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The Borda-Condorcet Social Evaluation Function

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Abstract

This paper presents a social evaluation function that combines the ideas of Borda and Condorcet by computing the support that each alternative receives on average, when confronted with any other in a series of tournaments. Even though the evaluation follows Condorcet's tournament approach and evaluates social alternatives in terms of pairwise comparisons, it ponders the outcomes of those comparisons differently, depending on how each alternative fares with respect to the others (a Borda count ingredient). The evaluation appears as the stable distribution of an iterative process in which each alternative competes randomly with any other, and results in a vector of positive numbers that tell us the relative social support of the different options.

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1 Introduction

The purpose of this paper is to present a cardinal social evaluation function that combines the ideas of Borda and Condorcet by computing the support that each alternative receives on average, when confronted with any other in a series of tournaments. We shall refer to this new rule, not very imaginatively, as the Borda-Condorcet social evaluation function.

Given a set of alternatives, or social states, and a society whose members have complete and transitive preferences over those alternatives, a social evaluation function (SEF for short) is a mapping that associates to each agents' preference profile a vector of numbers, one for each alternative, so that a higher number means a socially preferred alternative. A SEF is cardinal when those numbers also convey information about how much better is one alternative than another.

It is well known, by Arrow's Impossibility Theorem, that there is no SEF that is general (universal domain), democratic (no dictatorial and Pareto inclusive), and informationally efficient (the social evaluation of two alternatives only depends on how agents rank those two alternatives). Escaping from Arrow's result implies paying a price because one has to renounce to some of those desirable properties. Both Borda and Condorcet proposed choice rules based on sound principles that nevertheless failed to satisfy one of those requisites: Borda fails to satisfy informational efficiency whereas Condorcet does not ensure the existence of a social ordering, unless we renounce to universal domain.

The Borda rule is a SEF that assigns "a score to each alternative based on how high the alternative stands in the voters' list" (Young, 1994, p. 37). The Condorcet criterion seeks for "an alternative that would receive a strict majority of the votes when compared pairwise with every other alternative" (Ibid. p. 38). Borda and Condorcet approaches are conceptually different and yield conflicting evaluations. The first one is based on a system of scores whereas the second one uses tournaments to elucidate the best option. Moreover, the Borda rule yields a complete ranking of the alternatives for all possible problems whereas the Condorcet method may fail to do so (or even to find a best option, the Condorcet winner), and usually generates a different ordering (see Young, 1975 for a discussion). Finally, even if there is a Condorcet winner the Borda rule may choose a different alternative as the one selected by Condorcet as the best option.

The Borda rule ranks social alternatives according to the place they occupy in agents' preferences, but disregards the number of agents who support them. The Condorcet criterion, on the contrary, focuses on the support received by each alternative vis-a-vis the others, but ignores the distance between them in the agents' rankings. Put it roughly, Borda cares for how much support an alternative has, and Condorcet for how many agents prefer that alternative when confronted with any other. So each procedure implements a sensible evaluation principle but also misses some relevant trait. Our proposal here can be regarded as a compromise between the Borda and the Condorcet procedures, which takes into account both the "distance" between the alternatives in the agents' preferences and the number of agents who support them. This idea is reminiscent of those in Morales (1798, 1805), a Spanish thinker contemporary to Borda and Condorcet, who claimed that the ranking of social alternatives should be related to the "amount of opinion" of the citizens in favour of each of them. By that he meant the sum of the number of times an alternative beats any other.[1]

The evaluation procedure we propose here adopts Condorcet's tournament strategy to evaluate alternatives in terms of pairwise comparisons, but ponders the outcomes of those comparisons differently, depending on how each alternative fares with respect to the others (a pinch of Borda count). This can be summarised by saying that the "*marginal score"* obtained by beating an alternative is higher the stronger this alternative, where the strength of each alternative is determined by the relative frequency with which it beats another when randomly matched, in an indefinite iterative procedure. The evaluation of the alternatives appears as the stable distribution of this iterative process and consists of a vector of positive numbers that tell us the relative social worth of the different alternatives.

