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Influence Networks and Public Goods

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Influence networks and public goods

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Abstract

We consider a model of social interactions in which agents are assumed to acquire information from others through a certain sampling process that generates an *influence network*. These networks comprise a wide array of options depending on the level of correlation assumed between agents' *in and out degree*. We study the provision of public goods in influence networks and show that the equilibrium (of the corresponding *best-shot game*) always exists and it is unique. We derive further insights for this problem by performing a comparative statics analysis.

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1 Introduction

The study of networks has been of significant importance in diverse academic fields such as sociology, physics and computer science (see, e.g., Wasserman and Faust 1994; Newman 2003, and the long list of references cited therein). The last two decades have witnessed how numerous phenomena of economic relevance have also been studied using the paradigm of networks. Instances are network formation (e.g., Jackson and Wolinsky 1996; Bala and Goyal 2000), diffusion of behaviors (Morris 2001; López-Pintado 2008a), labor markets (Calvó-Armengol 2004; Calvó-Armengol and Jackson 2007), peer effects in education (Calvó-Armengol et al.

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2009), crime activities (Ballester et al. 2006; Goyal and Virgier 2014), financial markets (Golub and Jackson 2014) and microfinance credits (Benerjee et al. 2013).[1]

This paper is concerned with the provision of public goods in a network context. Such a problem was first analyzed by Bramoullé and Kranton (2007) who characterized the (multiple) Nash equilibria of the associated game in a static framework. Boticinelli and Pin (2012) introduced a dynamic model and addressed the issue of equilibrium selection. Galeotti et al. (2010) studied public good games in networks with incomplete information, whereas Golub and Elliot (2014) focussed on the efficiency properties of public good games in a general weighted network.

Our approach is related to the literature on random networks originated to study the diffusion (or contagion) of an infectious disease in a population.[2] We depart from this literature in two main ways. On the first hand, we consider public good games instead of contagion games, i.e., actions are strategic substitutes instead of strategic complements. On the second hand, we consider a dynamic sampling process (the *influence network*) which allows us to obtain theoretical predictions. This is in contrast with the related literature on random networks in which the network is assumed to be fixed throughout the dynamics. As a consequence, the results must rely on mean-field approximations and therefore simulations are required to test such approximations.

We shall be concerned with *directed* networks, i.e., networks in which the benefits from interacting with an agent providing the public good is only one way. We define the *out-degree* of an agent as the number of agents he observes, whereas his *in-degree* indicates how many agents observe this agent. The population will typically be heterogenous with respect to degrees and thus will be characterized by an (out-)degree distribution. We then propose a sampling process (the *influence network*) comprising a wide array of options depending on the level of correlation assumed between agents' in and out degree. Two extreme cases can be singled out: The case where the in and out-degree coincide, and the case where the in and out-degree are completely uncorrelated. Our objective in this paper is to encompass these two extreme cases in a more general framework and to explore the effect that the in/out-degree correlation has on the predictions of the model.

[1] See Goyal (2007), Vega-Redondo (2007) and Jackson (2008) for comprehensive treatments of the general topic of social and economic networks.

[2] See, e.g., the seminal paper by Pastor-Satorrás and Vespignani (2001) and subsequent research on the issue by Jackson and Rogers (2007), López-Pintado (2008a) and Galeotti and Goyal (2009), among others.

We study a particular class of public good games known as the best-shot game. In this game agents have to make a binary decision (invest, or not, in the public good). Moreover, an agent has incentives to invest in the public good only if no other agent observed by him has already done so. We consider the best-shot game in an influence network context and show that the dynamics describing the evolution of the provision of the public good over time has a unique globally stable *state*. In other words, the dynamics will always converge to a certain *fraction* of individuals providing the public good. Thus, comparative statics on such unique equilibrium outcome can be provided.

There are two outcomes of interest: (1) the fraction of public good providers in equilibrium, and (2) the fraction of samples that reach a public good provider in equilibrium. The comparative statics results lead to the following conclusions. On the one hand, for any given out-degree distribution, an increase in the in/out-degree correlation increases measure (1) and decreases measure (2). On the other hand, for any in/out-degree correlation, if the network is sufficiently dense, a First Order Stochastic Dominance shift (Mean Preserving Spread) of the out-degree distribution decreases (increases) measure (2).

