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JEL Classification: D63, C71, Z20.



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On the axiomatic approach to sharing the revenues from broadcasting sports leagues*

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Abstract

We take the axiomatic approach to uncover the structure of the revenue-sharing problem from broadcasting sports leagues. We formalize two notions of impartiality, depending on the stance one takes with respect to the revenue generated in the games involving each pair of teams. We show that the resulting two axioms lead towards two broad categories of rules, when combined with additivity and some other basic axioms. We complement those results strengthening the impartiality notions to consider axioms of order preservation.

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1 Introduction

In a recent paper (Bergantiños and Moreno-Ternero, 2020a), we have introduced a formal model to analyze the problem of sharing the revenues from broadcasting sports leagues among participating teams (clubs), based on the audiences they generate. It is a stylized and simple model, yet rich enough to obtain numerous interesting insights. The model's input is simply a square matrix (A) whose entries (a_{ij}) indicate the audience of the game the two corresponding teams play, with the convention that the row team (i) plays home and the column team (j) plays away.¹ The goal is to (axiomatically) derive rules that associate for each matrix an allocation (among participating teams) of the overall audience in the tournament (i.e., the aggregation of all entries in the matrix).

In this paper, we uncover the structure of this stylized model further, thanks to the axiomatic approach. Our starting point is the principle of *impartiality*, with a long tradition in the theory of justice (e.g., Moreno-Ternero and Roemer, 2006). The basic formulation of such a principle is typically the axiom of *equal treatment of equals*. In our setting, this axiom can take two forms, depending on how we define the concept of *equals*. One says that, to consider two teams as equals, they should generate the same audience each time they play a third team. Another says that, moreover, the (two) games played by those two teams also generate the same audience. We shall refer to both axioms as *equal treatment of equals* and *weak equal treatment of equals*, respectively. Depending on the stance one takes with respect to the audience in the games involving each pair of teams, one might consider one or the other axiom more appropriate. To wit, if we consider each pair of teams accountable for the total audience they generate at the two games involving them, *equal treatment of equals* seems to be the right one. If, instead, teams should be held accountable for the audience generated at their own stadiums, so that $a_{ij} \neq a_{ji}$ justifies to treat i and j differently, then *weak equal treatment of equals* seems to be the right one.

As the names suggest, *equal treatment of equals* implies *weak equal treatment of equals*. We also formalize an axiom that fills the gap between both axioms: *pairwise reallocation proofness*. This axiom says that a redistribution between the two audiences of the games involving a pair of teams does not affect the amounts obtained by the teams in the pair.

¹Thus, we assume a double round robin tournament. The model could be easily extended to accommodate other tournament forms.

We explore the implications of the previous axioms, when combined with some other basic axioms: *additivity*, *maximum aspirations*, *non-negativity* and *weak upper bound*. The first one, which is standard in axiomatic work and can be traced back to Shapley (1953), says that awards are additive on audiences. The other three formalize reasonable (lower or upper) bounds, which are principles with a long tradition in the literature on fair allocation (e.g., Thomson, 2011; 2019). More precisely, *maximum aspirations* says that no team can receive an amount higher than its *claim* (i.e., the aggregate audience at all the games in which the team was involved). *Non-negativity* says that no team receives negative awards. Finally, *weak upper bound* (which is weaker than the previous two axioms) says that individual awards are bounded above by the aggregate audience obtained at the whole tournament.

We show that *equal treatment of equals*, *additivity* and *maximum aspirations* characterize the so-called *EC*-family of rules, which is made of compromises between two rules that stand out as focal to solve this problem.² They are the so-called *equal-split* rule, which splits the revenue generated from each game equally among the participating teams, and *concede-and-divide*, which concedes each team the revenues generated from its fan base and divides equally the residual. Each rule within the *EC*-family of rules is defined by a parameter that establishes the specific convex combination between the solutions the *equal-split* rule and *concede-and-divide* yield (for each problem). The *equal-split* rule can be seen itself as a convex combination between the so-called *uniform* rule (another focal rule in this model, which allocates an equal portion of the whole audience to each participating team) and *concede-and-divide*. We shall refer to the resulting family of rules compromising between the *uniform* rule and *concede-and-divide* via convex combinations as the *UC*-family of rules. Alternatively, one could consider linear (but not necessarily convex) combinations between those rules. And that would give rise to the *generalized UC*-family. It turns out that if we replace *maximum aspirations* by *non-negativity* or *weak upper bound* then we characterize two other subfamilies of the *generalized UC*-family of rules.

The three characterization results mentioned above can be formulated alternatively upon replacing *equal treatment of equals* by the pair of axioms made of *weak equal treatment of equals* and *pairwise reallocation proofness*. More interestingly, it turns out that replacing the latter

²This result is almost equivalent to the characterization in Bergantiños and Moreno-Ternero (2021a). Therein, we use a stronger notion than *equal treatment of equals* indicating that two teams with the same claims receive the same awards.

by a somewhat similar axiom yields completely different outcomes. More precisely, we consider the axiom *stand-alone pair*, which states that in situations where only the games involving a pair of teams have a positive audience, the total audience should be allocated to such a pair of teams. The combination of this axiom with *additivity*, *weak equal treatment of equals* and any axiom from the group made of *maximum aspirations*, *non-negativity* and *weak upper bound* characterizes a new family of rules. We call them *split rules*, as they generalize the *equal-split* rule to allow for unequal (but fixed) splits of the audience of each game between the two playing teams.

Somewhat related to the above, we also consider the axiom of *homogeneous effect of stand-alone pair*. This axiom says that, in situations where only games played by a pair of teams have a positive audience, the remaining teams obtain a homogeneous amount. To complement the results mentioned above, we characterize the family of rules satisfying *homogeneous effect of stand-alone pair*, *additivity*, and *weak equal treatment of equals*. In such a family, the amount each team receives depends on three numbers: the overall home audience of the team, the overall away audience of the team, and the total audience of the whole tournament. There is a weighted aggregation of these three numbers and the weights are the same for each team.

Finally, our axioms of equal treatment of equals can naturally be strengthened to axioms of *order preservation*. If we do so, we can obtain additional characterization results. To wit, we show that *order preservation*, *additivity* and *maximum aspirations* also characterize the *EC*-family of rules (thus, being an “inferior” result to the one mentioned above with *equal treatment of equals*). On the other hand, we show that *order preservation*, *additivity* and *non-negativity* characterize the so-called *UE*-family of rules, which is made of compromises (in the form of convex combinations) between the *uniform* rule and the *equal-split* rule. And *order preservation*, *additivity* and *weak upper bound* characterize the *UC*-family of rules mentioned above, which is precisely the union of the previous two families.

The last three characterization results can also be formulated alternatively upon replacing *order preservation* by the pair of axioms made of *home* (or *away*) *order preservation* and *pair-wise reallocation proofness*. And replacing the latter by *stand-alone pair*, we obtain completely different outcomes too. More precisely, the combination of *additivity*, *home* (respectively, *away*) *order preservation*, *stand-alone pair* and any of the three bounds axioms, characterizes one half of the family of *split rules*: those that impose a fixed split of the audience of each game between

the two playing teams, but guaranteeing at least (respectively, at most) one half to the local team. We conclude our analysis dismissing the bounds axioms in these last results. That is, we characterize the rules that satisfy *additivity*, *home (or away) order preservation* or *weak equal treatment of equals*, and *pairwise reallocation proofness* or *stand-alone pair*.

The rest of the paper is organized as follows. We introduce the model, axioms and rules in Section 2. In Section 3, we provide the characterization results. We conclude in Section 4. For a smooth passage we defer all proofs to an appendix.

2 The model

We consider the model introduced by Bergantiños and Moreno-Ternerero (2020a). Let N describe a finite set of teams. Its cardinality is denoted by n . We assume $n \geq 3$. For each pair of teams $i, j \in N$, we denote by a_{ij} the broadcasting audience (number of viewers) for the game played by i and j at i 's stadium. We use the notational convention that $a_{ii} = 0$, for each $i \in N$. Let $A \in \mathcal{A}_{n \times n}$ denote the resulting matrix of broadcasting audiences generated in the whole tournament involving the teams within N .³ Each matrix $A \in \mathcal{A}_{n \times n}$ with zero entries in the diagonal will thus represent a *problem* and we shall refer to the set of problems as \mathcal{P} .⁴

Let $\alpha_i(A)$ denote the total audience achieved by team i , i.e.,

$$\alpha_i(A) = \sum_{j \in N} (a_{ij} + a_{ji}).$$

Without loss of generality, we normalize the revenue generated from each viewer to 1 (to be interpreted as the “pay per view” fee). Thus, we sometimes refer to $\alpha_i(A)$ by the *claim* of team i . When no confusion arises, we write α_i instead of $\alpha_i(A)$. We define $\bar{\alpha}$ as the average audience of all teams. Namely,

$$\bar{\alpha} = \frac{\sum_{i \in N} \alpha_i}{n}.$$

³We are therefore assuming a round-robin tournament in which each team plays in turn against each other team twice: once home, another away. This is the usual format of the main European football leagues. Our model could also be extended to leagues in which some teams play other teams a different number of times and play-offs at the end of the regular season, which is the usual format of North American professional sports. In such a case, a_{ij} is the broadcasting audience in all games played by i and j at i 's stadium.