Similar ideas appear in the study of centrality in networks (see Freeman, 1977, Wasserman & Faust, 1994, Newmann, 2003), the relevance of the journals in citation analysis (Pinsky & Narin, 1976, Liebowitz & Palmer, 1984, Palacios-Huerta & Volij, 2004, Albarrán et al, 2018), or the ranking of teams in competitions (Keener, 1993, Slutzski & Volij, 2006, Anderson et al, 2009). See also Chebotarev & Shamis (1998) and Herrero & Villar (2017).

The rest of the paper is organized as follows. We present the model in Section 2 and discuss its main features in Section 3. The paper concludes by means of an illustration of the working of this evaluation protocol in a real life problem (the 2017 Solar Decathlon competition).

[1] Morales defended the Borda rule but he was concerned for the conditions under which this declared best the Condorcet winner, when there was one.

2. The BC social evaluation function

Let A = {*1, 2, …, m*} be a finite set of alternatives, with cardinal m, and N = {1, 2, …, n} a finite set of agents. Let us assume that agents' preferences, defined over the set A of social alternatives, are complete and transitive. Notice that we admit indifferences in agents' preferences. Each agent $h \in N$ will provide a ranking of the alternatives in A, $R_h(A)$, according to her tastes. A preferences profile $R(A) = [R_1(A), R_2(A), \dots, R_n(A)]$, is a collection of n of those rankings, one for each agent. Our task is to provide a social evaluation of the alternatives in A based on the information in $R(A)$. More precisely, we look for a social evaluation function F such that, for each possible preferences profile, $R(A)$, yields a vector $F[R(A)] \in \boldsymbol{R}_+^m$ such that $F_i[R(A)] \geq F_j[R(A)]$ implies that alternative i is regarded by society as better than or equal to alternative j.

Consider now the following evaluation protocol. For a given problem (*A, N*) and a given preferences profile, $R(A)$, select arbitrarily a pair of alternatives, *i* and *j* in *A*, and an agent at random. If this agent prefers *i* to *j,* alternative *i* is declared the winner; if it happens that *j* is preferred to *i, the winner is j;* if the agent is indifferent between both alternatives, a coin is tossed and each alternative is declared the winner with equal probability. Then, a new alternative is chosen to compete with the previous winner, according to the preferences of a new agent, also randomly chosen. Then, by running this process infinitely many times, we can evaluate each alternative by the fraction of time that it keeps the floor as the winner in the long run.[2]

Let n_{ij} the number of agents who prefer alternative *i* to alternative *j*, let n_{ji} the number of agents who prefer *j* to *i*, and let $e_{ij} = e_{ji}$ the number of agents who are indifferent between both options. As $n = n_{ij} + n_{ji} + e_{ij}$ the probability that alternative *i* beats alternative *j* in a given confrontation, conditional on *j* keeping the floor in the former one is given by:

$$p_{ij} = \frac{n_{ij}+(e_{ij}/2)}{n(m-1)} \qquad [1]$$

that is, the probability that *i* beats *j* times the probability that *i* be randomly chosen.

Similarly, the probability that alternative *j* keeps the floor for one more round is:

$$p_{jj} = \frac{\sum_{i\neq j}(n_{ji}+(e_{ij}/2))}{n(m-1)} \qquad [2]$$

[2] This procedure is similar to the "ping-pong" protocol in tournaments (see Laslier, 1997).

Note that, for the case of strict orderings (no indifferences), we have:

$$p_{jj} = \frac{\sum_{i \neq j} n_{ji}}{n(m-1)} = \frac{B(j)}{n(m-1)} \quad [2']$$

where $B(j)$ is simply the *Borda score* of alternative *j*. Then, by slightly abusing the language, we call $\sum_{i \neq j}(n_{ji} + (e_{ij}/2)) = GB(j)$, the *generalized Borda score* of alternative *j.* Similarly, notice that in the case of linear orderings, n_{ij} and n_{ji} are the Condorcet numbers to compare alternatives i, j.

