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# Compensation and sacrifice in the probabilistic rationing of indivisible units

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## Abstract

We consider the problem of randomly allocating indivisible units of a resource among agents with conflicting claims on the resource. An axiom reflecting a principle of *compensation* allows us to characterize the focal *probabilistic uniform awards* rule. A dual axiom reflecting a principle of *sacrifice* allows us to characterize the (dual) *probabilistic uniform losses* rule. The combination of two (other) axioms reflecting both principles (of *compensation* and *sacrifice*) allows us to characterize the (compromise) *probabilistic concede-and-divide* in the two-agent case. There is, however, no consistent extension of this rule to the general case of an arbitrary number of agents.

**Keywords**: game theory, resource allocation, axioms, probabilistic, discrete goods.

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# 1 Introduction

The problem of adjudicating conflicting claims over a resource is ubiquitous in economics. Instances are already documented in ancient sources such as Aristotle's Ethics, the Talmud or Maimonides' Book of Judgements. But it was not until the early 1980's that a formal (yet simple) model to analyze this problem was developed. The model gave rise to an important literature in the last four decades.[1] This literature mostly concentrated on the case in which the resource is perfectly divisible and the allocation is deterministic. But real-life situations in which the resource comes in indivisible units and so do claims abound in daily life: e.g., organ transplants, immigration visas, rationing food in refugee camps, or ventilators as COVID-19 spreads worldwide.[2] Two options can be considered to solve rationing problems in which the resource comes in indivisible units: deterministic or probabilistic allocations.[3] In this paper, we concentrate on the latter case. That is, we look for rules that associate with each problem a random variable over the set of all possible deterministic allocations for such a problem.[4]

We believe the probabilistic approach is particularly relevant for cases in which rotation (time-sharing) or randomization (each claimant has a certain probability of getting the indivisible good) are natural and appealing solutions. Rotation, for instance, is customary in resolving child custody disputes and frequent in solving some other disputes associated to divorce settlements (such as those referring to estate properties). Randomization is not only widespread in gambling, but also in more relevant settings such as school lotteries or the so-called Diversity Immigrant Visa Program. In all those cases, the aim is to obtain the probabilities associated to each possible outcome, to be interpreted as the portion of time in which the good will be enjoyed by each claimant, or the chances each claimant has of getting the good.

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[1] The seminal paper is O'Neill (1982). For an excellent and extremely comprehensive survey, the reader is referred to Thomson (2019). Other partial surveys are Moulin (2002b), Thomson (2003, 2015) and Moreno-Ternero (2018). The field of operations research has devoted considerable attention to O'Neill's model (e.g., Young, 1987; Lahiri, 2001; van den Brink et al., 2013; Giménez-Gómez and Peris, 2014), some of its applications (e.g., Casas-Méndez et al., 2011; Gutiérrez et al., 2018), or several of its generalizations (e.g., Calleja et al., 2005; Bergantiños and Vidal-Puga, 2006; Bergantiños and Lorenzo, 2008; Moreno-Ternero and Vidal-Puga, 2021; Estévez-Fernández et al., 2021).

[2] For a recent example considering the allocation of ventilators, using a reserve system instead of objective claims, the reader is referred to Pathak et al., (2020).

[3] The *discrete* model (with deterministic allocations) was introduced by Moulin (2000). See also Herrero and Martínez (2008), Chen (2015), and Estañ et al. (2021). The *probabilistic* model (in random allocations) was introduced by Moulin (2002a). See also Moulin and Stong (2002, 2003) and Tasnadi (2002). Fragnelli et al., (2014, 2016) analyze an intermediate case, with perfectly divisible claims but a discrete resource.

[4] Our approach will thus be reminiscent to another class of claims problems with probabilities, which are those with uncertain needs (e.g., Xue, 2018; Long et al., 2021; Long and Xue, 2021).