⁴As the set N will be fixed throughout our analysis, we shall not explicitly consider it in the description of each problem.

For each $A \in \mathcal{A}_{n \times n}$, let $\|A\|$ denote the total audience of the tournament. Namely,

$$\|A\| = \sum_{i,j \in N} a_{ij} = \frac{1}{2} \sum_{i \in N} \alpha_i = \frac{n\bar{\alpha}}{2}.$$

2.1 Rules

A (sharing) **rule** is a mapping that associates with each problem the list of the amounts the teams get from the total revenue, which we have normalized to 1 per viewer. Thus, formally, $R : \mathcal{P} \rightarrow \mathbb{R}^n$ is such that, for each $A \in \mathcal{P}$,

$$\sum_{i \in N} R_i(A) = \|A\|.$$

The following three rules have been highlighted as focal for this problem (e.g., Bergantiños and Moreno-Ternero, 2020a; 2020b; 2021a; 2021b). The *uniform rule* divides equally among all teams the overall audience of the whole tournament. The *equal-split rule* divides the audience of each game equally, among the two participating teams. *Concede-and-divide*, which can be rationalized from a simple form of statistical estimation (e.g., Bergantiños and Moreno-Ternero, 2020a), compares the performance of a team with the average performance of the other teams.⁵ Formally,

Uniform, U : for each $A \in \mathcal{P}$, and each $i \in N$,

$$U_i(A) = \frac{\|A\|}{n} = \frac{\bar{\alpha}}{2}.$$

Equal-split rule, ES : for each $A \in \mathcal{P}$, and each $i \in N$,

$$ES_i(A) = \frac{\alpha_i}{2}.$$

Concede-and-divide, CD : for each $A \in \mathcal{P}$, and each $i \in N$,

$$CD_i(A) = \alpha_i - \frac{\sum_{j,k \in N \setminus \{i\}} (a_{jk} + a_{kj})}{n-2} = \frac{(n-1)\alpha_i - \|A\|}{n-2} = \frac{2(n-1)\alpha_i - n\bar{\alpha}}{2(n-2)}.$$

The following family of rules encompasses the above three rules.

UC-family of rules $\{UC^\lambda\}_{\lambda \in [0,1]}$: for each $\lambda \in [0,1]$, each $A \in \mathcal{P}$, and each $i \in N$,

$$UC_i^\lambda(A) = (1-\lambda)U_i(A) + \lambda CD_i(A).$$

⁵This interesting principle of comparing individual performance with average performance is often used in other contexts (e.g., Arrfelt et al., 2013; Allen et al., 2017; Ju et al., 2021).

Equivalently,

$$UC_i^\lambda(A) = (1 - \lambda) \frac{\|A\|}{n} + \lambda \frac{(n-1)\alpha_i - \|A\|}{n-2} = \frac{\bar{\alpha}}{2} + \lambda \frac{n-1}{n-2} (\alpha_i - \bar{\alpha}).$$

At the risk of stressing the obvious, note that, when $\lambda = 0$, UC^λ coincides with the *uniform* rule, whereas, when $\lambda = 1$, UC^λ coincides with *concede-and-divide*. That is, $UC^0 \equiv U$ and $UC^1 \equiv CD$. Bergantiños and Moreno-Ternerero (2020a) prove that, for each $A \in \mathcal{P}$,

$$ES(A) = \frac{n}{2(n-1)}U(A) + \frac{n-2}{2(n-1)}CD(A).$$

That is, $UC^\lambda \equiv ES$, where $\lambda = \frac{n-2}{2(n-1)}$.⁶

Consequently, the *UC-family of rules* can be split in two.

On the one hand, the family of rules compromising between the *uniform* rule and the *equal-split* rule. Formally,

UE-family of rules $\{UE^\lambda\}_{\lambda \in [0,1]}$: for each $\lambda \in [0, 1]$, each $A \in \mathcal{P}$, and each $i \in N$,

$$UE_i^\lambda(A) = (1 - \lambda)U_i(A) + \lambda ES_i(A) = \frac{\bar{\alpha}}{2} + \frac{\lambda}{2}(\alpha_i - \bar{\alpha}).$$

On the other hand, the family of rules compromising between the *equal-split* rule and *concede-and-divide*.⁷ Formally,

EC-family of rules $\{EC^\lambda\}_{\lambda \in [0,1]}$: for each $\lambda \in [0, 1]$, each $A \in \mathcal{P}$, and each $i \in N$,

$$EC_i^\lambda(A) = (1 - \lambda)ES_i(A) + \lambda CD_i(A) = \frac{\alpha_i}{2} + \lambda \frac{n}{2(n-2)} (\alpha_i - \bar{\alpha}).$$

As Figure 1 illustrates, the family of *UC* rules is indeed the union of the family of *UE* rules and *EC* rules. Note that $UE^0 \equiv UC^0 \equiv U$, $EC^1 \equiv UC^1 \equiv CD$, whereas $ES \equiv UE^1 \equiv EC^0 \equiv UC^{\frac{n-2}{2(n-1)}}$ is the unique rule belonging to both families.

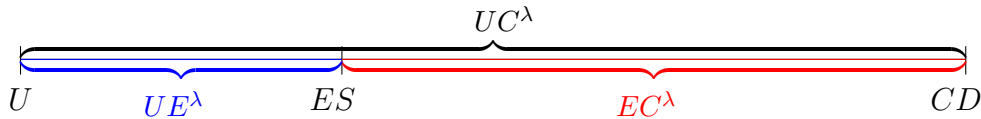


Figure 1. Illustration of the first three families. $\{UC^\lambda\} = \{UE^\lambda\} \cup \{EC^\lambda\}$

We could generalize the previous families by considering any linear (but not necessarily convex) combination between *U* and *CD*, which we shall sometimes refer as *generalized compromise* rules. Formally,

⁶Note that λ approaches 0.5 (from below) for n arbitrarily large.

⁷We studied this family independently in Bergantiños and Moreno-Ternerero (2021a).

GUC-family of rules $\{GUC^\lambda\}_{\lambda \in \mathbb{R}}$: for each $\lambda \in \mathbb{R}$, each $A \in \mathcal{P}$, and each $i \in N$,

$$GUC_i^\lambda(A) = (1 - \lambda)U_i(A) + \lambda CD_i(A).$$

We also consider another category of rules arising from generalizing the *equal-split* rule to allow for unequal (but fixed) splits of the audience of each game between the two playing teams. Formally,

Split rules $\{S^\lambda\}_{\lambda \in [0,1]}$: for each $\lambda \in [0, 1]$, each $A \in \mathcal{P}$, and each $i \in N$,

$$S_i^\lambda(A) = (1 - \lambda) \sum_{j \in N \setminus \{i\}} a_{ij} + \lambda \sum_{j \in N \setminus \{i\}} a_{ji}.$$

The *equal-split* rule corresponds to the case where $\lambda = 0.5$. The case $\lambda = 0$ corresponds to the rule assigning to each team its home audience and the case $\lambda = 1$ corresponds to the rule assigning to each team its away audience. Thus, we shall refer to each half of the family as *home-biased split* rules ($\{S^\lambda\}_{\lambda \in [0, \frac{1}{2}]}$) and *away-biased split* rules ($\{S^\lambda\}_{\lambda \in [\frac{1}{2}, 1]}$), respectively. As before, we could also consider a generalization of the *split rules*.

Generalized split rules $\{GS^\lambda\}_{\lambda \in \mathbb{R}}$: for each $\lambda \in \mathbb{R}$, each $A \in \mathcal{P}$, and each $i \in N$,

$$GS_i^\lambda(A) = (1 - \lambda) \sum_{j \in N \setminus \{i\}} a_{ij} + \lambda \sum_{j \in N \setminus \{i\}} a_{ji}.$$

To conclude with this inventory, we consider the most general family of rules encompassing all the rules introduced above. The allocation received by each team i depends on three data: the home audience ($\sum_{j \in N \setminus \{i\}} a_{ij}$), the away audience ($\sum_{j \in N \setminus \{i\}} a_{ji}$), and the total audience of the tournament ($\|A\|$). The relative importance of each data is the same for each team. Formally,

General rules $\{G^{xyz}\}_{x+y+nz=1}$. For each trio $x, y, z \in \mathbb{R}$ with $x + y + nz = 1$, each $A \in \mathcal{P}$, and each $i \in N$,

$$G_i^{xyz}(A) = x \sum_{j \in N \setminus \{i\}} a_{ij} + y \sum_{j \in N \setminus \{i\}} a_{ji} + z \|A\|.$$

Note that the *uniform* rule corresponds to the case in which $x = 0$, $y = 0$ and $z = \frac{1}{n}$. The *equal-split* rule corresponds to the case in which $x = \frac{1}{2}$, $y = \frac{1}{2}$ and $z = 0$. *Concede-and-divide* corresponds to the case in which $x = \frac{n-1}{n-2}$, $y = \frac{n-1}{n-2}$ and $z = \frac{-1}{n-2}$. Thus, all *GUC* rules belong to this family. Besides, generalized split rules also belong to this family (they are obtained when $x = 1 - \lambda$, $y = \lambda$ and $z = 0$).

