_l} \frac{\pi}{k_l}.$$

Note that the above rule neither takes into account p nor v , and then its definition would also be valid for \mathcal{P} and \mathcal{P}^p . Nevertheless, the previous rule can also be generalized, so that p and v play a role in the allocation process. For instance, the *p-Shapley rule* allocates the price of each pass among the museums visited by the user of such pass, proportionally to the entrance price of the museums. Namely, for each $(M, N, \pi, K, p, v) \in \mathcal{P}^{pv}$, and $i \in M$,

$$S_i^p(M, N, \pi, K, p, v) = \sum_{l \in N, i \in K_l} \frac{p_i}{\sum_{j \in K_l} p_j} \pi.$$

Note that the previous rule can also be defined on \mathcal{P}^p , but not on \mathcal{P} .

Finally, the *pv-Shapley rule* allocates the price of each pass among the museums visited by the user of such pass, proportionally to the product of the entrance price and the number of visits without the pass of the museums. Namely, for each $(M, N, \pi, K, p, v) \in \mathcal{P}^{pv}$, and $i \in M$,

$$S_i^{pv}(M, N, \pi, K, p, v) = \sum_{l \in N, i \in K_l} \frac{p_i v_i}{\sum_{j \in K_l} p_j v_j} \pi.$$

Note that the last rule can neither be defined on \mathcal{P} , nor on \mathcal{P}^p .

2.3 Axioms

We now present some axioms in this model. For ease of exposition, we only define the axioms in \mathcal{P}^{pv} . The counterpart definitions in \mathcal{P}^p and \mathcal{P} are straightforwardly obtained.

Equal treatment of equals says that if two museums have the same visitors with the pass, the same number of visits without the pass, and the same entrance price, then they should receive the same amount.

ETE. For each $(M, N, \pi, K, p, v) \in \mathcal{P}^{pv}$, and $i, j \in M$ such that $(p_i, v_i, U_i(K)) = (p_j, v_j, U_j(K))$,

$$R_i(M, N, \pi, K, p, v) = R_j(M, N, \pi, K, p, v).$$

⁵The reader is referred to Section 2.5 for a plausible reason to name this rule after Shapley.

Dummy says that if nobody visits a given museum with the pass, then such museum gets no revenue.

DUM. For each $(M, N, \pi, K, p, v) \in \mathcal{P}^{pv}$, and $i \in M$, such that $U_i(K) = \emptyset$,

$$R_i(M, N, \pi, K, p, v) = 0.$$

Additivity on visitors says that, given two groups of customers acquiring the museum pass, it is equivalent to consider them separately, or as the same group.

ADV. For each (M, N, π, K, p, v) , (M, N^1, π, K^1, p, v) , $(M, N^2, \pi, K^2, p, v) \in \mathcal{P}^{pv}$, such that $N = N^1 \cup N^2$ and $K = (K^1, K^2)$,

$$R(M, N, \pi, K, p, v) = R(M, N^1, \pi, K^1, p, v) + R(M, N^2, \pi, K^2, p, v).$$

The next axiom is *proportionality to independent visits*, which refers to the effect that the number of visits without the pass should have on the allocation process. In order to motivate it, consider two museums i and j such that the only difference between them is that $v_i = 2v_j$. In such a case, it seems natural that the revenue of museum i be twice the revenue of museum j . More generally, the axiom says the following:

PIV. For each $(M, N, \pi, K, p, v) \in \mathcal{P}^{pv}$ and $i, j \in M$ such that $U_i(K) = U_j(K)$, $p_i = p_j$ and $v_i = \lambda v_j$, for some $\lambda \in \mathbb{Z}_{++}$,

$$R_i(M, N, \pi, K, p, v) = \lambda R_j(M, N, \pi, K, p, v).$$

Notice that *PIV* is a stronger axiom than *ETE*.

The last axiom we consider is *splitting-proofness*. To motivate this axiom, think of the situation in which a museum considers splitting into several museums (for instance, each floor of the initial museum is considered as a new independent museum), assuming that the customers that would had visited the original museum (with or without the pass) would also visit each new museum, and that the entrance price of the original museum is the sum of the entrance prices of the new museums. The axiom requires that, under these specific circumstances, the revenue obtained by each of the other pre-existing museums does not change.

SP. Let $(M, N, \pi, K, p, v) \in \mathcal{P}^{pv}$, $i \in M$, and (M', N, π, K', p', v') be such that:

- $M' = (M \setminus \{i\}) \cup \{i^1, \dots, i^r\}$.
- For each $l \in N$, $K'_l = K_l$ if $i \notin K_l$ and $K'_l = (K_l \setminus \{i\}) \cup \{i^1, \dots, i^r\}$ otherwise.
- For each $j \in M \setminus \{i\}$, $p'_j = p_j$ and $p'_{i^1} + p'_{i^2} + \dots + p'_{i^r} = p_i$.
- For each $j \in M \setminus \{i\}$, $v'_j = v_j$ and $v'_{i^1} = v'_{i^2} = \dots = v'_{i^r} = v_i$.

Then,

$$R_j(M, N, \pi, K, p, v) = R_j(M', N, \pi, K', p', v'), \text{ for each } j \in M \setminus \{i\}.$$

Notice that SP implies that

$$R_i(M, N, \pi, K, p, v) = \sum_{s=1}^r R_{i^s}(M', N, \pi, K', p', v').$$

This fact will be used often in the proofs of some results.

2.4 Axiomatic results

We start this section by presenting a straightforward characterization of the Shapley rule in \mathcal{P} , which partly motivates the remaining results of the paper.