The rest of the paper is organized as follows. In Section 2 we describe the model. In Section 3 we characterize the equilibrium. Section 4 addresses comparative statics results regarding the in/out-degree correlation and the out-degree distribution. In Section 5 we provide some final remarks.

2 The model

Consider a finite set of agents $N = \{1, ..., n\}$, with n sufficiently large. Each agent $i \in N$ chooses an action indicating whether or not to provide a (costly) local public good. More precisely, an agent chooses $a_i \in \{0, 1\}$, where $a_i = 1$ (0) is interpreted as the decision of (not) providing a public good. Let $a = (a_1, ..., a_n) \in \{0, 1\}^n$ denote an action profile. Each agent $i \in N$ is characterized by his out-degree d_i which is given exogenously and determines his sampling size. The sampling process is directed. That is, an agent i observes those agents that he samples, but he is not observed by them. Different agents might have different out-degrees, and thus, the population is described by $P(d)$, namely the out-degree distribution. In particular, $P(d)$ stands for the fraction of agents in the population with out-degree d. We consider a continuous-time dynamics to describe the sampling process as well as the evolution of agents' choices through time. At each time t an agent, say i, revises his action at a rate $\lambda \geq 0$.

Moreover, before revising his action, this agent selects from the population a certain sample of agents, say $N_i \subseteq N$ (of cardinality d_i), and chooses whether or not to provide the public good depending on the behavior of agents in N_i. We consider the best shot game to describe the incentive scheme. In particular, agent i would decide to provide the public good (i.e., choose $a_i = 1$) if and only if nobody in his sample N_i is providing it.[3] Notice that this simple behavioral rule could correspond with the optimal choice of agents under different specifications of their utility function. Our results would hold for any such specification.

We assume that sampling is not done completely at random, but instead, it is biased by out-degree. More precisely, let $\alpha \in \mathbb{R}$ determine the nature of such bias. In particular, the probability that a sample reaches an agent with out-degree d is proportional to $d^\alpha P(d)$. After a simple normalization, necessary to construct a new distribution, the probability of selecting an agent with out-degree d is equal to

$$\frac{1}{\langle d^\alpha \rangle_P} d^\alpha P(d),$$

where $\langle d^\alpha \rangle_P = \sum_d d^\alpha P(d)$. There are two special cases that could be singled out. Notice that if $\alpha = 0$ the sampling process is unbiased and, thus, agents select who to sample uniformly at random. If, instead, $\alpha = 1$ then an agent with out-degree d is selected d times more often than an agent with out-degree 1. [4]

The sampling process described above determines a well defined in-degree distribution denoted by $P_I(d)$, where $P_I(d)$ is the fraction of agents with (expected) in-degree d. The in-degree of an agent in this model corresponds with the expected number of agents that would potentially observe this agent if the whole population were to sample simultaneously. Intuitively, whereas the out-degree describes how much information an agent obtains before making a choice, the in-degree determines an agent's visibility. Consider a certain out-degree d. If all agents sampled simultaneously, the total number of samples that would reach the set of agents with out degree d would correspond with:

$$s_d = \frac{1}{\langle d^\alpha \rangle_P} d^\alpha P(d) \sum_k nP(k)k.$$

Notice that $\sum_k nP(k)k$ is the total number of samples performed by the overall population and $\frac{1}{\langle d^\alpha \rangle_P} d^\alpha P(d)$ is the probability that a sample reaches an agent with out-degree d. In order

[3]A best-shot game in networks has also been analyzed in other related papers such as Galeotti et al. (2010) and Botcinelli and Pin (2012), etc. These papers focus on the case where the network is undirected and fixed.

[4]López-Pintado (2008b) analyzes the case $\alpha = 1$, which could be interpreted as an approximation of an undirected network framework. López-Pintado (2013), however, studies the case $\alpha = 0$.

to compute the in-degree of the agent with out-degree d we now simply divide s_d by the total number of individuals with out-degree d. Thus,

$$d_{in} = \frac{s_d}{np(d)} = \frac{d^\alpha}{\langle d^\alpha \rangle_P} \langle d \rangle_P,$$

where d_{in} stands for the in-degree of an agent with out-degree d.