The family of *general* rules also contains rules outside the previous families. An interesting example is the following:

$$R_i(A) = \alpha_i - \frac{\|A\|}{n},$$

which corresponds to $(x, y, z) = (1, 1, \frac{-1}{n})$. This rule is a kind of *dual* rule of the *uniform* rule.

Another less intuitive rule is the following:

$$R_i(A) = \|A\| - \frac{n-1}{2}\alpha_i,$$

which corresponds to $(x, y, z) = (\frac{-n+1}{2}, \frac{-n+1}{2}, 1)$. With this rule, the larger the audience of a team, the smaller the revenue it receives. In particular, a team with null audience receives all the revenue.

2.2 Axioms

We now introduce the axioms we consider in this paper.

The first axiom says that if two teams have the same audiences, when facing each of the other teams, then they should receive the same amount.

Equal treatment of equals (ETE): For each $A \in \mathcal{P}$, and each pair $i, j \in N$ such that $a_{ik} = a_{jk}$, and $a_{ki} = a_{kj}$, for each $k \in N \setminus \{i, j\}$,

$$R_i(A) = R_j(A).$$

Now, we may want to require something more to consider two teams as truly equals. To wit, the two teams have the same audiences, not only when facing each of the other teams, but also when facing themselves at each stadium. The next axiom only guarantees that they receive the same amount when this extra condition holds.

Weak equal treatment of equals (WETE): For each $A \in \mathcal{P}$, and each pair $i, j \in N$ such that $a_{ij} = a_{ji}$, $a_{ik} = a_{jk}$, and $a_{ki} = a_{kj}$, for each $k \in N \setminus \{i, j\}$,

$$R_i(A) = R_j(A).$$

Obviously, this axiom is weaker than the previous one. The following axiom fills the gap. It says that a redistribution between the audiences of the two games involving a pair of teams does not affect the revenues obtained by the teams in the pair.

Pairwise reallocation proofness (PRP): For each pair $A, A' \in \mathcal{P}$, and each pair $i_0, j_0 \in N$, such that $a_{ij} = a'_{ij}$, for each pair $\{i, j\} \neq \{i_0, j_0\}$, and $a_{i_0 j_0} + a_{j_0 i_0} = a'_{i_0 j_0} + a'_{j_0 i_0}$,

$$R_k(A) = R_k(A') \text{ for each } k = i_0, j_0.$$

A somewhat related axiom refers to situations where only games played by a pair of teams have a positive audience. It says that the total audience should be allocated to such two teams.

Stand-alone pair (SAP): For each $A \in \mathcal{P}$ and each pair $i, j \in N$ such that $a_{kl} = 0$ for each pair $\{k, l\} \in N$ with $(k, l) \neq (i, j)$ and $(k, l) \neq (j, i)$,

$$R_i(A) + R_j(A) = \|A\|.$$

We could instead consider that, for those situations, the remaining teams obtain a homogeneous amount.

Homogeneous effect of stand-alone pair (HSAP): For each $A \in \mathcal{P}$ and each pair $i, j \in N$ such that $a_{kl} = 0$ for each pair $\{k, l\} \in N$, such that $(k, l) \neq (i, j)$ and $(k, l) \neq (j, i)$,

$$R_k(A) = R_l(A) \text{ for each } k, l \in N \setminus \{i, j\}.$$

A strengthening of *stand-alone pair* says that if a team has null audience, then it gets no revenue. Formally,

Null team (NT): For each $A \in \mathcal{P}$, and each $i \in N$, such that for each $j \in N$, $a_{ij} = 0 = a_{ji}$,

$$R_i(A) = 0.$$

The following axiom strengthens *equal treatment of equals* by saying that if the audience of team i is, game by game, not smaller than the audience of team j , then that team i should not receive less than team j .

Order preservation (OP): For each $A \in \mathcal{P}$ and each pair $i, j \in N$, such that, for each $k \in N \setminus \{i, j\}$, $a_{ik} \geq a_{jk}$ and $a_{ki} \geq a_{kj}$,

$$R_i(A) \geq R_j(A).$$

Alternatively, we can consider the natural weaker versions, aligned with *weak equal treatment of equals*, when also paying attention to the two games involving the pair.

Home order preservation (HOP): For each $A \in \mathcal{P}$ and each pair $i, j \in N$, such that, for each $k \in N \setminus \{i, j\}$, $a_{ik} \geq a_{jk}$, $a_{ki} \geq a_{kj}$, and $a_{ij} \geq a_{ji}$,

$$R_i(A) \geq R_j(A).$$

Away order preservation (AOP): For each $A \in \mathcal{P}$ and each pair $i, j \in N$, such that, for each $k \in N \setminus \{i, j\}$, $a_{ik} \geq a_{jk}$, $a_{ki} \geq a_{kj}$, and $a_{ji} \geq a_{ij}$,

$$R_i(A) \geq R_j(A).$$

The next axiom provides a natural lower bound as it says that no team should receive negative awards. Formally,

Non-negativity (NN): For each $A \in \mathcal{P}$ and $i \in N$,

$$R_i(A) \geq 0.$$

A natural upper bound would say that each team should receive, at most, the total audience of the games played by the team. Formally,

Maximum aspirations (MA): For each $A \in \mathcal{P}$ and each $i \in N$,

$$R_i(A) \leq \alpha_i.$$

Alternatively, one could consider a weaker upper bound set by the total audience of all games in the tournament.

Weak upper bound (WUB): For each $A \in \mathcal{P}$ and each $i \in N$,

$$R_i(A) \leq \|A\|.$$

Finally, we consider the axiom saying that revenues should be additive on A . Formally,

Additivity (ADD): For each pair A and $A' \in \mathcal{P}$

$$R(A + A') = R(A) + R(A').$$

The next proposition, whose straightforward proof we omit, summarizes the relations between the axioms introduced above.

Proposition 1 *The following implications among axioms hold:*

1. *Equal treatment of equals implies weak equal treatment of equals.*
2. *The combination of weak equal treatment of equals and pairwise reallocation proofness implies equal treatment of equals.*
3. *Order preservation implies home order preservation and away order preservation.*
4. *The combination of home order preservation (or away order preservation) and pairwise reallocation proofness implies order preservation.*
5. *Order preservation implies equal treatment of equals.*
6. *Home order preservation (or away order preservation) implies weak equal treatment of equals.*
7. *Null team implies stand-alone pair.*
8. *Non negativity implies weak upper bound.*
9. *Maximum aspirations implies weak upper bound.*

3 Characterization results

We consider two subsections. In the first one, we combine *weak equal treatment of equals* with some other axioms. In the second one, we perform a similar analysis replacing *weak equal treatment of equals* with *weak order preservation*.

3.1 With weak equal treatment of equals

Our first result provides several characterizations of the *split rules*. More precisely, Theorem 1 states that those rules are characterized combining *additivity*, *weak equal treatment of equals* and *stand-alone pair* with one of the three bounds axioms: *maximum aspirations*, *non-negativity* or *weak upper bound*.

Theorem 1 *A rule satisfies additivity, weak equal treatment of equals, stand-alone pair, and either maximum aspirations, non-negativity or weak upper bound if and only if it is a split rule.*

As all *split rules* satisfy *null team*, and *null team* implies *stand-alone pair* (the seventh statement of Proposition 1), we obtain the following corollary from Theorem 1.

Corollary 1 *A rule satisfies additivity, weak equal treatment of equals, null team, and either maximum aspirations, non-negativity or weak upper bound if and only if it is a split rule.*

As the next result states, to replace *stand-alone pair* by *pairwise reallocation proofness* in Theorem 1 yields a completely different outcome. Instead of *split rules*, we obtain *generalized compromise rules*. As a matter of fact, each of the bounds axioms leads to a subfamily of GUC^λ , arising from considering that λ ranges within a certain interval. More precisely, *maximum aspirations* leads to the interval $\left[\frac{n-2}{2(n-1)}, 1\right]$, *non-negativity* to the interval $\left[\frac{-1}{n-1}, \frac{n-2}{2(n-1)}\right]$ and *weak upper bound* to the interval $\left[1 - \frac{n}{2}, 1\right]$.

Theorem 2 *The following statements hold:*

1. *A rule satisfies additivity, weak equal treatment of equals, pairwise reallocation proofness and maximum aspirations if and only if it belongs to the EC-family of rules.*
2. *A rule satisfies additivity, weak equal treatment of equals, pairwise reallocation proofness and non-negativity if and only if it belongs to the GUC-family of rules for $\lambda \in \left[\frac{-1}{n-1}, \frac{n-2}{2(n-1)}\right]$.*
3. *A rule satisfies additivity, weak equal treatment of equals, pairwise reallocation proofness and weak upper bound if and only if it belongs to the GUC-family of rules for $\lambda \in \left[1 - \frac{n}{2}, 1\right]$.*

It turns out that dismissing all bounds axioms (*maximum aspirations*, *non-negativity* or *weak upper bound*) in the above results leads to the characterization of the two complete families of *generalized (split or compromise) rules*.