Proposition 1 *A rule defined on \mathcal{P} satisfies equal treatment of equals, dummy and additivity if and only if it is the Shapley rule.*

Proof. It is obvious that the Shapley rule satisfies the axioms. Let R be a rule satisfying the three axioms in the statement, and let $(M, N, \pi, K) \in \mathcal{P}$ and $l \in N$. By *DUM*, $R_i(M, \{l\}, \pi, K) = 0$ for each $i \notin K_l$. By *ETE*, $R_i(M, \{l\}, \pi, K) = R_j(M, \{l\}, \pi, K)$ for each $i, j \in K_l$. As $\sum_{i \in M} R_i(M, \{l\}, \pi, K) = \pi$, we deduce that $R_i(M, \{l\}, \pi, K) = \frac{\pi}{k_l}$ for each $i \in K_l$. Consequently, it follows, by *ADV*, that $R_i(M, N, \pi, K) = \sum_{l \in N, i \in K_l} \frac{\pi}{k_l}$, for each $i \in M$, as desired.

■

Remark 1 *The axioms used in Proposition 1 are independent.*

Let R^1 be the rule in which, for each customer j , the amount he pays goes to the museum with the lowest number he visited. Namely, for each problem $(M, N, \pi, K) \in \mathcal{P}$, and each $i \in M$,

$$R_i^1(M, N, \pi, K) = \sum_{j \in N: i = \min_{i' \in K_j} \{i'\}} \pi.$$

R^1 satisfies *DUM* and *ADV*, but fails to satisfy *ETE*.

Let R^2 be the equal split rule. Namely, for each problem $(M, N, \pi, K) \in \mathcal{P}$, and $i \in M$,

$$R_i^2(M, N, \pi, K) = \frac{E}{n}.$$

R^2 satisfies *ETE* and *ADV*, but fails to satisfy *DUM*.

Let R^3 be the rule that divides the total amount among all museums proportionally to their total number of visits. Namely, for each problem $(M, N, \pi, K) \in \mathcal{P}$, and $i \in M$,

$$R_i^3(M, N, \pi, K) = \frac{v_i}{\sum_{j \in M} v_j} E.$$

R^3 satisfies *ETE* and *DUM*, but fails to satisfy *ADV*. ■

Proposition 1 crucially relies on its restricted domain assumption. For instance, it is straightforward to see that the p -Shapley and pv -Shapley rules also satisfy equal treatment of equals, dummy and additivity on their respective domains. Actually, the next proposition shows that if we strengthen equal treatment of equals to proportionality to independent visits, and add splitting-proofness to the axioms used in Proposition 1, we single-out the pv -Shapley rule in the general domain \mathcal{P}^{pv} .

Proposition 2 *A rule defined on \mathcal{P}^{pv} satisfies proportionality to independent visits, dummy, additivity, and splitting-proofness if and only if it is the pv -Shapley rule.*

Proof. It is straightforward to prove that the pv -Shapley rule satisfies the axioms.

Let R be a rule, defined on \mathcal{P}^{pv} , satisfying *PIV*, *DUM*, *ADV*, and *SP*.

Let $(M, N, \pi, K, p, v) \in \mathcal{P}^{pv}$ and $i \in M$. By *ADV*,

$$R_i(M, N, \pi, K, p, v) = \sum_{l \in N} R_i(M, \{l\}, \pi, K_l, p, v).$$

By *DUM*, $R_i(N, \{l\}, \pi, K_l, p, v) = 0$, for each $i \notin K_l$. Thus,

$$R_i(M, N, \pi, K, p, v) = \sum_{l \in N, i \in K_l} R_i(M, \{l\}, \pi, K_l, p, v).$$

Thus, it suffices to prove that $R_i(M, \{l\}, \pi, K_l, p, v) = S_i^{pv}(M, \{l\}, \pi, K_l, p, v)$, for each $l \in N$ and $i \in K_l$.

Fix then $l \in N$ and $i \in M$ such that $i \in K_l$. We assume that p_i is not a rational number, whereas, for each $j \in K_l \setminus \{i\}$, p_j is a rational number.⁶

Let us denote $P = (M, \{l\}, \pi, K_l, p, v)$. For each $\varepsilon > 0$, and each $j \in K_l \setminus \{i\}$, let $q_j^\varepsilon, q^{*\varepsilon}, q_i^\varepsilon \in \mathbb{Z}_{++}$ be such that $\frac{1}{q^{*\varepsilon}} \leq \varepsilon$, $p_j = \frac{q_j^\varepsilon}{q^{*\varepsilon}}$ and $\frac{q_i^\varepsilon}{q^{*\varepsilon}} < p_i < \frac{q_i^\varepsilon + 1}{q^{*\varepsilon}}$.

For each $T \subset K_l \setminus \{i\} = \{h_1, \dots, h_{k_l-1}\}$ let $P^T = (M^T, \{l\}, \pi, K_l^T, p^T, v^T)$ be the problem in which each museum $j \in T$ splits into q_j^ε museums $\{j^1, \dots, j^{q_j^\varepsilon}\}$ and, for each $r = 1, \dots, q_j^\varepsilon$, $p_{j^r} = \frac{1}{q^{*\varepsilon}}$ and $v_{j^r} = v_j$. Besides, the museums in $M \setminus T$ remain the same. Formally,

- $M^T = (M \setminus T) \cup (\cup_{j \in T} \{j^1, \dots, j^{q_j^\varepsilon}\})$,
- $K_l^T = \cup_{j \in T} ((K_l \setminus \{j\}) \cup \{j^1, \dots, j^{q_j^\varepsilon}\})$,
- $p^T = (p_k^T)_{k \in M^T} \in \mathbb{R}_+^{|M^T|}$ is such that $p_{j^r}^T = p_{j^r} = \frac{1}{q^{*\varepsilon}}$, for each $j \in T$, and each $r = 1, \dots, q_j^\varepsilon$, whereas $p_k^T = p_k$, for each $k \in M \setminus T$,
- $v^T = (v_k^T)_{k \in M^T} \in \mathbb{Z}_+^{|M^T|}$ is such that $v_{j^r}^T = v_j$, for each $j \in T$, and each $r = 1, \dots, q_j^\varepsilon$, whereas $v_k^T = v_k$, for each $k \in M \setminus T$.