In order to evaluate the overall likelihood that one alternative beats another in the long run, we build a (column) stochastic matrix **P***,* where the (i, j) entry is p_{ij}, and the diagonal entries are given by p_{jj}, i.e., the Borda score divided by $n(m-1)$. The repeated random confrontation between alternatives corresponds to a Markov chain associated with this matrix **P***.* The fraction of time that each alternative keeps the floor, w_i, is thus given by the components of the positive eigenvector associated to the dominant eigenvalue of matrix **P**, according to the equation: $\mathbf{P}\boldsymbol{w} = \boldsymbol{w}$. More explicitly,

$$\frac{1}{n(m-1)} \begin{bmatrix} GB(1) & n_{12} + (e_{12}/2) & \ldots & n_{1m} + (e_{1m}/2) \\ n_{21} + (e_{21}/2) & GB(2) & \ldots & n_{2m} + (e_{2m}/2) \\ \ldots & \ldots & \ldots & \ldots \\ n_{m1} + (e_{m1}/2) & n_{m2} + (e_{m2}/2) & \ldots & GB(m) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \ldots \\ w_m \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ \ldots \\ w_m \end{bmatrix}$$

We have, therefore,

$$w_i = \frac{\sum_{j \neq i} \left(n_{ij} + (e_{ij}/2)\right) w_j}{\sum_{i \neq j} \left(n_{ji} + (e_{ji}/2)\right)} \quad [3]$$

Note that this eigenvector has one degree of freedom, so that by imposing the condition $\sum_{i=1}^{m} w_i = 1$ we can interpret the values in equation [3] as the fraction of time that each alternative keeps competing against the others. This is a measure of its social relevance, as those values can be identified with the relative support accrued by the different options, and it is the evaluation we propose here. We call this mapping the Borda-Condorcet Social Evaluation Function (or the BC rule, for short).

Remark: *Young (1994) introduces the* ***vote matrix*** *as the square m matrix whose (i, j) entry is the number of voters that rank i over j. The vote matrix coincides with* **P** *in all off-diagonal elements for the case of linear orders (no indifference) and it has zeroes in the diagonal.*

Note that, given the structure of matrix **P** (a semi-positive matrix all whose rows add-up to one), such eigenvector always exists and it is non-negative. Indeed, under very general conditions this vector is unique and strictly positive.[3]

3. Discussion

The Borda-Condorcet social evaluation function is a criterion that provides a complete and transitive evaluation of social states, which combines some of the principles informing the Borda and Condorcet choice rules. Indeed, the Borda count is a structural part of the evaluation, as can be deduced from the structure of matrix **P**, even though such an evaluation is obtained by making each alternative compete with the others, as in the Condorcet approach. Note that this SEF satisfies all standard properties in social choice, other that informational efficiency (i.e. universal domain, non-dictatorial, monotonicity, symmetry, anonymity). Also observe that the evaluation not only provides a ranking of the alternatives but also a cardinal measure of their social relevance. The BC rule always provides a solution that is generically unique and strictly positive.

3.1 Borda, Condorcet, and the Borda-Condorcet SEF

In spite of containing elements of Borda and Condorcet procedures, the BC rule is different from both of them, as shown in the following two examples. In the first one the BC rule yields the same ranking that the Borda rule, whereas in the second one coincides with that of Condorcet.

Example 1

Let $A = \{a, b, c, d\}$ and suppose the society is made of 21 agents whose preferences are described in Table 1 (the first row indicates the number of individuals who support the ranking described in the corresponding column).

[3] The uniqueness and strict positiveness of the dominant eigenvector of matrix **P** can be ensured when this is an irreducible matrix (e.g. Berman & Plemmons, 1994). A sufficient condition for that is that all of its entries are positive. This amounts to saying that there is no alternative that every agent ranks as the worst one. Interestingly enough when there exists such an alternative the corresponding value in the eigenvector is zero and deleting the alternative from the list does not change the evaluation of the remaining ones. We discuss later on the case of reducible matrices.