Theorem 3 *The following statements hold:*

1. *A rule satisfies additivity, weak equal treatment of equals and either stand-alone pair or null team if and only if it belongs to the family of generalized split rules.*
2. *A rule satisfies additivity, weak equal treatment of equals and pairwise reallocation proofness if and only if it belongs to the GUC-family of rules.*

The *equal-split* rule is the unique rule satisfying *additivity*, *equal treatment of equals* and *null team* (e.g., Bergantiños and Moreno-Ternero, 2020a). The first statement of Theorem 3 says that if we replace *equal treatment of equals* by *weak equal treatment of equals* in that result we obtain the family of *split rules* instead of the *equal-split* rule. It also follows from the second statement of Theorem 3 that the only rule satisfying *null team* within the *GUC-family of rules* is precisely the *equal-split* rule.

The next proposition is obtained by replacing *weak equal treatment of equals* and *pairwise reallocation proofness* by *equal treatment of equals*, in Theorem 2 and the second statement of Theorem 3.

Proposition 2 *The following statements hold:*

1. *A rule satisfies additivity, equal treatment of equals and maximum aspirations if and only if it belongs to the EC-family of rules.*
2. *A rule satisfies additivity, equal treatment of equals and non-negativity if and only if it belongs to the GUC-family of rules for $\lambda \in \left[\frac{-1}{n-1}, \frac{n-2}{2(n-1)} \right]$.*
3. *A rule satisfies additivity, equal treatment of equals and weak upper bound if and only if it belongs to the GUC-family of rules for $\lambda \in \left[1 - \frac{n}{2}, 1 \right]$.*
4. *A rule satisfies additivity and equal treatment of equals if and only if it belongs to the GUC-family of rules.*

The first statement of Proposition 2 is a refinement of Theorem 1 in Bergantiños and Moreno-Ternero (2021a), obtained by replacing *equal treatment of equals* with an axiom dubbed *symmetry*, which says that if two teams have the same audiences they receive the same amounts.

In the next theorem we focus on *homogenous effect of stand-alone pair*. We replace in the statement of Theorem 1 *stand-alone pair* by this new axiom, and we dismiss the bounds axioms. We obtain a characterization of the general rules.

Theorem 4 *A rule satisfies additivity, weak equal treatment of equals and homogenous effect of stand-alone pair if and only if it is a general rule.*

In Theorem 1, we obtain the same family of rules when adding either one of the three bounds axioms. This does not happen in Theorem 4. For instance, the rule corresponding to $(x, y, z) = (1, 1, \frac{-1}{n})$ satisfies maximum aspirations (and hence weak upper bound) but violates non-negativity. The rule corresponding to $(x, y, z) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2n})$, which is $\frac{1}{2}U + \frac{1}{2}ES$, satisfies non negativity (and hence weak upper bound) but violates maximum aspirations. The rule corresponding to $(\frac{-n+1}{2}, \frac{-n+1}{2}, 1)$ satisfies weak upper bound but violates maximum aspirations and non-negativity.

We conclude providing two tables summarizing the results obtained in this section. First, in Table 1, we summarize the results obtained for *generalized split* rules. We use capital letters for necessary axioms in the characterizations and small letters for axioms that can be exchanged in the characterization (only one of them is necessary). Second, in Table 2, we summarize the results obtained for *generalized compromise* rules. When we write $[a, b]$ we refer to the family of rules $\{GUC^\lambda : \lambda \in [a, b]\}$.

Rules / Axioms	<i>ADD</i>	<i>WETE</i>	<i>SAP</i>	<i>NT</i>	<i>MA</i>	<i>NN</i>	<i>WUB</i>
$\{S^\lambda\}_{\lambda \in [0,1]}$ (Th. 1)	X	X	X		x	x	x
$\{S^\lambda\}_{\lambda \in [0,1]}$ (Cor. 1)	X	X		X	x	x	x
$\{GS^\lambda\}_{\lambda \in \mathbb{R}}$ (Th. 3.1)	X	X	x	x			

Table 1. Characterizations of generalized split rules.

Rules / Axioms	<i>ADD</i>	<i>ETE</i>	<i>WETE</i>	<i>PRP</i>	<i>MA</i>	<i>NN</i>	<i>WUB</i>
$[\frac{n-2}{2(n-1)}, 1]$ (Th. 2.1)	X		X	X	X		
$[\frac{-1}{n-1}, \frac{n-2}{2(n-1)}]$ (Th. 2.2)	X		X	X		X	
$[1 - \frac{n}{2}, 1]$ (Th. 2.3)	X		X	X			X
$(-\infty, +\infty)$ (Th. 3.2)	X		X	X			
$[\frac{n-2}{2(n-1)}, 1]$ (Pr. 2.1)	X	X			X		
$[\frac{-1}{n-1}, \frac{n-2}{2(n-1)}]$ (Pr. 2.2)	X	X				X	
$[1 - \frac{n}{2}, 1]$ (Pr. 2.3)	X	X					X
$(-\infty, +\infty)$ (Pr. 2.4)	X	X					

Table 2. Characterizations of generalized compromise rules.

3.2 With weak order preservation

We now explore the implications of strengthening *(weak) equal treatment of equals* to become *(weak) order preservation* in the above results. On the one hand, we observe that the three characterizations of the *split rules* become now characterizations of one half of the family; depending on whether they bias the splitting in favor of the home team or the away team.

Theorem 5 *The following statements hold:*

1. *A rule satisfies additivity, home order preservation, stand-alone pair, and either maximum aspirations, non-negativity, or weak upper bound if and only if it is a home-biased split rule.*
2. *A rule satisfies additivity, away order preservation, stand-alone pair, and either maximum aspirations, non-negativity, or weak upper bound if and only if it is an away-biased split rule.*

As all *split rules* satisfy *null team*, and *null team* implies *stand-alone pair* (the seventh statement of Proposition 1), we obtain the following corollary from Theorem 5.

Corollary 2 *The following statements hold:*

1. *A rule satisfies additivity, home order preservation, null team, and either maximum aspirations, non-negativity or weak upper bound if and only if it is a home-biased split rule.*
2. *A rule satisfies additivity, away order preservation, null team, and either maximum aspirations, non-negativity or weak upper bound if and only if it is an away-biased split rule.*

On the other hand, we obtain characterizations of each of the three families involving convex combinations between the three focal rules; namely, the *uniform* rule, *equal-split* rule, and *concede-and-divide*.

Theorem 6 *The following statements hold:*

1. *A rule satisfies additivity, home order preservation or away order preservation, pairwise reallocation proofness and maximum aspirations if and only if it belongs to the EC-family of rules.*

2. A rule satisfies additivity, home order preservation or away order preservation, pairwise reallocation proofness and non-negativity if and only if it belongs to the UE-family of rules.
3. A rule satisfies additivity, home order preservation or away order preservation, pairwise reallocation proofness and weak upper bound if and only if it belongs to the UC-family of rules

As the next result states, dismissing all bounds axioms (*maximum aspirations, non-negativity* or *weak upper bound*) in the above results leads to the characterization of half of each of the two complete families of *generalized (split or compromise)* rules.

Theorem 7 *The following statements hold:*

1. A rule satisfies additivity, home order preservation and either stand-alone pair or null team if and only if it belongs to the GS-family of rules for $\lambda \leq \frac{1}{2}$.
2. A rule satisfies additivity, away order preservation and either stand-alone pair or null team if and only if it belongs to the GS-family of rules for $\lambda \geq \frac{1}{2}$.
3. A rule satisfies additivity, home order preservation or away order preservation and pairwise reallocation proofness if and only if it belongs to the GUC-family of rules for $\lambda \geq 0$.

The next proposition is obtained by replacing in Theorem 6, and the last statement of Theorem 7, the combination of *home/away order preservation* and *pairwise reallocation proofness* by *order preservation*.

Proposition 3 *The following statements hold:*

1. A rule satisfies additivity, order preservation and maximum aspirations if and only if it belongs to the EC-family of rules.
2. A rule satisfies additivity, order preservation and non-negativity if and only if it belongs to the UE-family of rules.
3. A rule satisfies additivity, order preservation and weak upper bound if and only if it belongs to the UC-family of rules.

4. A rule satisfies additivity and order preservation if and only if it belongs to the GUC-family of rules for $\lambda \geq 0$.

We conclude providing two tables summarizing the results obtained in this section. First, in Table 3, we summarize the results obtained for *generalized split* rules. When we write $[a, b]$ we refer to the family of rules $\{GS^\lambda : \lambda \in [a, b]\}$. We use capital letters for necessary axioms in the characterizations and small letters for axioms that can be exchanged in the characterization (only one of them is necessary). Second, in Table 4, we summarize the results obtained for *generalized compromise* rules. When we write $[a, b]$ we refer to the family of rules $\{GUC^\lambda : \lambda \in [a, b]\}$.

Rules / Axioms	ADD	HOP	AOP	SAP	NT	MA	NN	WUB
$[0, \frac{1}{2}]$ (Th. 5.1)	X	X		X		x	x	x
$[\frac{1}{2}, 1]$ (Th. 5.2)	X		X	X		x	x	x
$[0, \frac{1}{2}]$ (Cor. 2.1)	X	X			X	x	x	x
$[\frac{1}{2}, 1]$ (Cor. 2.2)	X		X		X	x	x	x
$(-\infty, \frac{1}{2}]$ (Th. 7.1)	X	X		x	x			
$[\frac{1}{2}, +\infty)$ (Th. 7.2)	X		X	x	x			

Table 3. Further characterizations of generalized split rules.