Let $P' = (M', \{l\}, \pi, K_l', p', v')$ be the problem arising from $P^{\{h_1, \dots, h_{k_l-1}\}}$ after museum i splits into $q_i^\varepsilon + 1$ museums $\{i^1, \dots, i^{q_i^\varepsilon+1}\}$ such that $p_{i^r} = \frac{1}{q^{*\varepsilon}}$ for each $r = 1, \dots, q_i^\varepsilon$, $p_{i^{q_i^\varepsilon+1}} = p_i - \frac{q_i^\varepsilon}{q^{*\varepsilon}}$, and $v_{i^r} = v_i$ for each $r = 1, \dots, q_i^\varepsilon + 1$. Formally,

- $M' = (M^{\{h_1, \dots, h_{k_l-1}\}} \setminus \{i\}) \cup \{i^1, \dots, i^{q_i^\varepsilon+1}\}$,
- $K_l' = (K_l^{\{h_1, \dots, h_{k_l-1}\}} \setminus \{i\}) \cup \{i^1, \dots, i^{q_i^\varepsilon+1}\}$,
- $p' = (p_k')_{k \in M'} \in \mathbb{R}_+^{|M'|}$ is such that $p_{i^r}' = \frac{1}{q^{*\varepsilon}}$, for each $r = 1, \dots, q_i^\varepsilon$, $p_{i^{q_i^\varepsilon+1}}' = p_i - \frac{q_i^\varepsilon}{q^{*\varepsilon}}$, whereas $p_k' = p_k$, for each $k \in M^{\{h_1, \dots, h_{k_l-1}\}} \setminus \{i\}$,
- $v' = (v_k')_{k \in M'} \in \mathbb{Z}_+^{|M'|}$ is such that $v_{i^r}' = v_i$ for each $r = 1, \dots, q_i^\varepsilon + 1$ whereas $v_k' = v_k$, for each $k \in M^{\{h_1, \dots, h_{k_l-1}\}} \setminus \{i\}$.

Consider the sequence of problems

$$P \rightarrow P^{\{h_1\}} \rightarrow P^{\{h_1, h_2\}} \rightarrow \dots \rightarrow P^{\{h_1, \dots, h_{k_l-1}\}} \rightarrow P'.$$

⁶Using an induction argument we can extend this proof to the general case in which a number of museums (not necessarily one) have prices that are not rational numbers. The case in which all prices are rational numbers is similarly obtained.

Notice that each problem is obtained from the previous one after some museum splits into several ones, as in the axiom of SP . Thus, we can apply such axiom to each pair of consecutive problems. More precisely:

- By SP , applied to P and $P^{\{h_1\}}$,

$$R_j(P) = R_j(P^{\{h_1\}}) \text{ for each } j \in K_l \setminus \{h_1\}, \text{ and}$$

$$R_{h_1}(P) = \sum_{r=1}^{q_{h_1}^\varepsilon} R_{h_1^r}(P^{\{h_1\}})$$

- By SP , applied to $P^{\{h_1\}}$ and $P^{\{h_1, h_2\}}$,

$$R_j(P^{\{h_1\}}) = R_j(P^{\{h_1, h_2\}}) \text{ for each } j \in K_l \setminus \{h_1, h_2\},$$

$$R_{h_1^r}(P^{\{h_1\}}) = R_{h_1^r}(P^{\{h_1, h_2\}}) \text{ for each } r = 1, \dots, q_{h_1}^\varepsilon, \text{ and}$$

$$R_{h_2}(P^{\{h_1\}}) = \sum_{r=1}^{q_{h_2}^\varepsilon} R_{h_2^r}(P^{\{h_1, h_2\}})$$

- Reiterating the previous argument, we end up obtaining, after applying SP to $P^{\{h_1, \dots, h_{k_l-1}\}}$ and P' ,

$$R_{h_\alpha^r}(P^{\{h_1, \dots, h_{k_l-1}\}}) = R_{h_\alpha^r}(P') \text{ for each } \alpha = 1, \dots, k_l - 1 \text{ and } r = 1, \dots, q_{h_\alpha}^\varepsilon, \text{ and}$$

$$R_i(P^{\{h_1, \dots, h_{k_l-1}\}}) = \sum_{r=1}^{q_i^\varepsilon+1} R_{i^r}(P').$$

Thus,

$$R_i(P) = R_i(P^{\{h_1\}}) = \dots = R_i(P^{\{h_1, \dots, h_{k_l-1}\}}) = \sum_{r=1}^{q_i^\varepsilon} R_{i^r}(P') + R_{i^{q_i^\varepsilon+1}}(P').$$

As R satisfies PIV , which implies ETE , it follows that

$$R_i(P) = q_i^\varepsilon R_{i^1}(P') + R_{i^{q_i^\varepsilon+1}}(P'). \quad (1)$$

Let $j \in K_l \setminus \{i\}$. Then $j = h_g$ for some $g = 1, \dots, k_l - 1$. Now,

$$R_j(P) = R_j(P^{\{h_1\}}) = \dots = R_j(P^{\{h_1, \dots, h_{g-1}\}})$$

$$= \sum_{r=1}^{q_{h_g}^\varepsilon} R_{h_g^r}(P^{\{h_1, \dots, h_g\}})$$

$$= \sum_{r=1}^{q_{h_g}^\varepsilon} R_{h_g^r}(P^{\{h_1, \dots, h_{g+1}\}}) = \dots = \sum_{r=1}^{q_{h_g}^\varepsilon} R_{h_g^r}(P').$$

As R satisfies PIV , which implies ETE , it follows that

$$R_j(P) = \sum_{r=1}^{q_j^\varepsilon} R_{h_r^j}(P') = q_j^\varepsilon R_{j^1}(P') = q_j^\varepsilon \frac{v_j}{v_i} R_{i^1}(P'). \quad (2)$$

Claim. $R_{i^{q_i^\varepsilon+1}}(P') \leq R_{i^1}(P')$.