Table 1

3	5	7	6
a	*a*	*b*	*c*
b	*c*	*d*	*b*
c	*b*	*c*	*d*
d	*d*	*a*	*a*

Denoting by n_{ij} the number of agents who rank i over j, we obtain the following: n_{ab} = 8, n_{ac} = 8, n_{ad} = 8, n_{ba} = 13, n_{bc} = 10, n_{bd} = 21, n_{ca} = 13, n_{cb} = 11, n_{cd} = 14, n_{da} = 13, n_{db} = 0, and n_{dc} = 7. The Condorcet ranking that derives from those numbers is: $c >_C b >_C d >_C a$.

The Borda count yields the following: $B(a)$ = 24, $B(b)$ = 44, $B(c)$ =38, and $B(d)$ = 20. That is, Borda's ranking is: $b \succ_B c \succ_B a \succ_B d$.

The corresponding **P** matrix and the associated dominant eigenvector **w** are given by:

$$\mathbf{P} = \frac{1}{21\times 3}\begin{pmatrix} 24 & 8 & 8 & 8 \\ 13 & 44 & 10 & 21 \\ 13 & 11 & 38 & 14 \\ 13 & 0 & 7 & 20 \end{pmatrix} \qquad \mathbf{w} = \begin{pmatrix} 0.17 \\ 0.40 \\ 0.32 \\ 0.10 \end{pmatrix}$$

Therefore, the ranking generated by the BC rule coincides with that of Borda, which shows that this rule is not Condorcet consistent.

Example 2

Take again A = {*a, b, c, d*} but suppose now that the society is made of 100 individuals whose preferences are described in Table 2.

Table 2

70	25	5
a	*b*	*c*
b	*c*	*b*
c	*d*	*a*
d	*a*	*d*

In this case alternative a is the Condorcet winner whereas the Borda count yields the following ranking: $b \succ_B a \succ_B c \succ_B d$. The corresponding matrix **P** and vector **w** are:

$$\mathbf{P} = \frac{1}{100 \times 3}\begin{pmatrix} 215 & 70 & 70 & 75 \\ 30 & 225 & 95 & 100 \\ 30 & 5 & 135 & 100 \\ 25 & 0 & 0 & 25 \end{pmatrix} \qquad \mathbf{w} = \begin{pmatrix} 0.45 \\ 0.39 \\ 0.12 \\ 0.04 \end{pmatrix}$$

So the ranking is: $a >_{BC} b >_{BC} c >_{BC} d$, which selects as the best option the Condorcet winner. So the BC rule disagrees with Borda.

Borda is a special case of the family of choice procedures known as *scoring rules*. A scoring rule is a choice procedure in which each agent assigns a given number of votes to each alternative, depending on the place it occupies in her own ranking. Young (1975) introduces the property of *reinforcement* to discuss the differences between scoring rules and those rules based on tournaments (more precisely rules that are Condorcet consistent, that is, those that always choose the Condorcet winner when there is one). Reinforcement can be described as follows. Suppose that **x** is the alternative chosen from the set *A* of social states both by society N_1 and society N_2. Then the society $N = N_1 \cup N_2$, resulting from merging those two societies, should also choose **x** as the best option in *A*. Young proves that all scoring rules verify the property of *reinforcement*, whereas no rule consistent with the Condorcet principle does it. One may wonder whether the BC rule satisfies reinforcement, given that it is not a Condorcet consistent rule, as illustrated in Example 1 above. The following example shows that the BC rule does not satisfy reinforcement. As a consequence, the BC rule *is not* a scoring rule.

Example 3

Let *A* = {*a, b, c*} and consider the evaluation of those alternatives by societies N_1 and N_2, given in Table 3.

Table 3

N_1		N_2	
100	**99**	**10**	**17**
a	*c*	*b*	*a*
b	*b*	*c*	*b*
c	*a*	*a*	*c*

The BC rule yields the following evaluation:

$$w(N_1)=\begin{pmatrix}0.335\\0.333\\0.331\end{pmatrix}\qquad w(N_2)=\begin{pmatrix}0.459\\0.436\\0.104\end{pmatrix}$$

We therefore conclude that $a \succ b \succ c$ in both societies.