Rules/Axioms	ADD	OP	WOP	PRP	MA	NN	WUB
$[\frac{n-2}{2(n-1)}, 1]$ (Th. 6.1)	X		X	X	X		
$[0, \frac{n-2}{2(n-1)}]$ (Th. 6.2)	X		X	X		X	
$[0, 1]$ (Th. 6.3)	X		X	X			X
$[0, +\infty)$ (Th. 7.3)	X		X	X			
$[\frac{n-2}{2(n-1)}, 1]$ (Pr. 3.1)	X	X			X		
$[0, \frac{n-2}{2(n-1)}]$ (Pr. 3.2)	X	X				X	
$[0, 1]$ (Pr. 3.3)	X	X					X
$[0, +\infty)$ (Pr. 3.4)	X	X					

Table 4. Further characterizations of generalized compromise rules.

4 Discussion

We have explored in this paper the axiomatic approach to the problem of sharing the revenues raised from the collective sale of broadcasting rights in sports leagues. We have mostly focussed on two alternative axioms formalizing the notion of *impartiality*: *equal treatment of equals* and *weak equal treatment of equals*. These two axioms, when combined with *additivity* and some other basic axioms, lead towards two alternative categories of rules. To wit, the former leads to several families of rules compromising between focal existing rules: the *uniform* rule, the *equal-split* rule and *concede-and-divide*. The latter instead leads to a unique family generalizing the *equal-split* rule to allow for unequal (but fixed) sharing of the audience of each game between the home team and the away team. The choice between *equal treatment of equals* and *weak equal treatment of equals* is ultimately a choice between disregarding the effect of $a_{ij} \neq a_{ji}$ or not and, as such, it is reflected in the characterization results just summarized.

We also characterize a more general family of rules (encompassing all of the above) by combining *additivity*, *weak equal treatment of equals* and a third axiom dubbed *homogeneous effect of stand-alone pair*, which states that in situations where only games played by a pair of teams have a positive audience, the remaining teams obtain a homogeneous amount. In the resulting family, the amount each team receives depends on three numbers: the overall home audience of the team, the overall away audience of the team, and the total audience of the whole tournament. There is a weighted aggregation of these three numbers and the weights are the same for each team.

We have also obtained further characterizations upon strengthening the axioms of *equal treatment of equals* to become axioms of *order preservation* in the previous results. In one case, that leads precisely towards convex combinations of the three focal rules. In the other case, to home-biased (or away-biased) split rules, which prioritize home teams (or away teams) in the splitting process within each pair of teams for the games they play.

Common to all of our characterization results is the axiom of *additivity*. This is an invariance requirement with a long tradition in axiomatic work (e.g., Shapley, 1953) but also considered strong under some circumstances. For results without *additivity* in this model, the reader is referred to Bergantiños and Moreno-Tertero (2020b).

As mentioned above, our analysis is restricted to a fixed-population setting. A generalization to a variable-population setting is left for further research. In such a setting, one would expect

that focal axioms such as *consistency*, or *population monotonicity*, would play a major role to obtain new characterization results.

To conclude, we mention that one could also be interested into approaching our problems with a (cooperative) game-theoretical approach, a standard approach in many related models of resource allocation (e.g., Littlechild and Owen, 1973; van den Nouweland et al., 1996; Ginsburgh and Zang, 2003; Bergantiños and Moreno-Ternero, 2015). In Bergantiños and Moreno-Ternero (2020a), we associate to our problems a natural *optimistic* cooperative TU game in which, for each subset of teams, we define its worth as the total audience of the games played by the teams in that subset.⁸ For such a resulting game, only two-player coalitions have nonzero dividends. Thus, it follows that the Shapley value coincides with the τ -value and the nucleolus (e.g., Deng and Papadimitriou, 1994; van den Nouweland et al., 1996). All those values yield the same solutions as the *equal-split* rule for the original problem.⁹ The egalitarian value (e.g., van den Brink, 2007) of that game yields the same solutions as the *uniform* rule. Casajus and Huettner (2013), van den Brink et al., (2013) and Casajus and Yokote (2019) characterize the family of values arising from the convex combination of the Shapley value and the egalitarian value. In our setting, this would correspond to the family of rules compromising (via convex combinations) between the *equal-split* rule and the *uniform* rule. Thus, the second statement of Proposition 3 in our paper could be considered as a parallel result to some of the results in that literature. One could conceivably go beyond convex combinations to consider linear combinations of the Shapley value and the egalitarian value. If so, the resulting generalized egalitarian Shapley values could be associated to our entire family of *generalized compromise* rules (including *concede-and-divide* as a member). Finally, the so-called egalitarian nonseparable contribution value (e.g., Driessen and Funaki, 1991) of that game yields the same solutions as the *dual* of the *uniform* rule, i.e.,

⁸Alternative natural options could also be considered in order to construct a characteristic function leading to a TU-game associated to our problems. To do so, one has to take a stance on what revenue a coalition can generate on its own. It is not obvious that any coalition of teams can generate the broadcasting revenues its members generated while being members of the league (the grand coalition). The above is an “optimistic” option assuming that each coalition of teams can generate the broadcasting revenues its members generated while being members of the league.

⁹The property that only two-player coalitions have nonzero dividends is used by Maniquet (2003), who shows that the so-called minimal transfer rule for queueing problems coincides with the Shapley value of the associated TU-game. As mentioned above, it follows from the property that coincidence with the nucleolus of such a game also holds, which is shown directly by Chun and Hokari (2007).

the rule that imposes to each team a uniform loss (where loss is understood as the difference between claim and obtained amount).¹⁰ Thus, the family of values arising from the convex combination of the egalitarian value and the egalitarian nonseparable contribution value (e.g., van den Brink et al., 2016) would correspond in our setting to the family of rules compromising (via convex combinations) between the *uniform* rule and its *dual*.

Nevertheless, the game-theoretical approach, although both natural and tractable, is not without loss of generality. For example, one may consider that being a member of a league also involves significant externalities, so that a subgroup of teams may not necessarily achieve the same audiences when part of a smaller league. This, together with the fact that there is not a unique way to associate a TU-game to our problems, leads us to prefer the axiomatic approach to the game-theoretical approach to analyze our problems. That is why we have endorsed the axiomatic approach in this paper.

5 Appendix. Proofs of the results.

For ease of exposition, we present the proofs of the results in a different order. In most of the proofs, we shall make use of the following notation. For each pair $i, j \in N$, with $i \neq j$, let $\mathbf{1}^{ij}$ denote the matrix with the following entries:

$$\mathbf{1}_{kl}^{ij} = \begin{cases} 1 & \text{if } (k, l) = (i, j) \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\mathbf{1}_{ji}^{ij} = 0$.

Proof of the first statement of Theorem 3

It is straightforward to show that each *generalized split* rule satisfies *additivity*, *weak equal treatment of equals*, *stand-alone pair* and *null team*. Conversely, let R be a rule satisfying *additivity*, *weak equal treatment of equals* and either *stand-alone pair* or *null team*. Let $i, j \in N$ with $i \neq j$. By *weak equal treatment of equals*, there exists $z^{ij} \in \mathbb{R}$ such that $R_k(\mathbf{1}^{ij}) = z^{ij}$ for all $k \in N \setminus \{i, j\}$. Let $R_i(\mathbf{1}^{ij}) = x^{ij}$ and $R_j(\mathbf{1}^{ij}) = y^{ij}$. Then, $x^{ij} + y^{ij} + (n-2)z^{ij} = \|\mathbf{1}^{ij}\| = 1$.

¹⁰This is actually a specific convex combination between the *equal-split* rule and *concede-and-divide*, thus satisfying *additivity*, *equal treatment of equals* (as a matter of fact, *order preservation*) and *maximum aspirations*. As the number of teams becomes large, the convex combination approaches *concede-and-divide*.

By *stand-alone pair* or *null team*, $z^{ij} = 0$ and $y^{ij} = 1 - x^{ij}$.

Let $k \in N \setminus \{i, j\}$. By *additivity*,

$$\begin{aligned} R_j(\mathbf{1}^{ij} + \mathbf{1}^{ik}) &= R_j(\mathbf{1}^{ij}) + R_j(\mathbf{1}^{ik}) = 1 - x^{ij}, \text{ and} \\ R_k(\mathbf{1}^{ij} + \mathbf{1}^{ik}) &= R_k(\mathbf{1}^{ij}) + R_k(\mathbf{1}^{ik}) = 1 - x^{ik}. \end{aligned}$$

By *weak equal treatment of equals*, $1 - x^{ij} = 1 - x^{ik}$. Hence, $x^{ij} = x^{ik}$.

Therefore, for each $i \in N$ there exists $x_i \in \mathbb{R}$ such that $R_i(\mathbf{1}^{ij}) = x_i$, for each $j \in N$. Thus,

$$\begin{aligned} R_i(\mathbf{1}^{ij} + \mathbf{1}^{ji}) &= x_i + 1 - x_j \text{ and} \\ R_j(\mathbf{1}^{ij} + \mathbf{1}^{ji}) &= 1 - x_i + x_j \end{aligned}$$

By *weak equal treatment of equals*, $x_i + 1 - x_j = 1 - x_i + x_j$. Hence, $x_i = x_j$. Thus, $R_i(\mathbf{1}^{ij}) = x$, for each pair $i, j \in N$.