We take the axiomatic approach for our enterprise. Key to our analysis will be axioms reflecting principles of *compensation* and *sacrifice*, formalizing (respectively) protective criteria forvagents with “very large” orv“very small” claims.

Specifically, the *conditional compensation* axiom states that agents who cannot be considered responsible for the scarcity should be exempted from rationing. Accountability of an agent is conditional on whether replacing all higher claims by her own claim, there would be enough to compensate everyone or not. For example, consider the problem in which there are only 10 units available but three agents claim 2, 30 and 50 units respectively. Here, the scarcity is mainly due to the high demands of the last two agents. From the first agent’s perspective, her claim is “reasonable” in the following sense: if the other two agents would claim no more than the amount claimed by Agent 1, then the aggregate demand could be satisfied. In this sense, agent 1 is not responsible for the resource shortage and the axiom states that she should not be rationed. In our probabilistic setting, any allocation for this problem will convey that Agent 1’s demand be fully honored.

Dually, one could adopt the viewpoint that agents with larger claims should be given priority so that agents with much smaller claims should not receive anything. In our probabilistic setting, this means that any allocation at which an agent with a (relatively) very small claim receives a positive amount gets zero probability. That kind of value judgement makes sense, for instance, when we want to compensate more to those who have risked larger amounts. More precisely, the *conditional sacrifice* axiom states that agents with “residual”claims are fully forgone in the rationing process. In this case, the condition is formalized as stating that the aggregate excess claim relative to the agent holding this claim exceeds the amount available. In the numerical example presented above, the claim of 2 units was residual, but so was the claim of 30 units.

The two previous axioms are somewhat extreme. They also provide conditional statements on the allocations a given agent obtains. Namely, in both cases, the fulfillment of a certain requirement (to qualify as either having a sufficiently high or sufficiently low claim) guarantees a certain feature of individual allocations (obtaining the whole claim, or nothing, with probability 1). For both reasons, we shall also consider alternative partial versions of the two axioms, aiming to compromise between the principles of *compensation* and *sacrifice*, as well as guaranteeing agents certain amounts without imposing any prior condition. More precisely, the *sure compensation* axiom states that agents will always get a sure lower bound, which only depends on the available amount, the number of claimants and the own claim of that agent. Similarly, the *sure sacrifice* axiom states that agents will always have to face at least a sure loss, which only depends on the overall loss, the number of claimants and the own claim of that agent. To give both axioms bite, the precise value of both bounds in the statements is set at the maximum possible level. This implies they are exactly the minimum of the integer part of one $n$-th of the claim and the overall amount, or the aggregate loss. In the numerical example presented above, the *sure compensation* axiom would imply that the two agents with large claims would get at least 3 units, whereas the *sure sacrifice* axiom would have no bite.

Apart from the previous axioms, we shall also consider a standard axiom (*claim monotonicity*) indicating that an increase in an agent's claim, ceteris paribus, cannot decrease (stochastically) that agent's (random) award. Its dual axiom (*linked claim-endowment monotonicity*) says that if an agent's claim and the amount to divide increase by the same amount, the agent's award should not increase by more than that amount (again, in stochastic terms). Equivalently, an agent's random loss does not decrease as her claim increases while the loss and the others' claims are unchanged. Both axioms are used previously in this setting by Moulin and Stong (2002).

We also consider an axiom in the two-agent case (*bilateral evenness*) stating that if only two allocations are chosen with positive probability, and they only differ in one unit, then they are equiprobable.

Finally, we consider the standard notion of *consistency*, which has played a crucial role in axiomatic research (e.g., Thomson, 2012). It relates the solution of a given problem to the solutions of the subproblems that appear when we consider a subgroup of agents as a new population and the sum of their assignments in the original problem as the available amount to be distributed. More precisely, *consistency* requires that the application of the rule to each subproblem produces precisely the allocation that the subgroup obtained in the original problem. In this setting, we follow Moulin (2002a) and Moulin and Stong (2002) in incorporating the random character of the allocation process in the formulation of the axiom.