In order to prove the claim, let P^α be the problem obtained from P' after splitting museum i^1 in two museums: α^1 , with price $p_i - \frac{q_i^\varepsilon}{q^{*\varepsilon}}$, and α^2 with price $\frac{1}{q^{*\varepsilon}} - \left(p_i - \frac{q_i^\varepsilon}{q^{*\varepsilon}}\right)$. As R is non-negative and satisfies SP ,

$$R_{\alpha^1}(P^\alpha) \leq R_{\alpha^1}(P^\alpha) + R_{\alpha^2}(P^\alpha) = R_{i^1}(P').$$

Furthermore, by SP ,

$$R_{i^{q_i^\varepsilon+1}}(P') = R_{i^{q_i^\varepsilon+1}}(P^\alpha) = R_{\alpha^1}(P^\alpha).$$

Thus,

$$R_{i^{q_i^\varepsilon+1}}(P') \leq R_{i^1}(P'),$$

as stated in the claim.

The claim implies that $R_i(P)$ is minimum when $R_{i^{q_i^\varepsilon+1}}(P') = 0$ and maximum when $R_{i^{q_i^\varepsilon+1}}(P') = R_{i^1}(P')$. We then compute it in both cases:

- Suppose $R_{i^{q_i^\varepsilon+1}}(P') = 0$. Then, by (1) and (2),

$$\begin{aligned} \pi &= R_i(P) + \sum_{j \in K_I \setminus \{i\}} R_j(P) \\ &= q_i^\varepsilon R_{i^1}(P') + \sum_{j \in K_I \setminus \{i\}} q_j^\varepsilon \frac{v_j}{v_i} R_{i^1}(P') \\ &= R_{i^1}(P') \left(q_i^\varepsilon + \sum_{j \in K_I \setminus \{i\}} q_j^\varepsilon \frac{v_j}{v_i} \right). \end{aligned}$$

Now,

$$\begin{aligned} q_i^\varepsilon + \sum_{j \in K_I \setminus \{i\}} q_j^\varepsilon \frac{v_j}{v_i} &= \frac{q_i^\varepsilon}{q^{*\varepsilon}} q^{*\varepsilon} + \sum_{j \in K_I \setminus \{i\}} \frac{q_j^\varepsilon}{q^{*\varepsilon}} \frac{v_j}{v_i} q^{*\varepsilon} \\ &= \frac{q^{*\varepsilon}}{v_i} \left(\frac{q_i^\varepsilon}{q^{*\varepsilon}} v_i + \sum_{j \in K_I \setminus \{i\}} p_j v_j \right). \end{aligned}$$

Thus,

$$R_{i^1}(P') = \frac{\pi v_i}{\left(\frac{q_i^\varepsilon}{q^{*\varepsilon}} v_i + \sum_{j \in K_I \setminus \{i\}} p_j v_j \right) q^{*\varepsilon}},$$

and, therefore,

$$R_i(M, \{l\}, \pi, K_l, p, v) = \frac{\pi v_i \frac{q_i^\varepsilon}{q^{*\varepsilon}}}{\frac{q_i^\varepsilon}{q^{*\varepsilon}} v_i + \sum_{j \in K_l \setminus \{i\}} p_j v_j}. \quad (3)$$

Likewise, for each $j \in K_l \setminus \{i\}$,

$$\begin{aligned} R_j(M, \{l\}, \pi, K_l, p, v) &= q_j^\varepsilon \frac{v_j}{v_i} R_{i^1}(P') \\ &= \frac{\pi v_j}{\left(\frac{q_i^\varepsilon}{q^{*\varepsilon}} v_i + \sum_{j \in K_l \setminus \{i\}} p_j v_j \right) q^{*\varepsilon}} q_j^\varepsilon \\ &= \frac{\pi p_j v_j}{\frac{q_i^\varepsilon}{q^{*\varepsilon}} v_i + \sum_{j \in K_l \setminus \{i\}} p_j v_j}. \end{aligned}$$

- Suppose $R_{i^{q_i^\varepsilon+1}}(P') = R_{i^1}(P')$. Using arguments similar to those used in the previous bullet item we can prove that

$$R_i(M, \{l\}, \pi, K_l, p, v) = \frac{\pi v_i \frac{q_i^\varepsilon+1}{q^{*\varepsilon}}}{\frac{q_i^\varepsilon+1}{q^{*\varepsilon}} v_i + \sum_{j \in K_l \setminus \{i\}} p_j v_j}, \quad (4)$$

and, for each $j \in K_l \setminus \{i\}$,

$$R_j(M, \{l\}, \pi, K_l, p, v) = \frac{\pi p_j v_j}{\frac{q_i^\varepsilon+1}{q^{*\varepsilon}} v_i + \sum_{j \in K_l \setminus \{i\}} p_j v_j}.$$

As $\frac{1}{q^{*\varepsilon}} \leq \varepsilon$, and $\varepsilon > 0$ is arbitrary, we deduce from (3) and (4) that, for each $i \in K_l$,

$$R_i(M, \{l\}, \pi, K_l, p, v) = \frac{p_i v_i}{\sum_{j \in K_l} p_j v_j} = S_i^{pv}(M, \{l\}, \pi, K_l, p, v),$$

as desired. ■

Remark 2 *The axioms used in Proposition 2 are independent.*

The p -Shapley rule S^p satisfies DUM, ADV and SP, but fails to satisfy PIV.