Therefore,

$$P_I(d_{in}) = P(d).$$

The case $\alpha = 0$ will be referred as the case with no in/out-degree correlation. Here, all agents are homogeneous with respect to their visibility and they have an in-degree equal to the average out-degree $\langle d \rangle_P$. Then,

$$P_I(d) = 1 \text{ if and only if } d = \langle d \rangle_P$$

If, however, $\alpha = 1$, this case is referred as the "perfect correlation case" as an agent's out-degree and his in-degree coincide. Then $d_{in} = d$ and

$$P_I(d) = P(d).$$

Hereafter the in/out-degree correlation can be roughly identified with parameter α when $\alpha \in [0, 1]$ as an increase in such parameter corresponds with an increase in the similarity between the out-degree and in-degree of agents.

If $\alpha > 1$ this would represent a case where the effect that having a high out-degree has on the in-degree is amplified. That is, having a high out-degree leads to an even higher in-degree, whereas having a low out-degree leads to an even lower in-degree. As a consequence, $P_I(d_{in})$ would typically have a larger variance than $P(d)$.

Finally, if $\alpha < 0$ this means that there is a negative correlation between out-degree and in-degree (unlike the cases where $\alpha > 0$, for which the correlation is always positive). That is, agents that are observed by many others (i.e., have a high in-degree), will typically have a very low out-degree (i.e., will not observe many others). The variance of the in-degree distribution in this case will increase as α decreases.

For most of the paper we will concentrate on the case $\alpha \in [0, 1]$, although we will point out which results can be extended to other values of α.

3 The equilibrium

We now formalize the dynamic process introduced above and study its long-run properties. It will become clear later in the text that the notion of stationary state in this context is not completely standard as it will not require that agents are choosing the same action over time. Let $\rho_d(t)$ denote the proportion of agents with out-degree d that are choosing action 1 at time t. A state is determined by the profile $\{\rho_d(t)\}_{d\geq 1}$, where we assume that all agents have at least sample size 1. There are two measures that will be of particular importance for our analysis.

First, the overall fraction of agents choosing 1 (non-conditional on degree). This measure is denoted by $\rho(t)$ and can be computed as follows:

$$\rho(t) = \sum_{d\geq 1} P(d)\rho_d(t).$$

Second, the probability that a samples reaches an agent choosing 1. This probability is represented by $\theta(t)$ and can be computed as follows:

$$\theta(t) = \frac{1}{\langle d^\alpha \rangle_P} \sum_{d\geq 1} d^\alpha P(d)\rho_d(t). \tag{1}$$

The computation of $\theta(t)$ is derived from the fact that $\frac{d^\alpha P(d)}{\langle d^\alpha \rangle}$ is the probability of selecting an agent with out-degree d and, conditional on having out-degree d, the probability of providing the public good is $\rho_d(t)$. Notice that, in the extreme case where $\alpha = 0$ (i.e., there is no correlation between in and out-degrees) then $\theta(t) = \rho(t)$ but, in general, these two measures will differ.

For each degree d, the deterministic approximation of the evolution of $\rho_d(t)$ is given by the following differential equation:

$$\frac{d\rho_d(t)}{dt} = (1-\rho_d(t))\lambda(1-\theta(t))^d - \rho_d(t)\lambda(1-(1-\theta(t))^d). \tag{2}$$

Notice that the positive term in the equation accounts for transitions from action 0 to 1, whereas the negative term accounts for the reverse transitions (i.e., from action 1 to 0). The first term can be interpreted as follows: $1-\rho_d(t)$ is the probability that an agent with out-degree d is choosing 0 at time t. This agent would revise at a rate λ and would switch to 1 if nobody in his sample is choosing 1, something which occurs with probability $(1-\theta(t))^d$. The second term can be interpreted in a similar way: $\rho_d(t)$ is the probability that an agent with out-degree d is choosing 1 at time t. This agent would revise at a rate λ and would switch to 0 if at least one agent in his sample is choosing 1, something which occurs with probability $1-(1-\theta(t))^d$.

After simplifications of equation (2) we find that, for each d,

$$\frac{d\rho_d(t)}{dt} = \lambda(-\rho_d(t) + (1-\theta(t))^d). \tag{3}$$

In a stationary state (or equilibrium) of this dynamics, $\frac{d\rho_d(t)}{dt} = 0$, for all d, which implies that:

$$\rho_d(\theta) = (1-\theta)^d. \tag{4}$$

Therefore, in equilibrium, and given equation (1), θ^* must be a solution of the following (fixed-point) equation:

$$\theta = H_{P,\alpha}(\theta), \tag{5}$$

where we define

$$H_{P,\alpha}(\theta) = \sum_{d \geq 1} \frac{d^\alpha P(d)}{\langle d^\alpha \rangle_P}(1-\theta)^d.$$

Once we know θ^* in equilibrium we can also determine ρ_d^* for each d and, consequently, the overall fraction of public good providers ρ^*.