Theorem 1 states that *conditional compensation*, *claim monotonicity*, *bilateral evenness* and *consistency* characterize the *probabilistic uniform awards* rule, which is the counterpart in this setting of the so-called *constrained equal awards* rule in the benchmark model of deterministic claims. Corollary 1 is the dual of Theorem 1. It states that *conditional sacrifice*, *linked claim-endowment monotonicity*, *bilateral evenness* and *consistency* characterize the *probabilistic uniform losses rule*, which is the counterpart in our setting of the so-called *constrained equal losses* rule in the benchmark model of deterministic claims.

Moulin and Stong (2002) provide alternative characterizations of the previous two rules. In their case, instead of *consistency* and axioms formalizing the principles of *compensation* and *sacrifice*, they consider *composition* axioms pertaining to the way a rule reacts with respect to tentative estimations of the endowment to be allocated. Instead of *bilateral evenness* they consider two impartiality notions: *equal treatment ex ante*, and *equal treatment ex post*. Table 1 in Section 5.3 provides the performance of the rules with respect to the axioms considered here, as well as those considered in Moulin and Stong (2002).

Theorem 2 states that, in the two-agent case, *sure compensation*, *sure sacrifice* and *bilateral evenness* characterize *probabilistic concede-and-divide*, which is the counterpart in our setting of the so-called *concede-and-divide*, a rule in the benchmark model of deterministic claims that can be traced back to the Talmud.[5] For endowments below the smallest claim, this rule behaves as the *probabilistic uniform awards rule*. For endowments above the highest claim, it behaves as the *probabilistic uniform losses rule*. This implies that, for intermediate endowments, the rule yields the lowest claimant (in probabilistic terms) one half of her claim, when possible.

It turns out that Theorem 2 does not extend to the general case of an arbitrary number of agents. In fact, Theorem 3 states that *sure compensation*, *sure sacrifice*, *bilateral evenness* and *consistency* are incompatible. In other words, there is no consistent extension of the *probabilistic concede-and-divide* rule.

The previous theorems have counterparts in the benchmark model with perfectly divisible endowments. The *constrained equal awards* rule is characterized by *conditional compensation* and *claim monotonicity* (Yeh, 2006).[6] By duality, the *constrained equal losses* rule is characterized by *conditional sacrifice* and *linked claim-endowment monotonicity*. In the counterpart results in our setting we need to add the axioms of *bilateral evenness* and *consistency* (otherwise, more rules would satisfy the previous pairs of axioms). *Concede-and-divide* is characterized by *sure compensation* and *sure sacrifice* (Moreno-Ternero and Villar, 2004).[7] The counterpart result in our setting adds the axiom of *bilateral evenness* (otherwise, more rules would exist). Also, the so-called *Talmud* rule, the unique consistent extension of *concede-and-divide*, is precisely characterized by *sure compensation*, *sure sacrifice* and *consistency* (Moreno-Ternero and Villar, 2004). In our setting, the axioms are incompatible. This exemplifies the striking difference between the probabilistic model of rationing considered here and the benchmark model.

Striking differences between the different models of rationing are not rare. Quite the opposite.

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[5] The term was coined by Thomson (2003) as it reflects the intuitive procedure underlying the rule: each claimant concedes to the other what she does not claim and the contested part is split equally.

[6] *Conditional compensation* is dubbed *sustainability* by Yeh (2006).

[7] These two axioms are dubbed *securement* and *the dual of securement* by Moreno-Ternero and Villar (2004).

For instance, Moulin (2000) characterizes the rules satisfying *consistency* and two invariance notions known as *composition down* and *composition up* both in the benchmark model and in the benchmark discrete model. In the latter model, only the so-called *priority* rules arise. In the former model, a much richer family arises. In the probabilistic model we consider in this paper, Moulin and Stong (2002) show that another family arises, but this family is not as rich as the one in the benchmark model.