Finally, let $A \in \mathcal{P}$ and $i \in N$. By *additivity*,

$$\begin{aligned} R_i(A) &= \sum_{j,k \in N} a_{jk} R_i(\mathbf{1}^{jk}) = \sum_{j \in N \setminus \{i\}} a_{ij} R_i(\mathbf{1}^{ij}) + \sum_{j \in N \setminus \{i\}} a_{ji} R_i(\mathbf{1}^{ji}) \\ &= x \sum_{j \in N \setminus \{i\}} a_{ij} + (1 - x) \sum_{j \in N \setminus \{i\}} a_{ji}. \end{aligned}$$

Let $\lambda = 1 - x$. Then, $R(A) = GS^\lambda(A)$, as desired.

Proof of Theorem 1

It is straightforward to show that each *split rule* satisfies all the axioms in the statement. Conversely, let R be a rule that satisfies *additivity*, *weak equal treatment of equals*, *stand-alone pair*, and either *maximum aspirations*, *non-negativity* or *weak upper bound*. By the first statement of Theorem 3, there exists $\lambda \in \mathbb{R}$ such that $R(A) = GS^\lambda(A)$.

Let $i, j \in N$ with $i \neq j$.

- By *maximum aspirations*, $1 - \lambda = R_i(\mathbf{1}^{ij}) \leq \alpha_i(\mathbf{1}^{ij}) = 1$, and $\lambda = R_j(\mathbf{1}^{ij}) \leq \alpha_j(\mathbf{1}^{ij}) = 1$.
- By *non-negativity*, $1 - \lambda = R_i(\mathbf{1}^{ij}) \geq 0$, and $\lambda = R_j(\mathbf{1}^{ij}) \geq 0$.
- By *weak upper bound*, $1 - \lambda = R_i(\mathbf{1}^{ij}) \leq \|\mathbf{1}^{ij}\| = 1$, and $\lambda = R_j(\mathbf{1}^{ij}) \leq \|\mathbf{1}^{ij}\| = 1$.

Thus, either way, $\lambda \in [0, 1]$, which concludes the proof.

Proof of the fourth statement of Proposition 2

It is obvious that U and CD satisfy *additivity* and *equal treatment of equals*. Then, each rule within the GUC -family also satisfies the two axioms. Conversely, let R be a rule satisfying *additivity* and *equal treatment of equals*. Let $i, j \in N$ with $i \neq j$. By *equal treatment of equals*, there exist $x^{ij}, z^{ij} \in \mathbb{R}$ such that $R_i(\mathbf{1}^{ij}) = R_j(\mathbf{1}^{ij}) = x^{ij}$ and $R_k(\mathbf{1}^{ij}) = z^{ij} = \frac{1-2x^{ij}}{n-2}$ for all $k \in N \setminus \{i, j\}$.

Let $k \in N \setminus \{i, j\}$. By *additivity*,

$$\begin{aligned} R_j(\mathbf{1}^{ij} + \mathbf{1}^{ik}) &= x^{ij} + z^{ik} \text{ and} \\ R_k(\mathbf{1}^{ij} + \mathbf{1}^{ik}) &= z^{ij} + x^{ik}. \end{aligned}$$

By *equal treatment of equals*, $R_j(\mathbf{1}^{ij} + \mathbf{1}^{ik}) = R_k(\mathbf{1}^{ij} + \mathbf{1}^{ik})$. Thus, $x^{ij} + \frac{1-2x^{ik}}{n-2} = x^{ik} + \frac{1-2x^{ij}}{n-2}$, which implies that $x^{ij} = x^{ik}$. Therefore, there exists $x \in \mathbb{R}$ such that for each $\{i, j\} \subset N$,

$$\begin{aligned} R_i(\mathbf{1}^{ij}) &= R_j(\mathbf{1}^{ij}) = x, \text{ and} \\ R_l(\mathbf{1}^{ij}) &= \frac{1-2x}{n-2} \text{ for each } l \in N \setminus \{i, j\}. \end{aligned}$$

Let $\lambda = \frac{nx-1}{n-1}$. Then,

$$(1-\lambda)U_k(\mathbf{1}^{ij}) + \lambda CD_k(\mathbf{1}^{ij}) = \begin{cases} (1-\lambda)\frac{1}{n} + \lambda = x & \text{if } k = i, j \\ (1-\lambda)\frac{1}{n} - \lambda\frac{1}{n-2} = \frac{1-2x}{n-2} & \text{otherwise.} \end{cases}$$

Thus, $R(\mathbf{1}^{ij}) = GUC^\lambda(\mathbf{1}^{ij})$. As both rules satisfy *additivity*, we deduce from here that $R(A) = GUC^\lambda(A)$, for each $A \in \mathcal{P}$, as desired.

Proof of the first statement of Proposition 2

By the fourth statement of Proposition 2, each rule within the EC -family satisfies *additivity* and *equal treatment of equals*. It is straightforward to show that they also satisfy *maximum aspirations*. Conversely, let R be a rule satisfying those three axioms. By the fourth statement of Proposition 2, there exists $\lambda \in \mathbb{R}$ such that $R = GUC^\lambda$.

Let $i, j \in N$ with $i \neq j$. By *maximum aspirations*,

$$R_i(\mathbf{1}^{ij}) = (1-\lambda)\frac{1}{n} + \lambda \leq \alpha_i(\mathbf{1}^{ij}) = 1,$$

and for each $k \in N \setminus \{i, j\}$,

$$R_k(\mathbf{1}^{ij}) = (1 - \lambda) \frac{1}{n} - \lambda \frac{1}{n-2} \leq \alpha_k(\mathbf{1}^{ij}) = 0.$$

It then follows that λ ranges from $\frac{n-2}{2(n-1)}$ to 1. As $UC^\lambda \equiv ES$, when $\lambda = \frac{n-2}{2(n-1)}$, this concludes the proof of this statement.

Proof of the second statement of Proposition 2

By the fourth statement of Proposition 2, each rule from the statement satisfies *additivity* and *equal treatment of equals*. As for *non-negativity*, one can also show (after some algebraic computations) that, for each $\lambda \in \left[\frac{-1}{n-1}, \frac{n-2}{2(n-1)} \right]$, GUC^λ satisfies it too. Conversely, let R be a rule satisfying those three axioms. By the fourth statement of Proposition 2, there exists $\lambda \in \mathbb{R}$ such that $R = GUC^\lambda$.

Let $i, j \in N$ with $i \neq j$. By *non-negativity*,

$$R_i(\mathbf{1}^{ij}) = (1 - \lambda) \frac{1}{n} + \lambda \geq 0,$$

and for each $k \in N \setminus \{i, j\}$,

$$R_k(\mathbf{1}^{ij}) = (1 - \lambda) \frac{1}{n} - \lambda \frac{1}{n-2} \geq 0.$$

It then follows that λ ranges from $\frac{-1}{n-1}$ to $\frac{n-2}{2(n-1)}$ to 1, which concludes the proof of this statement.

Proof of the third statement of Proposition 2

By the fourth statement of Proposition 2, each rule from the statement satisfies *additivity* and *equal treatment of equals*. As for *weak upper bound*, one can also show (after some algebraic computations) that, for each $\lambda \in \left[1 - \frac{n}{2}, 1 \right]$, GUC^λ satisfies it too. Conversely, let R be a rule satisfying those three axioms. By the fourth statement of Proposition 2, there exists $\lambda \in \mathbb{R}$ such that $R = GUC^\lambda$.

Let $i, j \in N$ with $i \neq j$. By *weak upper bound*,

$$R_i(\mathbf{1}^{ij}) = (1 - \lambda) \frac{1}{n} + \lambda \leq \|\mathbf{1}^{ij}\| = 1,$$

and for each $k \in N \setminus \{i, j\}$,

$$R_k(\mathbf{1}^{ij}) = (1 - \lambda) \frac{1}{n} - \lambda \frac{1}{n-2} \leq \|\mathbf{1}^{ij}\| = 1.$$

It then follows that λ ranges from $1 - \frac{n}{2}$ to 1, which concludes the proof of this statement.

Proof of Theorem 2

It is straightforward to show that each rule within the *GUC*-family satisfies *pairwise reallocation proofness*. Then, each statement of Theorem 2 follows from the corresponding statement at Proposition 2, together with the first two statements of Proposition 1.

Proof of the second statement of Theorem 3

It follows from the fourth statement of Proposition 2, together with the first two statements of Proposition 1.

Proof of Theorem 4

It is straightforward to show that each general rule satisfies the three axioms in the statement. Conversely, let R be a rule satisfying the three axioms. Let $i, j \in N$ with $i \neq j$. By *weak equal treatment of equals*, there exists $z^{ij} \in \mathbb{R}$ such that $R_k(\mathbf{1}^{ij}) = z^{ij}$ for all $k \in N \setminus \{i, j\}$. Let $R_i(\mathbf{1}^{ij}) = x^{ij}$ and $R_j(\mathbf{1}^{ij}) = y^{ij}$. Then, $x^{ij} + y^{ij} + (n-2)z^{ij} = \|\mathbf{1}^{ij}\| = 1$.