Notice that, in the dynamics described above, agents can switch actions (from 0 to 1 and vice-versa) in equilibrium, given that the network is randomly generated every period. Therefore, the concept of stationary state, only refers to stationary values for θ, $\{\rho_d\}_d$ and ρ. The next result addresses the issue of existence and uniqueness of the equilibrium.

Proposition 1 *Given an out-degree distribution P and a sampling process characterized by $\alpha \in [0,1]$, there exists a unique equilibrium of the dynamic model. Furthermore, this equilibrium is globally stable.*

Proof. Note that $\frac{dH_{P,\alpha}(\theta)}{d\theta} = -\frac{1}{\langle d^\alpha \rangle_P} \sum_{d \geq 1} d^\alpha P(d) d (1-\theta)^{d-1} \leq 0$ for all $\theta \in [0,1]$. Therefore, $H_{P,\alpha}(\theta)$ is a (continuous and) decreasing function of θ. Furthermore, $H_{P,\alpha}(0) = 1$ and $H_{P,\alpha}(1) = 0$. Thus, there exists a unique solution $\theta^* \in (0,1)$ of equation (5) and therefore a unique value for θ (and ρ) in equilibrium. To conclude, let us show that θ^* is globally stable, i.e., starting from any initial fraction of agents choosing 1 ($\rho = \rho_0$), the dynamics converges to a state where $\theta = \theta^*$. To do so, notice that

$$\frac{d\theta(t)}{dt} = \frac{1}{\langle d^\alpha \rangle_P} \sum_{d \geq 1} d^\alpha P(d) \frac{d\rho_d(t)}{dt},$$

and, substituting $\frac{d\rho_d(t)}{dt}$ for its value determined by (3), we obtain that

$$\frac{d\theta(t)}{dt} = \frac{1}{\langle d^\alpha \rangle_P} \sum_{d \geq 1} d^\alpha P(d) \lambda(-\rho_d + (1-\rho(t))^d);$$

or, equivalently,

$$\frac{d\theta(t)}{dt} = \lambda(H_{P,\alpha}(\theta(t)) - \theta(t)),$$

from where the desired conclusion follows. ■

Notice that Proposition 1 actually holds for all $\alpha \in \mathbb{R}$.

4 Comparative Statics

In the next section we develop comparative statics results with respect to the in/out-degree correlation α and the out-degree distribution P (the two primitives of the sampling process). For this purpose, let $\rho^*(P, \alpha)$ and $\theta^*(P, \alpha)$ denote the equilibrium values for ρ and θ, respectively, given P and α.

Consider first the comparative statics with respect to α.

Proposition 2 *Let P be a given out-degree distribution. Also, let $\alpha_1 \in [0, 1]$ and $\alpha_2 \in [0, 1]$ be two levels of in/out-degree correlation. If $\alpha_1 \leq \alpha_2$ then $\theta^*(P, \alpha_1) \geq \theta^*(P, \alpha_2)$ and $\rho^*(P, \alpha_1) \leq \rho^*(P, \alpha_1)$.*

The intuition for such a result is the following. It is always the case that, in equilibrium, the fraction of public good providers decreases as the out-degree increases, given that the chance of sampling an agent that is already providing the public good is lower for agents with smaller out-degrees (see equation 4). If α increases, agents with low out-degree are observed by relatively fewer agents which is why the probability of sampling an agent choosing 1 (i.e., θ) would also decrease. To compensate for such a decrease in θ, in equilibrium, the fraction of agents providing the public good (i.e., ρ) increases with α.

The formal proof is the following.

Proof. Recall that the fixed point equation characterizing $\theta^*(P, \alpha)$ is

$$\theta = \sum_{d \geq 1} \frac{d^\alpha P(d)}{\langle d^\alpha \rangle_P} (1 - \theta)^d.$$