Consider now society $N = N_1 \cup N_2$, which presents the preferences described in Table 4.

Table 4

N		
117	**99**	**10**
a	*c*	*b*
b	*b*	*c*
c	*a*	*a*

The corresponding evaluation by the BC rule is given by:

$$w(N)=\begin{pmatrix}0.349\\0.351\\0.298\end{pmatrix}$$

which implies that $b \succ a \succ c$.

3.2 An alternative interpretation of the BC rule

It is interesting to note that the case in which we have only two social alternatives the BC rule becomes an elementary evaluation procedure in which the alternatives are given values that are proportional to the number of agents that give priority to one over the other. That is,

$$\frac{w_i}{w_j}=\frac{n_{ij}+(e_{ij}/2)}{n_{ji}+(e_{ji}/2)} \qquad [4]$$

The BC rule can be regarded as an extension of this proportionality principle that is reminiscent of Morales' idea of "the amount of support". To see this observe first that when there are just two alternatives we are applying majority voting (Borda, Condorcet and the BC rule rankings coincide, provided $n_{12} \neq n_{21}$). Nevertheless, applying this criterion to any pair of alternatives when $m \geq 3$ is not going to work, because of the presence of intransitivities. Now observe that equation [4] can be rewritten as follows:

$$w_i = \frac{\left[n_{ij} + \left(e_{ij}/2\right)\right] w_j}{n_{ji} + \left(e_{ji}/2\right)} \qquad [4']$$

This expression says that the evaluation of alternative *i* is given by the ratio of two interesting variables: (a) The number of agents who prefer *i* to *j*, weighted by the evaluation of *j*; and (b) The number of agents who prefer *j* to *i*. The numerator can be regarded as a measure of the strength of *i* over *j* and the denominator as a measure of the disadvantage of *i* with respect to *j*. Then the evaluation formula [3] corresponds to the ratio between the average value of the strength of *i* over any other alternative and the corresponding average disadvantage.

3.3 Reducible matrices

We have already mentioned that the uniqueness and strict positiveness of the eigenvector associated to the dominant eigenvalue of matrix **P** relies on the irreducibility of such matrix. Those properties may fail when matrix **P** is reducible (let us recall here that the eigenvector still exists and it is nonnegative). Dealing with a reducible matrix should not be regarded as an inconvenience, though. Because what a reducible matrix implies is the existence of independent "subsystems" which are globally ranked (that is, one subsystem is better than another). In other words, whenever the matrix is reducible we can divide the alternatives into different "divisions", in such a way that only the alternatives within the same division are actually comparable, but those divisions are ranked, that is, all alternatives in an inferior division are dominated by those in a higher one.

Let us illustrate this feature by means of a numerical example.

Example 4

Let a society made out of 10 individuals with the preferences over the set of alternatives *{a,b,c,d,e,f}* given in Table 5:

Table 5

5	2	3
a	*b*	*ab*
b	*a*	*c*
c	*f*	*def*
d	*d*	
ef	*ec*	

In this case, the **P** matrix is obtained from Table 6 by dividing each entry by 50:

Table 6

46.5	6.5	10	10	10	10
3.5	43.5	10	10	10	10
0	0	25	8	9	8
0	0	2	17	8.5	6.5
0	0	1	1.5	6.5	4
0	0	2	3.5	6	11.5

Here we find two alternatives, *a,b* that are above the rest. The eigenvector associated to the dominant eigenvalue is (0.64993; 0.35007; 0; 0; 0; 0), reflecting the division of alternatives into two separate groups, {*a, b*}, and {*c, d, e, f*}. The first two components of the eigenvector provide and estimate of the relative support of alternatives *a, b,* while the zeroes indicate that they fully dominate the remaining alternatives.

We may now evaluate the relative support accrued by the alternatives in the dominated group, {*c, d, e, f*}, considered as a different problem (a competition within the second division, so to speak). We thus consider the submatrix obtained from dividing Table 7 by 30

Table 7

25	8	9	8
2	17	8.5	6.5
1	1.5	6.5	4
2	3.5	6	11.5

whose eigenvector is (0.61952; 0.19588; 0.061089; 0.12351), that expresses the relative support of this alternatives within their subgroup.