Let R^4 be the rule that divides the total amount proportionally to the product of the price and the number of independent visits. Namely, for each problem $(M, N, \pi, K, p, v) \in \mathcal{P}^{pv}$ and $i \in M$,

$$R_i^4(M, N, \pi, K, p, v) = \frac{p_i v_i}{\sum_{j \in M} p_j v_j} E.$$

R^4 satisfies PIV, ADV, and SP, but fails to satisfy DUM.

Let R^5 be the rule that divides the total amount among the museums proportionally to the product of the price, the number of independent visits, and the number of visits with pass. Namely, for each problem $(M, N, \pi, K, p, v) \in \mathcal{P}^{pv}$ and $i \in M$,

$$R_i^5(M, N, \pi, K, p, v) = \frac{p_i v_i \nu_i}{\sum_{j \in M} p_j v_j \nu_j} E.$$

R^5 satisfies PIV, DUM, and SP, but fails to satisfy ADV.

Let R^6 be such that for each problem $(M, N, \pi, K, p, v) \in \mathcal{P}^{pv}$ and $i \in M$,

$$R_i^6(M, N, \pi, K, p, v) = \sum_{j \in M, i \in K_j} \frac{p_i^2 v_i}{\sum_{i' \in K_j} p_{i'}^2 v_{i'}} E.$$

R^6 satisfies PIV, DUM, and ADV, but fails to satisfy SP. ■

We conclude this section with a counterpart characterization of the p -Shapley in \mathcal{P}^p .

Proposition 3 *A rule defined on \mathcal{P}^p satisfies dummy, additivity, and splitting-proofness if and only if it is the p -Shapley rule.*

Proof. It is straightforward to prove that S^p satisfies the axioms in the statement. Conversely, let R be a rule defined on \mathcal{P}^p satisfying DUM, ADV, and SP. Let $(M, N, \pi, K, p) \in \mathcal{P}^p$ and $i \in M$. By ADV,

$$R_i(M, N, \pi, K, p) = \sum_{l \in N} R_i(M, \{l\}, \pi, K_l, p).$$

By DUM, $R_i(M, \{l\}, \pi, K_l, p) = 0$, for each $i \notin K_l$. Thus,

$$R_i(M, N, \pi, K, p) = \sum_{l \in N, i \in K_l} R_i(M, \{l\}, \pi, K_l, p).$$

Fix $l \in N$ and consider the problem $(M, \{l\}, \pi, K_l, p)$. We show next that R satisfies ETE in $(K_l, \{l\}, \pi, K_l, p)$.⁷ Suppose, by contradiction, that there exist $i, j \in K_l$ such that $p_i = p_j$, and $R_i(K_l, \{l\}, \pi, K_l, p) \neq R_j(K_l, \{l\}, \pi, K_l, p)$. Consider then the two new problems in which all the remaining museums in K_l merge with i or j , respectively, to appear as a single museum, whose price is the sum of the former individual prices, i.e.,

$$P_i \equiv \left(\{i, h\}, \{l\}, \pi, \{i, h\}, \left(p_i, \sum_{k \in K_l \setminus \{i\}} p_k \right) \right),$$

⁷The proof of this claim is inspired by a similar argument developed by de Frutos (1999).

and

$$P_j \equiv \left(\{j, h\}, \{l\}, \pi, \{j, h\}, \left(p_j, \sum_{k \in K_i \setminus \{j\}} p_k \right) \right).$$

Then, by *SP*,

$$R(P_i) = (R_i(K_l, \{l\}, \pi, K_l, p), \pi - R_i(K_l, \{l\}, \pi, K_l, p)),$$

and

$$R(P_j) = (R_j(K_l, \{l\}, \pi, K_l, p), \pi - R_j(K_l, \{l\}, \pi, K_l, p)).$$

We now consider two new problems arising from P_i and P_j , respectively, after splitting museums i and j in two new museums each with half price, i.e.,

$$\hat{P}_i \equiv \left(\{i, j, h\}, \{l\}, \pi, \{i, j, h\}, \left(\frac{p_i}{2}, \frac{p_i}{2}, \sum_{k \in K_i \setminus \{i\}} p_k \right) \right),$$

and

$$\hat{P}_j \equiv \left(\{i, j, h\}, \{l\}, \pi, \{i, j, h\}, \left(\frac{p_j}{2}, \frac{p_j}{2}, \sum_{k \in K_i \setminus \{j\}} p_k \right) \right).$$

Then, by *SP*,

$$R_i(K_l, \{l\}, \pi, K_l, p) = R_i(\hat{P}_i) + R_j(\hat{P}_i),$$

and

$$R_j(K_l, \{l\}, \pi, K_l, p) = R_i(\hat{P}_j) + R_j(\hat{P}_j).$$

Now, note that $\hat{P}_i \equiv \hat{P}_j$. Thus, the above implies that $R_i(K_l, \{l\}, \pi, K_l, p) = R_j(K_l, \{l\}, \pi, K_l, p)$, a contradiction.

As R satisfies *ADV*, it follows from the above claim that R also satisfies *ETE* for each problem $(M, N, \pi, K, p) \in \mathcal{P}^p$.

The rest of the proof is almost analogous to the corresponding part of the proof of Proposition 2 and, thus, we omit it. ■

Remark 3 *The axioms used in Proposition 3 are independent.*

Let R^7 be the rule that divides the total amount proportionally to the price. Namely, for each problem $(M, N, \pi, K, p) \in \mathcal{P}^p$ and $i \in M$,

$$R_i^7(M, N, \pi, K, p) = \frac{p_i}{\sum_{j \in N} p_j} E.$$

R^7 satisfies *ADV* and *SP*, but fails to satisfy *DUM*.