Let $k \in N \setminus \{i, j\}$. By *additivity*,

$$\begin{aligned} R_j(\mathbf{1}^{ij} + \mathbf{1}^{ik}) &= R_j(\mathbf{1}^{ij}) + R_j(\mathbf{1}^{ik}) = y^{ij} + z^{ik}, \text{ and} \\ R_k(\mathbf{1}^{ij} + \mathbf{1}^{ik}) &= R_k(\mathbf{1}^{ij}) + R_k(\mathbf{1}^{ik}) = z^{ij} + y^{ik} \end{aligned}$$

By *weak equal treatment of equals*, $y^{ij} + z^{ik} = z^{ij} + y^{ik}$. Hence, $y^{ij} - z^{ij} = y^{ik} - z^{ik}$. Thus, $y^i = y^{ij} - z^{ij}$ is well defined (does not depend on j). Now, for each $i, j \in N$, $y^{ij} = z^{ij} + y^i$.

By *additivity*,

$$\begin{aligned} R_j(\mathbf{1}^{ji} + \mathbf{1}^{ki}) &= R_j(\mathbf{1}^{ji}) + R_j(\mathbf{1}^{ki}) = x^{ji} + z^{ki}, \text{ and} \\ R_k(\mathbf{1}^{ji} + \mathbf{1}^{ki}) &= R_k(\mathbf{1}^{ji}) + R_k(\mathbf{1}^{ki}) = z^{ji} + x^{ki}. \end{aligned}$$

By *weak equal treatment of equals*, $x^{ji} + z^{ki} = z^{ji} + x^{ki}$. Hence, $x^{ji} - z^{ji} = x^{ki} - z^{ki}$. Thus, $x^i = x^{ij} - z^{ij}$ is well defined (does not depend on j). Now, for each $i, j \in N$, $x^{ij} = z^{ij} + x^i$.

Let $i, j, k \in N$ with $i \neq j$ and $i \neq k$. Then,

$$\begin{aligned} 1 &= \|\mathbf{1}^{ij}\| = \sum_{l \in N} R_l(\mathbf{1}^{ij}) = nz^{ij} + x^i + y^j \text{ and} \\ 1 &= \|\mathbf{1}^{ik}\| = \sum_{l \in N} R_l(\mathbf{1}^{ik}) = nz^{ik} + x^i + y^i. \end{aligned}$$

Hence, $z^{ij} = z^{ik}$ for each $j, k \in N \setminus \{i\}$.

By *homogeneous effect of stand-alone pair*, for each $i, j \in N$, $z^{ij} = z^{ji}$.

Let $i, j, k, l \in N$ with $i \neq j$ and $k \neq l$. Then,

$$z^{ij} = z^{il} = z^{li} = z^{lk} = z^{kl}.$$

Thus, there exists z such that $z^{ij} = z$ for all $i, j \in N$.

By *weak equal treatment of equals*, $R_i(\mathbf{1}^{ij} + \mathbf{1}^{ji}) = R_j(\mathbf{1}^{ij} + \mathbf{1}^{ji})$. Since

$$\begin{aligned} R_i(\mathbf{1}^{ij} + \mathbf{1}^{ji}) &= R_i(\mathbf{1}^{ij}) + R_i(\mathbf{1}^{ji}) = z^{ij} + x^i + z^{ji} + y^j = 2z + x^i + y^j \text{ and} \\ R_j(\mathbf{1}^{ij} + \mathbf{1}^{ji}) &= R_j(\mathbf{1}^{ij}) + R_j(\mathbf{1}^{ji}) = z^{ij} + y^i + z^{ji} + x^j = 2z + y^i + x^j, \end{aligned}$$

we conclude that

$$x^i + y^j = y^i + x^j.$$

As $1 = \|\mathbf{1}^{ij}\| = nz + x^i + y^i$, and $1 = \|\mathbf{1}^{ji}\| = nz + x^j + y^j$, we deduce that

$$x^i + y^i = x^j + y^j.$$

Now,

$$x^i = y^i + x^j - y^j \text{ and } x^i = x^j + y^j - y^i.$$

Thus, $y^i = y^j$, and hence $x^i = x^j$. It follows that $x = x^i$ and $y = y^i$ are well defined (they do not depend on $i \in N$).

The above implies that R is characterized by three numbers x , y , and z , satisfying that $nz + x + y = 1$, and that, for each $i, j \in N$,

$$R_k(\mathbf{1}^{ij}) = \begin{cases} z + x & \text{if } k = i \\ z + y & \text{if } k = j \\ z & \text{otherwise} \end{cases}$$

Finally, by *additivity*, for each $i \in N$, and $A \in \mathcal{P}$,

$$\begin{aligned} R_i(A) &= \sum_{j \in N \setminus \{i\}} a_{ij} R_i(\mathbf{1}^{ij}) + \sum_{j \in N \setminus \{i\}} a_{ji} R_i(\mathbf{1}^{ji}) + \sum_{j,k \in N \setminus \{i\}} a_{jk} R_i(\mathbf{1}^{jk}) \\ &= (z+x) \sum_{j \in N \setminus \{i\}} a_{ij} + (z+y) \sum_{j \in N \setminus \{i\}} a_{ji} + z \sum_{j,k \in N \setminus \{i\}} a_{jk} \\ &= x \sum_{j \in N \setminus \{i\}} a_{ij} + y \sum_{j \in N \setminus \{i\}} a_{ji} + z \|A\|, \end{aligned}$$

as desired.

Proof of the first two statements of Theorem 7

By the first statement of Theorem 3, each *generalized split rule* satisfies *additivity*, *stand-alone pair* and *null team*. As for *home order preservation*, let $A \in \mathcal{P}$, and $i, j \in N$ as in the definition of the axiom. Note that

$$\begin{aligned} GS_i^\lambda(A) &= a_{ij} GS_i^\lambda(\mathbf{1}^{ij}) + a_{ji} GS_i^\lambda(\mathbf{1}^{ji}) + \sum_{k \in N \setminus \{i,j\}} a_{ik} GS_i^\lambda(\mathbf{1}^{ik}) \\ &\quad + \sum_{k \in N \setminus \{i,j\}} a_{ki} GS_i^\lambda(\mathbf{1}^{ki}) + \sum_{k,l \in N \setminus \{i,j\}} a_{kl} GS_i^\lambda(\mathbf{1}^{kl}), \end{aligned}$$

and

$$\begin{aligned} GS_j^\lambda(A) &= a_{ij} GS_j^\lambda(\mathbf{1}^{ij}) + a_{ji} GS_j^\lambda(\mathbf{1}^{ji}) + \sum_{k \in N \setminus \{i,j\}} a_{jk} GS_j^\lambda(\mathbf{1}^{jk}) \\ &\quad + \sum_{k \in N \setminus \{i,j\}} a_{kj} GS_j^\lambda(\mathbf{1}^{kj}) + \sum_{k,l \in N \setminus \{i,j\}} a_{kl} GS_j^\lambda(\mathbf{1}^{kl}). \end{aligned}$$

Now, for each $k \in N \setminus \{i, j\}$, $GS_i^\lambda(\mathbf{1}^{ik}) = GS_j^\lambda(\mathbf{1}^{jk}) = 1 - \lambda$ and $GS_i^\lambda(\mathbf{1}^{ki}) = GS_j^\lambda(\mathbf{1}^{kj}) = \lambda$. Similarly, $GS_i^\lambda(\mathbf{1}^{kl}) = GS_j^\lambda(\mathbf{1}^{kl}) = 0$, for each pair $k, l \in N \setminus \{i, j\}$. Finally, $a_{ij} GS_i^\lambda(\mathbf{1}^{ij}) + a_{ji} GS_i^\lambda(\mathbf{1}^{ji}) \geq a_{ij} GS_j^\lambda(\mathbf{1}^{ij}) + a_{ji} GS_j^\lambda(\mathbf{1}^{ji})$ if and only if $(a_{ij} - a_{ji})(1 - 2\lambda) \geq 0$. As $\lambda \leq \frac{1}{2}$, $GS_i^\lambda(A) \geq GS_j^\lambda(A)$, as desired. Likewise, for each $\lambda \geq \frac{1}{2}$, GS^λ satisfies *away order preservation*.

Conversely, let R be a rule satisfying *additivity*, either *home order preservation* or *away order preservation*, and either *stand-alone pair* or *null team*. By the sixth statement of Proposition 1, R also satisfies *weak equal treatment of equals*. Thus, by the first statement of Theorem 3, there exists $\lambda \in \mathbb{R}$ such that $R = GS^\lambda$. Then, for each pair $i, j \in N$ such that $i \neq j$, $R_i(\mathbf{1}^{ij}) = 1 - \lambda$ and $R_j(\mathbf{1}^{ij}) = \lambda$. Now, if R satisfies *home order preservation*, $1 - \lambda = R_i(\mathbf{1}^{ij}) \geq R_j(\mathbf{1}^{ij}) = \lambda$. Thus, $\lambda \in (-\infty, \frac{1}{2}]$, which concludes the proof of the first statement. Likewise, if R satisfies

away order preservation, $1 - \lambda = R_i(\mathbf{1}^{ij}) \leq R_j(\mathbf{1}^{ij}) = \lambda$. Thus, $\lambda \in [\frac{1}{2}, +\infty)$, which concludes the proof of the second statement.