Moulin and Stong (2002) also characterize the so-called family of *standard-of-gains* rules by the combination of *composition up*, *claim monotonicity* and *consistency*. Dually, the combination of *composition down*, *linked claim-endowment monotonicity* and *consistency*, characterizes the so-called family of *standard-of-losses methods*. Recently, Chambers and Moreno-Ternero (2017) characterize in the benchmark model the (continuous and symmetric) rules satisfying *composition down* and *consistency*. They comprise a large family of rules (*generalized equal-sacrifice rules*) between the so-called *parametric* rules (characterized by Young, 1987) and the so-called *equal-sacrifice* rules (characterized by Young, 1988). These families are not related in an obvious way, although the *probabilistic uniform losses rule* belongs to the family of *standard-of-losses methods* and the *constrained equal losses* belongs to the family of *generalized equal-sacrifice rules*.

Our analysis ignores the *proportional* rule, which has played a central role both in the benchmark model (e.g., Thomson, 2019) as well as in the probabilisitic model (e.g., Moulin, 2002b).[8] We emphasize that the *proportional* rule does not have a unique analogue in the probabilisitic model. In particular, the simplest approach is to achieve ex-ante proportionality, and subject to this constraint minimize variance, as characterized by Tasnadi (2002).[9] An alternative rule that also achieves ex-ante proportionality is characterized by Moulin (2002b). This rule satisfies a variety of axioms including *consistency*, and even has a characterization without *consistency* (e.g., Moulin and Stong, 2002). But, obviously, the rule violates our central notions of *compensation* and *sacrifice* (and that is the reason why it is ignored by our analysis). This is probably the most disturbing feature, from a distributive justice perspective, of this otherwise central rule.

The rest of the paper is organized as follows. In Section 2, we present the model. We present the results in Section 3. We conclude in Section 4. For a smooth passage, most of the proofs are deferred to an appendix (where we also include the independence of the axioms used in our characterizations).

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[8] Obviously, the rule cannot be properly defined in the *discrete* model, in which allocations are deterministic and the endowment comes in indivisible units.

[9] Note that this minimal variance approach could also be followed with other rules from the benchmark model (not necessarily the *proportional* rule). In particular, doing so with the *constrained equal awards* and the *constrained equal losses* rules would allow us to obtain different axiomatizations for the *probabilistic uniform rewards/losses* rules.

## 2 The model

Let $N = \{1, \ldots, n\}$ be a set of claimants, or **agents**. For each $i \in N$, let $c_i \in \mathbb{N}$ be $i$'s **claim** and $c \equiv (c_i)_{i \in N}$ the claims profile (without loss of generality we assume that $c_1 \leq c_2 \leq \cdots \leq c_n$).[10] An endowment $E \in \mathbb{N}$ is to be allocated among agents in $N$. A (discrete rationing) **problem** is a pair $(c, E) \in \mathbb{N}^N \times \mathbb{N}$, such that $\sum_{i \in N} c_i \geq E$. Let $C \equiv \sum_{i \in N} c_i$ and $L \equiv C - E$. Let $\mathcal{D}^N$ be the domain of all problems. We shall also consider a variable-population generalization of the model. Then, there is a set of potential claimants, which are indexed by the natural numbers (excluding 0).

Given a problem $(c, E) \in \mathcal{D}^N$, a **deterministic allocation** is an integer vector $x \in \mathbb{Z}^N$ satisfying the following two conditions: (i) for each $i \in N$, $0 \leq x_i \leq c_i$, and (ii) $\sum_{i \in N} x_i = E$. Let $X(c, E)$ denote the set of all deterministic allocations for the problem $(c, E)$. A **deterministic rule** is a mapping $r : \mathcal{D} \longrightarrow \bigcup_{N \in \mathcal{N}} \mathbb{Z}^N$ that selects, for each problem $(c, E) \in \mathcal{D}^N$, a unique deterministic allocation $r(c, E) \in X(c, E)$.