Let $Q_{\alpha,P}(d) = \frac{d^\alpha P(d)}{\langle d^\alpha \rangle_P}$. We can interpret $Q_{\alpha,P}(d)$ as the out-degree distribution of sampled agents. We show next that $Q_{\alpha_2,P}(d)$ First Order Stochastic Dominates $Q_{\alpha_1,P}(d)$ if and only if $\alpha_1 \leq \alpha_2$. Intuitively, a higher in/out-degree correlation implies that agents with high out-degree are sampled more often which also implies that the out-degree distribution of sampled agents must take larger values. Formally, we find that the cumulative distribution function of $Q_{\alpha_2,P}(d)$ is always below the cumulative distribution function of $Q_{\alpha_1,P}(d)$. That is, for all $D > 1$

$$\sum_{d\geq 1}^{D} \frac{d^{\alpha_2} P(d)}{\langle d^{\alpha_2}\rangle_P} \leq \sum_{d\geq 1}^{D} \frac{d^{\alpha_1} P(d)}{\langle d^{\alpha_1}\rangle_P},$$

or, analogously, that

$$\sum_{d\geq 1}^{D} d^{\alpha_2} P(d) \langle d^{\alpha_1}\rangle_P \leq \sum_{d\geq 1}^{D} d^{\alpha_1} P(d) \langle d^{\alpha_2}\rangle_P . \tag{6}$$

Condition (6) can be written as follows:

$$\sum_{d_1\geq 1}^{D} \sum_{d_2\geq 1} d_1^{\alpha_2} d_2^{\alpha_1} P(d_1)P(d_2) \leq \sum_{d_1\geq 1}^{D} \sum_{d_2\geq 1} d_1^{\alpha_1} d_2^{\alpha_2} P(d_1)P(d_2).$$

Note that the two expressions coincide, as long as d_2 is bounded below D. That is, we know that

$$\sum_{d_1\geq 1}^{D} \sum_{d_2\geq 1}^{D} d_1^{\alpha_2} d_2^{\alpha_1} P(d_1)P(d_2) = \sum_{d_1\geq 1}^{D} \sum_{d_2\geq 1}^{D} d_1^{\alpha_1} d_2^{\alpha_2} P(d_1)P(d_2).$$

For the part of the sum where d_2 exceeds D, however, this is no longer the case (since a permutation of the indices no longer appears in the sum). For such cases, notice that, as $d_1 < d_2$ and $\alpha_1 < \alpha_2$, then $d_1^{\alpha_2} d_2^{\alpha_1} < d_1^{\alpha_1} d_2^{\alpha_2}$. Thus,

$$\sum_{d_1\geq 1}^{D} \sum_{d_2\geq D} d_1^{\alpha_2} d_2^{\alpha_1} P(d_1)P(d_2) < \sum_{d_1\geq 1}^{D} \sum_{d_2\geq D} d_1^{\alpha_1} d_2^{\alpha_2} P(d_1)P(d_2)$$

which proves condition (6).

To complete the proof we use that $\rho_d = (1-\theta)^d$ is decreasing as a function of d (for all $\theta \in [0,1]$), which implies that

$$\sum_{d\geq 1} \frac{d^{\alpha_2} P(d)}{\langle d^{\alpha_2}\rangle_P}(1-\theta)^d \leq \sum_{d\geq 1} \frac{d^{\alpha_1} P(d)}{\langle d^{\alpha_1}\rangle_P}(1-\theta)^d,$$

and thus that $\theta^*(P,\alpha_2) \leq \theta^*(P,\alpha_1)$. Then, $\rho_d^*(P,\alpha_1) \leq \rho_d^*(P,\alpha_2)$ for all d and $\rho^*(P,\alpha_1) \leq \rho^*(P,\alpha_2)$. ■

Notice that the result is true for all possible values of $\alpha \in \mathbb{R}$, not just $\alpha \in [0,1]$.

Consider now the comparative statics with respect to P. To do so, we consider a fix value of α and analyze how different out-degree distributions lead to different outcomes. We first study the effect of a First Order Stochastic Dominance shift of the out-degree distribution, and then analyze the effect of a Mean Preserving Spread.

Proposition 3 *Let $\alpha \in [0,1]$ and let d_m denote the minimum degree in the network. If $\overline{P}(d)$ First Order Stochastic Dominates $P(d)$ and*

$$1 - e^{\frac{-\alpha}{d_m}} \leq \frac{1}{\langle d^\alpha \rangle} \sum_{d \geq d_m} d^\alpha \overline{P}(d) e^{-\frac{d\alpha}{d_m}}, \tag{7}$$

then $\theta^*(\overline{P}, \alpha) \leq \theta^*(P, \alpha)$.