Let R^8 be the rule that divides the total amount proportionally to the product of the price and the number of visits with pass. Namely, for each problem $(M, N, \pi, K, p) \in \mathcal{P}^p$ and $i \in M$,

$$R_i^8(M, N, \pi, K, p) = \frac{p_i \nu_i}{\sum_{j \in N} p_j \nu_j} E.$$

R^8 satisfies DUM and SP, but fails to satisfy ADV.

Let R^9 be such that, for each problem $(M, N, \pi, K, p) \in \mathcal{P}^p$ and $i \in N$,

$$R_i^9(M, N, \pi, K, p) = \sum_{j \in M, i \in K_j} \frac{p_i^2}{\sum_{j \in K_j} p_j^2} E.$$

R^9 satisfies DUM and ADV, but fails to satisfy SP. ■

2.5 Game-theoretical results

We revisit in this section the game-theoretical approach to the museum pass problem, as initially considered by Ginsburgh and Zang (2003).

A cooperative game with transferable utility, *TU* game, is a pair (Γ, u) where Γ denotes a set of agents and $u : 2^\Gamma \rightarrow R$ satisfies that $u(\emptyset) = 0$. For each coalition $S \subset \Gamma$, the unanimity game of the coalition S , u_S , is defined by $u_S(T) = 1$ if $T \supseteq S$ and $u_S(T) = 0$ otherwise. The Shapley value ϕ is the linear function that, for each unanimity game u_S , is defined by $\phi_i(u_S) = \frac{1}{|S|}$ if $i \in S$, and 0 otherwise. A *weighted Shapley value* generalizes the Shapley value by allowing different ways to split one unit between the members of S in u_S . More precisely, a vector of positive weights $\lambda = \{\lambda_i\}_{i \in \Gamma}$ is prescribed and in each u_S players split proportionally to their weights.⁸

Ginsburgh and Zang (2003) associate with each problem $P = (M, N, \pi, K) \in \mathcal{P}$ a cooperative game with transferable utility (M, u_P) where, for each $S \subset M$, $u_P(S)$ denotes the amount paid by the customers that only visited museums in S . Namely,

$$u_P(S) = \sum_{l \in N, K_l \subset S} \pi_l.$$

Ginsburgh and Zang (2003) proved that the Shapley rule is the Shapley value of the game (M, u_P) . The next proposition shows that the remaining rules we have presented here are weighted Shapley values of (M, u_P) .

Proposition 4 *The following statements hold:*

⁸See, e.g., Kalai and Samet (1987) for further details.

1. For each problem $P = (M, N, \pi, K, p, v) \in \mathcal{P}^{pv}$, $S^{pv}(P)$ coincides with the weighted Shapley value of (M, u_P) where the weight system is $(p_i v_i)_{i \in M}$.
2. For each problem $P = (M, N, \pi, K, p) \in \mathcal{P}^p$, $S^p(P)$ coincides with the weighted Shapley value of (M, u_P) where the weight system is $(p_i)_{i \in M}$.

Proof. We only prove the first statement, as the second can be similarly proved.

Let $P = (M, N, \pi, K, p, v) \in \mathcal{P}$. For each $l \in N$, we denote $P_l = (M, \{l\}, \pi, K_l, p, v)$. Then, for each $S \subset M$,

$$u_{P_l}(S) = \begin{cases} \pi & \text{if } K_l \subset S \\ 0 & \text{otherwise,} \end{cases}$$

and, therefore,

$$u_P(S) = \sum_{l \in N, K_l \subset S} \pi = \sum_{l \in N} u_{P_l}(S).$$

Thus, by the additivity of weighted Shapley values (e.g., Kalai and Samet, 1987), it follows that

$$Sh^w(M, u_P) = \sum_{l \in N} Sh^w(M, u_{P_l}),$$

for any weight system w .

Let $w = (p_i v_i)_{i \in M}$. Then, it is straightforward to see that

$$Sh_i^w(M, u_{P_l}) = \begin{cases} \frac{p_i v_i}{\sum_{j \in K_l} p_j v_j} \pi & \text{if } i \in K_l \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, $Sh^w(M, u_P) = S^{pv}(P)$, as desired. ■

3 The second model

We now consider a second way of modeling the museum pass problem. The difference with respect to the first model is that now we do not observe the museums visited by each holder of a museum pass. We only know the total number of pass holders that visited each museum. The resulting model has been already considered by Casas-Méndez *et al.*, (2011) and Estévez-Fernández *et al.*, (2012).

3.1 Preliminaries

Let us assume that the notation and mathematical conventions set at the beginning of Section 2 also hold here. A (museum) **problem** is now defined by a 6-tuple (M, n, π, ν, p, v) where:

- $M \in \mathcal{M}$ is the set of museums.
- $n \in \mathbb{Z}_+$ is the number of (museum) passes sold.
- $\pi \in \mathbb{R}_+$ is the price of a pass.
- $\nu \in \mathbb{Z}_+^m$ is the profile of pass visits (to each museum).
- $p \in \mathbb{R}_+^m$ is the profile of entrance prices.
- $v \in \mathbb{Z}_+^m$ is the profile of visits without the pass.

The revenue generated from selling passes is also denoted by E , i.e., $E = n\pi$. The family of all the problems described as above is now denoted by $\widehat{\mathcal{P}}^{pv}$. Similarly, we also denote in this setting by $\widehat{\mathcal{P}}^p$ the corresponding set of problems after ignoring information about the number of visits without pass, and by $\widehat{\mathcal{P}}$ the corresponding set of problems after also ignoring information about entrance prices.