Proof of Theorem 5

Except for the *weak order preservation* axioms, we already know from Theorem 1 that each *split* rule satisfies all the axioms in the statement. By the first two statements of Theorem 7, each *home-biased split* rule satisfies *home order preservation* and each *away-biased split* rule satisfies *away order preservation*. Conversely, let R be a rule satisfying *additivity*, either *home order preservation* or *away order preservation*, *stand-alone pair*, and either *maximum aspirations*, *non-negativity*, or *weak upper bound*. By the sixth statement of Proposition 1, R also satisfies *weak equal treatment of equals*. Thus, by Theorem 1, there exists $\lambda \in [0, 1]$ such that $R = S^\lambda$. In particular, for each pair $i, j \in N$ such that $i \neq j$, $R_i(\mathbf{1}^{ij}) = 1 - \lambda$ and $R_j(\mathbf{1}^{ij}) = \lambda$. Now, if R satisfies *home order preservation*, $1 - \lambda = R_i(\mathbf{1}^{ij}) \geq R_j(\mathbf{1}^{ij}) = \lambda$. Thus, $\lambda \in [0, \frac{1}{2}]$, which concludes the proof of the first statement. Likewise, if R satisfies *away order preservation*, $1 - \lambda = R_i(\mathbf{1}^{ij}) \leq R_j(\mathbf{1}^{ij}) = \lambda$. Thus, $\lambda \in [\frac{1}{2}, 1]$, which concludes the proof of the second statement.

Proof of the fourth statement of Proposition 3

We know each rule within the *GUC-family* satisfies *additivity*. As *concede-and-divide* satisfies *order preservation*, and the *uniform rule* assigns the same amount to all teams, it follows that $(1 - \lambda)U + \lambda CD$ also satisfies *order preservation* for each $\lambda \in [0, \infty)$. Conversely, let R be a rule satisfying *additivity* and *order preservation*. By the fifth statement of Proposition 1, R also satisfies *equal treatment of equals*. Thus, by the fourth statement of Proposition 2, there exists $\lambda \in \mathbb{R}$ such that $R = GUC^\lambda$. Besides, given $i, j \in N$ with $i \neq j$,

$$R_k(\mathbf{1}^{ij}) = \begin{cases} x & \text{if } k = i, j \\ \frac{1-2x}{n-2} & \text{otherwise.} \end{cases}$$

By *order preservation*, $x \geq \frac{1-2x}{n-2}$ which implies that $x \geq \frac{1}{n}$. As $\lambda = \frac{nx-1}{n-1}$, it follows that $\lambda \in [0, +\infty)$.

Proof of the first statement of Proposition 3

As mentioned above, each rule within the *EC-family* satisfies *additivity* and *maximum aspirations*. It is obvious that *ES* and *CD* satisfy *order preservation*. Then, each rule within the *EC-family* satisfies *order preservation*. Conversely, let R be a rule satisfying those three axioms. By the fourth statement of Proposition 3, there exists $\lambda \in [0, +\infty)$ such that $R = GUC^\lambda$. By *maximum aspirations*, similarly to the proof of the first statement of Proposition 2, we can conclude that $\lambda \in \left[\frac{n-2}{2(n-1)}, 1 \right]$. Namely, R belongs to the *EC-family*, as desired.

Proof of the second statement of Proposition 3

As mentioned above, each rule within the *UE-family* satisfies *additivity* and *non-negativity*. As U and *ES* satisfy *order preservation*, each rule within the *UE-family* satisfies it too. Conversely, let R be a rule satisfying those three axioms. By the fourth statement of Proposition 3, there exists $\lambda \in [0, +\infty)$ such that $R = GUC^\lambda$. By *non negativity*, similarly to the proof of the second statement of Proposition 2, we can conclude that $\lambda \in \left[\frac{-1}{n-1}, \frac{n-2}{2(n-1)} \right]$. Then, $\lambda \in \left[0, \frac{n-2}{2(n-1)} \right]$. Namely, R belongs to the *UE-family*, as desired.

Proof of the third statement of Proposition 3

As mentioned above, each rule within the *UC-family* satisfies *additivity* and *weak upper bound*. It is obvious that U and *CD* satisfy *order preservation*. Then, each rule within the *UC-family* satisfies *order preservation*. Conversely, let R be a rule satisfying those three axioms. By the fourth statement of Proposition 3, there exists $\lambda \in [0, +\infty)$ such that $R = GUC^\lambda$. By *weak upper bound*, similarly to the proof of the third statement of Proposition 2, we can conclude that $\lambda \in \left[1 - \frac{n}{2}, 1 \right]$. Then, $\lambda \in [0, 1]$. Namely, R belongs to the *UC-family*, as desired.

Proof of Theorem 6

Each statement of Theorem 6 follows from the corresponding statement at Proposition 3, together with the third and fourth statements of Proposition 1.

Proof of the third statement of Theorem 7

It follows from the fourth statement of Proposition 3, together with the third and fourth statements of Proposition 1.

References

- [1] Allen, E., Dechow, P., Pope, D., Wu, G., 2017. Reference-Dependent Preferences: Evidence from Marathon Runners. *Management Science* 63, 1657-1672.
- [2] Arrfelt, M., Wiseman, R., Hult, G., 2013. Looking backward instead of forward: Aspiration-driven influences on the efficiency of the capital allocation process. *Academy of Management Journal* 56, 1081-1103.
- [3] Bergantiños, G., Moreno-Tertero, J.D., 2015. The axiomatic approach to the problem of sharing the revenue from museum passes. *Games and Economic Behavior* 89, 78-92.
- [4] Bergantiños, G., Moreno-Tertero, J.D., 2020a. Sharing the revenues from broadcasting sport events. *Management Science* 66, 2417-2431.
- [5] Bergantiños, G., Moreno-Tertero, J.D., 2020b. Allocating extra revenues from broadcasting sports leagues. *Journal of Mathematical Economics* 90, 65-73.
- [6] Bergantiños, G., Moreno-Tertero, J.D., 2021a. Compromising to share the revenues from broadcasting sports leagues. *Journal of Economic Behavior and Organization* 183, 57-74.
- [7] Bergantiños, G., Moreno-Tertero, J.D., 2021b. Monotonicity in sharing the revenues from broadcasting sports leagues. *European Journal of Operational Research*. Forthcoming.
- [8] van den Brink, R., 2007. Null or nullifying players: The difference between the Shapley value and equal division solutions. *Journal of Economic Theory* 136, 767-775.
- [9] van den Brink, R., Funaki Y., Ju, Y., 2013. Reconciling marginalism with egalitarianism: consistency, monotonicity, and implementation of egalitarian Shapley values. *Social Choice and Welfare* 40, 693-714.
- [10] van den Brink, R., Chun, Y., Funaki Y., Park, B., 2016. Consistency, Population Solidarity, and Egalitarian Solutions for TU-games. *Theory and Decision* 81, 427-447.

- [11] Casajus A., Huettner F., 2013. Null players, solidarity, and the egalitarian Shapley values. *Journal of Mathematical Economics* 49, 58-61.
- [12] Casajus A., Yokote, K., 2019. Weakly differentially monotonic solutions for cooperative games. *International Journal of Game Theory* 48, 979-997.
- [13] Chun, Y., Hokari, T., 2007. On the Coincidence of the Shapley Value and the Nucleolus in Queueing Problems. *Seoul Journal of Economics* 20, 223-237.
- [14] Deng, X., Papadimitriou, C., 1994. On the Complexity of Cooperative Solution Concepts. *Mathematics of Operations Research* 19, 257-266.
- [15] Driessen, T., Funaki, Y., 1991. Coincidence of and collinearity between game theoretic solutions, *OR Spektrum* 13, 15-30.
- [16] Ginsburgh, V., Zang, I., 2003. The museum pass game and its value. *Games and Economic Behavior* 43, 322-325.
- [17] Ju, B-G., Kim, M., Kim, S., Moreno-Tertero, J.D., 2021. Fair international protocols for the abatement of GHG emissions. *Energy Economics* 94, 105091.
- [18] Littlechild, S., Owen. G., 1973, A simple expression for the Shapley value in a special case. *Management Science* 20, 370-372.
- [19] Maniquet F., 2003. A characterization of the Shapley value in queueing problems. *Journal of Economic Theory* 109, 90-103.
- [20] Moreno-Tertero, J., Roemer, J., 2006. Impartiality, priority and solidarity in the theory of justice. *Econometrica* 74, 1419-1427.
- [21] van den Nouweland, A., Borm, P., van Golstein Brouwers, W., Groot Bruinderink, R., Tijs, S., 1996. A Game Theoretic Approach to Problems in Telecommunication. *Management Science* 42, 294-303.
- [22] Shapley, L., 1953. A value for n-person games, in *Contributions to the Theory of Games II* (*Annals of Mathematics Studies* 28), ed. by H.W. Kuhn and A.W. Tucker, Princeton: Princeton University Press, 307-317.

- [23] Thomson, W., 2011. Fair allocation rules, Chapter 21 in the Handbook of Social Choice and Welfare Vol. 2, Arrow, K., Sen, A., Suzumura, K., eds. North Holland, 393-506.
- [24] Thomson W., 2019. How to divide when there isn't enough: from Aristotle, the Talmud, and Maimonides to the axiomatics of resource allocation, Econometric Society Monograph. Cambridge University Press.