Notice that condition (7) is satisfied as long as d_m is sufficiently high. Roughly speaking, Proposition 3 implies that if the network becomes denser then, in equilibrium, the fraction of samples reaching the public good decreases. This result also implies that $\rho_d^*(\overline{P}, \alpha) \geq \rho_d^*(P, \alpha)$ for all d. The comparative statics, however, with respect to ρ is not transparent and it might depend on further properties of the out-degree distributions.[5]

The proof of the result is provided next.

Proof. We aim to show that $H_{\overline{P},\alpha}(\theta) \leq H_{P,\alpha}(\theta)$ for all $\theta \in [\theta^*(\overline{P}, \alpha), 1]$ as this would imply that $\theta^*(\overline{P}, \alpha) \leq \theta^*(P, \alpha)$. We first rewrite $H_{P,\alpha}(\theta)$ as

$$H_{P,\alpha}(\theta) = \frac{1}{\langle d^\alpha \rangle_P} \sum_{d \geq 1} P(d) g_\theta(d),$$

where

$$g_\theta(d) = d^\alpha (1-\theta)^d.$$

Let us show next that if $\theta \in [1 - e^{\frac{-\alpha}{d_m}}, 1]$ then $g_\theta(d)$ is decreasing for all $d \geq d_m$. Notice that

$$g'_\theta(d) = d^{\alpha-1}(1-\theta)^d [\alpha + d \ln(1-\theta)],$$

which is negative if and only if

$$d \geq \frac{-\alpha}{\ln(1-\theta)}.$$

Let θ_m be such that

$$d_m = \frac{-\alpha}{\ln(1-\theta_m)}.$$

That is, $\theta_m = 1 - e^{\frac{-\alpha}{d_m}}$. We can easily check that $\frac{-\alpha}{\ln(1-\theta)}$ is a decreasing function of θ which then implies that $g(d)$ is decreasing for all $d \geq d_m$, provided that $\theta \geq \theta_m$. Therefore, as $\overline{P}(d)$ First Order Stochastic Dominates $P(d)$,

$$\sum_{d \geq d_m} \overline{P}(d) g(d) \leq \sum_{d \geq d_m} P(d) g(d),$$

[5] Note that the result is straightforward if we considered the simpler framework where we compare two homogeneous populations, one with a low out-degree and another one with a high out-degree. In this case we find that the fraction of public good providers is higher for the low density network, as expected.

for all $\theta \in [\theta_m, 1]$. In addition, as d^α is an increasing function of d we know that $\langle d^\alpha \rangle_{\overline{P}} \geq \langle d^\alpha \rangle_P$. Thus,

$$H_{\overline{P},\alpha}(\theta) \leq H_{P,\alpha}(\theta) \text{ for all } \theta \in [\theta_m, 1].$$

To complete the proof we must show that $\theta_m \leq \theta^*(\overline{P}, \alpha)$, which is the case if $\theta_m \leq H_{\overline{P},\alpha}(\theta_m)$, or, analogously, if the next condition holds:

$$1 - e^{\frac{-\alpha}{d_m}} \leq \frac{1}{\langle d^\alpha \rangle_{\overline{P}}} \sum_{d \geq d_m} d^\alpha \overline{P}(d) e^{-\frac{d\alpha}{d_m}}.$$

■

It is straightforward to show that Proposition 3 holds for all values of $\alpha > 0$, but not for values of $\alpha < 0$, as d^α would be an increasing function of d and thus $\langle d^\alpha \rangle_{\overline{P}} \leq \langle d^\alpha \rangle_P$.

The last result compares two out-degree distributions with the same average out-degree, but with different variance. Which case would lead to a larger fraction of public good providers in equilibrium? The next result partially addresses this question.

Proposition 4 *Let $\alpha \in [0, 1]$ and let d_m denote the minimum degree in the network. If $\overline{P}(d)$ is a Mean Preserving Spread of $P(d)$, and*

$$1 - e^{-\frac{\alpha+\sqrt{\alpha}}{d_m}} \leq \frac{1}{\langle d^\alpha \rangle} \sum_{d \geq d_m} d^\alpha P(d) e^{-d\frac{\alpha+\sqrt{\alpha}}{d_m}}, \tag{8}$$

then $\theta^*(P, \alpha) \leq \theta^*(\overline{P}, \alpha)$.