3.2 Rules

A **rule** is a mapping that associates with each problem an allocation indicating the amount each museum gets from the revenue generated by passes sold. Formally, $R : \widehat{\mathcal{P}}^{pv} \rightarrow \mathbb{R}_+^m$ is such that, for each $(M, n, \pi, \nu, p, v) \in \widehat{\mathcal{P}}^{pv}$

$$\sum_{i \in M} R_i(M, n, \pi, \nu, p, v) = E.$$

We now give some examples of rules.

The *proportional rule* allocates the revenue proportionally to the number of pass visits. Namely, for each $(M, n, \pi, \nu, p, v) \in \widehat{\mathcal{P}}^{pv}$ and $i \in M$,

$$P_i(M, n, \pi, \nu, p, v) = \frac{\nu_i}{\sum_{j \in M} \nu_j} E.$$

Notice that, in this rule, neither prices nor independent visits play any role and, therefore, the definition of the rule would also apply for $\widehat{\mathcal{P}}$.

The *p-proportional rule* allocates the revenue proportionally to the amount each museum would receive if all pass holders visiting the museum would had paid the (full-fledged) entrance price. Namely, for each $(M, n, \pi, \nu, p, v) \in \widehat{\mathcal{P}}^{pv}$, and $i \in M$,

$$P_i^p(M, n, \pi, \nu, p, v) = \frac{p_i \nu_i}{\sum_{j \in M} p_j \nu_j} E.$$

Estévez-Fernández *et al.*, (2012) consider the *p-proportional rule* in $\widehat{\mathcal{P}}^p$.⁹

Finally, the *pv-proportional rule* allocates the revenue proportionally to the amount each museum would receive if all pass holders visiting the museum would had paid its (full-fledged) entrance price, weighted by the number of visits without pass. Namely, for each $(M, n, \pi, \nu, p, v) \in \widehat{\mathcal{P}}^{pv}$, and $i \in M$,

$$P_i^{pv}(M, n, \pi, \nu, p, v) = \frac{p_i \nu_i v_i}{\sum_{j \in M} p_j \nu_j v_j} E.$$

The *pv-proportional rule* has been considered in Casas-Méndez *et al.*, (2011).¹⁰

The three proportional rules defined above are natural applications of the general principle of proportionality to the three different classes of problems we have considered.

3.3 Axioms

We now present some axioms in this model. As in the previous section, and for ease of exposition, we only define the axioms in $\widehat{\mathcal{P}}^{pv}$. The counterpart definitions in $\widehat{\mathcal{P}}^p$ and $\widehat{\mathcal{P}}$ are straightforwardly obtained.

We first adapt the axioms of equal treatment of equals, dummy and additivity on visitors to this new model.

Equal treatment of equals (ETE). For each $(M, n, \pi, \nu, p, v) \in \widehat{\mathcal{P}}^{pv}$, and $i, j \in M$ such that $(\nu_i, p_i, v_i) = (\nu_j, p_j, v_j)$,

$$R_i(M, n, \pi, \nu, p, v) = R_j(M, n, \pi, \nu, p, v).$$

Dummy (DUM). For each $(M, n, \pi, \nu, p, v) \in \widehat{\mathcal{P}}^{pv}$, and $i \in M$, such that $\nu_i = 0$,

$$R_i(M, n, \pi, \nu, p, v) = 0.$$

⁹They actually endorse a two-stage modification of it, in which first *minimal rights*, i.e., the amount that is left, if positive, after granting each museum their *claim* (to be interpreted as the full-fledged entrance price for each pass visit they had) are assigned to each museum, and then the remaining revenue is allocated proportionally to the claims adjusted down by those minimal rights.

¹⁰They actually add the proviso that per capita rewards cannot exceed entrance prices.

Additivity on visitors (ADV). For each (M, n, π, ν, p, v) , $(M, n^1, \pi, \nu^1, p, v)$, $(M, n^2, \pi, \nu^2, p, v) \in \widehat{\mathcal{P}}^{pv}$, such that $n = n^1 + n^2$ and $\nu = \nu^1 + \nu^2$,

$$R(M, n, \pi, \nu, p, v) = R(M, n^1, \pi, \nu^1, p, v) + R(M, n^2, \pi, \nu^2, p, v).$$

We now introduce two other related axioms. First, an alternative form of additivity referring to the price of the museum pass. Formally,

Additivity on pass price (ADP). For each (M, n, π, ν, p, v) , (M, n, π^1, ν, p, v) , $(M, n, \pi^2, \nu, p, v) \in \widehat{\mathcal{P}}^{pv}$, such that $\pi = \pi^1 + \pi^2$,

$$R(M, n, \pi, \nu, p, v) = R(M, n, \pi^1, \nu, p, v) + R(M, n, \pi^2, \nu, p, v).$$

We conclude this inventory of axioms with a new (minimalistic) condition. *Compatibility* states that if the collected amount, via selling passes, equals the aggregate amount museums would had obtained charging their (full-fledged) entrance price to each pass visitor, weighted by the number of visits without the pass, then each museum should get such amount. Formally,

Compatibility (COMP). For each $(M, n, \pi, \nu, p, v) \in \widehat{\mathcal{P}}^{pv}$ such that $\sum_{i \in M} p_i \nu_i v_i = n\pi$,

$$R_i(M, n, \pi, \nu, p, v) = p_i \nu_i v_i,$$

for each $i \in M$.

3.4 Axiomatic results

Our first result says that the axioms of equal treatment of equals, dummy and additivity on visitors are incompatible in $\widehat{\mathcal{P}}$, and hence in $\widehat{\mathcal{P}}^{pv}$ and $\widehat{\mathcal{P}}^p$. Notice that this result is in stark contrast with those in the previous model, which showed that the three axioms were compatible (actually characterizing the Shapley rule).