The case of a reducible matrix is interesting because it illustrates well two relevant properties of the BC rule. On the one hand, that the evaluation of each alternative is relative to the alternatives with which it is compared. This is obvious from the very beginning but example 4 puts it in perspective. Alternatives {*c, d, e, f*} all appear as socially indifferent when compared with {a, b} and yet they are not when considered as a subgroup on its own. On the other hand reducibility not only informs about the existence of different "divisions" within the set of societies being compared, but provides an endogenous way of identifying them. This is a relevant aspect in many evaluation problems (e.g. the division of countries according to their human development level).

4. The Solar Decathlon competition 2017

We conclude by providing an empirical illustration of the working of the BC social evaluation function. It refers to the *Solar Decathlon competition* 2017. The Solar Decathlon is an architectonic competition of university teams aimed at promoting efficient houses, in particular with respect to their environmental impact: consuming less energy and producing less wastes. It is called Decathlon because the competing projects are evaluated with respect to 10 different aspects: (1) architecture; (2) market potential; (3) engineering; (4) communications; (5) innovation; (6) water; (7) health & comfort; (8) appliances; (9) home life, and (10) energy. Each of the teams is ranked independently for each of those 10 aspects. In some cases, by using some specific monitoring techniques, in some others a jury classifies the proposals by giving a score to the competitors.

Let us call T = {A, B, C, D, E, F, G, H, I, J, K} the 11 teams that were competing in the 2017 edition of the contest. The order the different teams obtained in the 10 aspects subject to evaluation appears in Table 8.

Table 8

Rank	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
1	A	H	A	H	G	A	A	C	A	A
2	G,J	F	G	I	B	I	C	A	C	C
3	D	K	H	A	D	F	I	D	E	E
4	B,F	E	I	D	J	B	E	B,H	I	J
5	H	B	F	G	I	J	F	F	F,H	B
6	E	G	B,C	C	C	D	B	I	B,D	D
7	C	C	D,E	F	E	H	H	G	G	I
8	I	I	J	B	F	C	K	K,E	J	F
9	K	D	K	K	H	G	G	J	K	G,H,K
10		J		E	A	E	D			
11		A		J	K	K	J			

Note that in this case we find indifferences in the evaluation, that is, there are teams that are ranked equally in some aspects. As there are 11 alternatives and ten "voters", the corresponding **P** matrix obtains as the product of 1/100 and the matrix described in Table 9.

Table 9

77	8	6	8	8	8	8	7	7	8	9
2	56	5	6,5	6	4	6	5,5	5	7	9
4	5	61	6	8	6	5	5	6	7	9
2	3,5	4	49,5	6	5	5	5	4	7	8
2	4	2	4	43	4	4	4	4	7	8
2	6	4	5	6	55,5	6	5,5	4	7	10
2	4	5	5	6	4	48,5	3,5	4	7,5	7,5
3	4,5	5	5	6	4,5	6,5	55,5	5,5	6	9,5
3	5	4	6	6	6	6	4,5	56,5	7	9
2	3	3	3	3	3	2,5	4	3	32,5	6
1	1	1	2	2	0	2,5	0,5	1	4	15

The dominant eigenvector associated to this matrix is given in Table 10. We have normalised the eigenvector by making the sum of the components equal to 100 so that the numbers provide a description of the percentage of time each team is the winner in the protocol.

Table 10

A	24,445
B	9,3886
C	12,103
D	7,3633
E	5,6515
F	9,0283
G	7,1399
H	9,5681
I	9,839
J	4,0542
K	1,4195

The ranking we obtain is, therefore: A, C, I, H, B, F, D, G, E, J, K. whereas the actual ranking that emerged in the 2017 competition was: A, B, C, D, E, F, G, H, I, J, K. We observe that even though the first and the two last positions of both rankings coincide, they differ in the remaining ones. Also note that the BC provides a clear estimate of the differences between competitors.

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