Note that condition (8) holds as long as d_m is sufficiently high. Proposition 4 indicates that the number of samples reaching a public good (i.e., θ) increases with the heterogeneity of the network. This result also implies that $\rho^*_d(\overline{P}, \alpha) \leq \rho^*_d(P, \alpha)$ for all d. The comparative statics, however, with respect to ρ is not transparent and it might depend on further properties of the out-degree distributions.

Proof. We aim to show that $H_{P,\alpha}(\theta) \leq H_{\overline{P},\alpha}(\theta)$ for all $\theta \in [\theta^*(P, \alpha), 1]$ as this would imply that $\theta^*(P, \alpha) \leq \theta^*(\overline{P}, \alpha)$. Consider again

$$g_\theta(d) = d^\alpha (1 - \theta)^d$$

and let us show next that if $\theta \in [1 - e^{-\frac{\alpha+\sqrt{\alpha}}{d_m}}, 1]$ then $g(d)$ is convex for all $d \geq d_m$. We have that

$$g''_\theta(d) = (1 - \theta)^d d^{\alpha-2} f_\theta(d),$$

where

$$f_\theta(d) = \left[(\ln(1 - \theta))^2 d^2 + 2\alpha \ln(1 - \theta)d + \alpha(\alpha - 1)\right].$$

Notice that $f_\theta(d)$ is a parabolic function. It is straightforward to see that $g''_\theta(d) \geq 0$ if and only if $f_\theta(d) \geq 0$. Moreover, the positive solution of $f_\theta(d) = 0$ is $\widehat{d} = \frac{\alpha+\sqrt{\alpha}}{-\ln(1-\theta)}$. Therefore, if $d_m = \widehat{d}$ then $g''_\theta(d)$ is positive for all $d \geq d_m$. Let θ_m be such that

$$d_m = \frac{\alpha + \sqrt{\alpha}}{-\ln(1-\theta_m)}.$$

That is, $\theta_m = 1 - e^{-\frac{\alpha+\sqrt{\alpha}}{d_m}}$. Since $\frac{\alpha+\sqrt{\alpha}}{-\ln(1-\theta)}$ is a decreasing function of θ then $g(d)$ is convex for all $d \geq d_m$, provided that $\theta \geq \theta_m$. Therefore,

$$\sum_{d \geq d_m} P(d)g(d) \leq \sum_{d \geq d_m} \overline{P}(d)g(d),$$

for all $\theta \in [\theta_m, 1]$. In addition, we know that $\langle d^\alpha \rangle_P \geq \langle d^\alpha \rangle_{\overline{P}}$, as d^α is a concave function of d. Thus,

$$H_{P,\alpha}(\theta) \leq H_{\overline{P},\alpha}(\theta) \text{ for all } \theta \in [\theta_m, 1].$$

To complete the proof we must show that $\theta_m \leq \theta^*(P,\alpha)$, which is the case if $\theta_m \leq H_{P,\alpha}(\theta_m)$ or, analogously, if the next condition holds:

$$1 - e^{-\frac{\alpha+\sqrt{\alpha}}{d_m}} \leq \frac{1}{\langle d^\alpha \rangle} \sum_{d \geq d_m} d^\alpha P(d) e^{-d\frac{\alpha+\sqrt{\alpha}}{d_m}}.$$

■

It is straightforward to show that Proposition 4 does not hold for $\alpha < 0$ or $1 < \alpha$, since in either case d^α would be a convex function of d and thus $\langle d^\alpha \rangle_P \leq \langle d^\alpha \rangle_{\overline{P}}$.

5 Discussion

In this paper we propose a stylized model of public good provision in a network context. We describe the influence structure by means of an explicit sampling process characterized by the correlation between the out-degree (information level) and in-degree (visibility level) of agents. We observe that, in equilibrium, an increase in such a correlation increases the fraction of public good providers (i.e., ρ), but, on the contrary, it decreases the fraction of samples that reach a public good provider (i.e., θ). Since we have not introduced a specific utility function, we cannot argue which level of correlation is preferable from a social planner point of view (i.e., which correlation level maximizes the population's aggregate utility). Intuitively, what this study suggests is that if the cost of providing the public good is sufficiently low a higher in/out-degree correlation would be more desirable for society. The reason for this is that the increase in efficiency provided by having a higher ρ would be enough to counterbalance the decrease in efficiency due to having a lower θ. If, on the contrary, the cost of providing the public good is high enough, then the opposite intuition would hold and, as a consequence, the social planner would prefer a network with a low in/out-degree correlation.