Proposition 5 *There is no rule defined on $\widehat{\mathcal{P}}$ satisfying equal treatment of equals, dummy and additivity on visitors.*

Proof. Assume that there exists a rule R satisfying *ETE*, *DUM* and *ADV*. Let $M = \{1, 2, 3, 4, 5\}$, $\nu^1 = (1, 1, 0, 0, 0)$, $\nu^2 = (0, 0, 1, 1, 1)$ and $\pi \in \mathbb{R}_+$. Then, by *ETE* and *DUM*,

$$\begin{aligned} R(M, 2, \pi, \nu) &= \left(\frac{2\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5} \right), \\ R(M, 1, \pi, \nu^1) &= \left(\frac{\pi}{2}, \frac{\pi}{2}, 0, 0, 0 \right), \text{ and} \\ R(M, 1, \pi, \nu^2) &= \left(0, 0, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3} \right), \end{aligned}$$

which contradicts *ADV*. ■

Propositions 1 and 5 illustrate the importance of setting the informational basis to analyze museum problems, as the same principles can lead to different predictions in the models resulting from different informational bases.

In the rest of the section, we show how to escape from the impossibility stated in Proposition 5 replacing the notion of additivity. More precisely, the following result provides a characterization of the three proportional rules in the three different domains considered, using additivity on pass price and compatibility.¹¹

Proposition 6 *The following statements hold:*

1. *A rule defined on $\widehat{\mathcal{P}}^{pv}$ satisfies additivity on pass price and compatibility if and only if it is the *pv*-proportional rule.*
2. *A rule defined on $\widehat{\mathcal{P}}^p$ satisfies additivity on pass price and compatibility if and only if it is the *p*-proportional rule.*
3. *A rule defined on $\widehat{\mathcal{P}}$ satisfies additivity on pass price and compatibility if and only if it is the proportional rule.*

Proof. We only prove part (a). Parts (b) and (c) are similar and, thus, we omit their proofs.

It is straightforward to prove that the *pv*-proportional rule satisfies the axioms in statement (a).

Let R be a rule that satisfies *ADP* and *COMP*, $(M, n, \nu, p, v) \in \widehat{\mathcal{P}}^{pv}$, and $i \in M$. By *ADP*, for each $\pi_1, \pi_2 \in \mathbb{R}_+$,

$$R_i(M, n, \pi_1, \nu, p, v) + R_i(M, n, \pi_2, \nu, p, v) = R_i(M, n, \pi_1 + \pi_2, \nu, p, v),$$

which is precisely one of Cauchy's canonical functional equations. By definition of a rule we know that $0 \leq R_i(M, n, \pi, \nu, p, v) \leq n\pi$. Thus, for each interval $[a, b]$ and each $\pi \in [a, b]$ we have that $R_i(M, n, \pi, \nu, p, v)$ is bounded. Now, it follows that the unique solutions to such equation are the linear functions (e.g., Aczel, 2006; page 34). More precisely, there exists a function $g_i : \mathcal{M} \times \mathbb{Z} \times \mathbb{Z}_+^m \times \mathbb{R}_+^m \times \mathbb{Z}_+^m \rightarrow \mathbb{R}$ such that

$$R_i(M, n, \pi, \nu, p, v) = g_i(M, n, \nu, p, v) \pi,$$

¹¹At the risk of stressing the obvious, let us recall that the axioms used in each of the statements of Proposition 6 are the corresponding ones for each domain. Thus, in spite of sharing their names, they are all different axioms.

for each $(M, n, \pi, \nu, p, v) \in \widehat{\mathcal{P}}^{pv}$.

Let $(M, n, \pi, \nu, p, v) \in \widehat{\mathcal{P}}^{pv}$ be such that $\sum_{j \in M} p_j \nu_j v_j = n\pi$. By *COMP*,

$$R_i(M, n, \pi, \nu, p, v) = p_i \nu_i v_i.$$

Thus,

$$g_i(M, n, \nu, p, v) = \frac{p_i \nu_i v_i}{\pi} = \frac{p_i \nu_i v_i}{\sum_{j \in M} p_j \nu_j v_j} n$$

and, hence,

$$R_i(M, n, \pi, \nu, p, v) = \frac{p_i \nu_i v_i}{\sum_{j \in M} p_j \nu_j v_j} n\pi = P_i^{pv}(M, n, \pi, \nu, p, v),$$

as desired. ■

Remark 4 *The axioms used in Proposition 6 are independent.*

The equal split rule satisfies ADP in each domain, but fails to satisfy COMP.

Let R^{10} be such that, for each problem $(M, n, \pi, \nu, p, v) \in \widehat{\mathcal{P}}^{pv}$, and $i \in M$,

$$R_i^{10}(M, n, \pi, \nu, p, v) = \begin{cases} p_i \nu_i v_i & \text{if } \sum_{i \in M} p_i \nu_i v_i = n\pi \\ \frac{n\pi}{m} & \text{otherwise.} \end{cases}$$

R^{10} satisfies COMP, but fails to satisfy ADP. It is straightforward to adapt the definition of R^{10} to obtain the counterpart behavior for the remaining domains. ■

4 Discussion

We have analyzed in this paper the problem of sharing the revenue from bundled pricing taking an axiomatic perspective. We have presented two different models for this problem depending on the informational basis of the setting. Both of them enrich in specific ways previous contributions in the literature. Nevertheless, we obtain qualitatively different results in both models, which highlights the importance of setting one specific informational basis to analyze the problem. In the first model we analyze, which is more in line with the seminal contribution by Ginsburgh and Zang (2013), axiomatic rationale is provided for several rules endorsing (in several, albeit related, ways) the decomposition principle that underlies the so-called *Shapley rule* proposed by Ginsburgh and Zang (2013) in their simpler setting. Even though we also provide (axiomatic) rationale for somewhat counterpart rules in the second model we analyze, we show how an incompatibility arises when combining the exact counterpart axioms to some of those considered in the first model.

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