We have also analyzed the effect that a variation of the out-degree distribution has on the equilibrium outcomes. Our results in this respect suggest that, if degrees are sufficiently large, an increase in the average level of information (i.e., an increase in the average degree) decreases the fraction of samples reaching a public good provider, whereas an increase in the dispersion of information (i.e., an increase in the variance of P) increases such fraction. In order to obtain results regarding the fraction of public good providers we would need to impose further properties on the out-degree distribution.

This paper contributes to the growing literature on public goods in networks by bridging the work developed in statistical physics (where random networks are commonly used) with the literature in economics, for which the problem of public good provision is a major topic of study. The assumption that the network is randomly generated every period is quite strong and thus, one possible direction for further study would be to enrich our model by allowing for clustering and community structures. This extension has already been addressed for contagion models with heterogenous agents and homphily (e.g., Jackson and López-Pintado, 2013).

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References

[1] Ballester C, Calvó-Armengol A, Zenou Y (2006) Who's Who in Networks. Wanted: the Key Player. Econometrica 75:1403-1418.

[2] Bramoullé Y, Kranton R (2007) Public Goods in Networks. Journal of Economic Theory 135:478-494.

[3] Boticinelli L, Pin P (2012) Stochastic Stability in the Best Shot Network Game. Games and Economic Behavior 75:538-554.

[4] Bala V, Goyal S (2000) A Non-Cooperative Model of Networks Formation. Econometrica 68:1181-1229.

[5] Calvó-Armengol A (2004) Job Contact Networks. Journal of Economic Theory 115:191-206.

[6] Calvó-Armengol A, Jackson MO (2007) Social Networks in Labor Markets: Wage and Employment Dynamics and Inequality. Journal of Economic Theory 132: 27-46.

[7] Calvó-Armengol A, Pattacchini E, Zenou Y (2009) Peer Effects and Social Networks in Education. Review of Economic Studies 76:1403-1417.

[8] Benerjee A, Chandrasekhar AG, Duflo E, Jackson MO (2013) The Diffusion of Microfinance. Science 341:1236498.

[9] Elliot ML, Golub B, Jackson MO (2014) Financial Networks and Contagion. American Economic Review 104:3115-3153.

[10] Elliot ML, Golub B (2013) A Network Approach to Public Goods. Proceedings of the Fourteenth ACM Conference on Electronic Commerce 377-378.

[11] Goyal S (2007) Connections. Princeton University Press.

[12] Goyal S, Virgier A (forthcoming) Attack, Defense and Contagion in Networks. Review of Economic Studies.

[13] Galeotti A, Goyal S (2009) Influencing the Influencers: A Theory of Strategic Diffusion. Rand Journal of Economics 40:509-532.

[14] Galeotti A, Goyal S, Jackson MO, Vega-Redondo F, Yariv L (2010) Network Games. Review of Economic Studies 77:218-244.

[15] Jackson MO (2008) Social and Economic Networks. Princeton University Press.

[16] Jackson MO, López-Pintado D (2013) Diffusion and Contagion in Networks with Heterogeneous Agents and Homophily. Network Science 1:49-67.

[17] Jackson MO, Rogers B (2007) Relating Network Structures to Diffusion Properties through Stochastic Dominance. The B. E. Journal of Theoretical Economics 7, Article 6.

[18] Jackson MO, Wolinsky A (1996) A Strategic Model of Social and Economic Networks. Journal of Economic Theory 71:44-74.

[19] López-Pintado D (2008a) Diffusion in Complex Social Networks. Games and Economic Behavior 62:573-590.

[20] López-Pintado D (2008b) The Spread of Free-Riding Behavior in a Social Network. Eastern Economic Journal 34:464-479.

[21] López-Pintado D (2012) Influence Networks. Games and Economic Behavior 75:776-787.

[22] López-Pintado D (2013) Public Goods in Directed Networks. Economic Letters 121:160-162.

[23] Morris S (2000) Contagion. Review of Economic Studies 67:57-78.

[24] Newman MEJ (2003) The Structure and Function of Complex Networks. SIAM Review 45:167-256.

[25] Pastor-Satorrás P, Vespignani A (2001) Epidemic Dynamics and Endemic States in Complex Networks. Physical Review E 63:066117.

[26] Vega-Redondo F (2007) Complex social networks. The Econometric Society Monograph Series. Cambridge University Press.

[27] Wasserman S, Faust K (1994) Social Network Analysis. Methods and Applications. Cambridge University Press.