Given a problem $(c, E) \in \mathcal{D}^N$, a **probabilistic allocation** is a random variable over $X(c, E)$. That is, it is a mapping $\theta : X(c, E) \longrightarrow [0, 1]$ such that $\sum_{x \in X(c,E)} \theta(x) = 1$. We denote by $\theta_i$ the marginal distribution of $\theta$ with respect to agent $i$. Let $\Theta(c, E)$ denote the set of all probabilistic allocations for the problem $(c, E) \in \mathcal{D}^N$. Formally,

$$\Theta(c, E) = \left\{ \theta : X(c, E) \longrightarrow [0, 1] : \sum_{x \in X(c,E)} \theta(x) = 1 \right\}.$$

A **probabilistic rule** $\rho$ is a mapping that selects, for each problem, a unique probabilistic allocation $\rho(c, E) \in \Theta(c, E)$. For a given probabilistic rule $\rho$ and deterministic allocation $x \in X(c, E)$, $\rho(c, E)(x)$ is the probability with which $x$ is selected under rule $\rho$. We denote by $\rho_i(c, E)$ the marginal distribution of $\rho(c, E)$ with respect to agent $i$.

Given a probabilistic rule $\rho$, its dual $\rho^d$ is defined as follows. For each $(c, E) \in \mathcal{D}^N$, and each $x \in X(c, E)$,

$$\rho^d(c, E)(x) = \rho(c, L)(c - x).$$

If a rule coincides with its dual then it is called self-dual.

Throughout this paper, we shall consider the standard notion of stochastic dominance to compare random variables. We say that $\theta_i$ **stochastically dominates** $\mu_i$ (written as $\theta_i \succeq_{\text{SD}} \mu_i$) if $\text{prob}(\theta_i \geq a) \geq \text{prob}(\mu_i \geq a)$ for each $a \in \mathbb{N}$. We shall also use the notation $(x_i', x_{-i})$, when the $i$-th coordinate of vector $x$ has been replaced by $x_i'$, and $\lfloor z \rfloor$ for the largest integer bounded above by $z$.

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[10] Let $\mathbb{N}$ denote the set of natural numbers $\mathbb{N} = \{0, 1, \cdots\}$.

## 2.1 Axioms

We now list the axioms for probabilistic rules we consider in this paper.

First, two axioms (already used in this setting by Moulin and Stong, 2002) respectively stating that an agent's award does not decrease (stochastically) if her claim increases, ceteris paribus, and that when her claim and the endowment increase by the same amount her share cannot increase (stochastically) more than such an amount.[11] Formally,

**Claim monotonicity**. For each $(c, E) \in \mathcal{D}^N$, each $i \in N$, and each $c'_i > c_i$, we have $\rho_i((c'_i, c_{-i}), E) \succeq_{\text{SD}} \rho_i(c, E)$.

**Linked claim-endowment monotonicity**. For each $(c, E) \in \mathcal{D}^N$, each $i \in N$, and each $d > 0$, we have $\rho_i(c, E) + d \succeq_{\text{SD}} \rho_i((c_i + d, c_{-i}), E + d)$.[12]

These two properties are dual.[13] So are the next two. The first one states that those agents with *sustainably small* claims should be exempted from rationing. That is, any allocation in which such a claimant does not receive her whole claim is chosen with zero probability. The second one says that agents with *residual* claims should be fully rationed. That is, each allocation in which such a claimant receives something is chosen with zero probability. Formally,

**Conditional compensation**. For each $(c, E) \in \mathcal{D}^N$ and each $x \in X(c, E)$ such that $x_i \neq c_i$, for some $i \in N$ such that $E \geq \sum_{j=1}^n \min\{c_i, c_j\}$, we have $\rho(c, E)(x) = 0$.

**Conditional sacrifice**. For each $(c, E) \in \mathcal{D}^N$ and each $x \in X(c, E)$ such that $x_i \neq 0$, for some $i \in N$ such that $E \leq \sum_{j=1}^n \max\{0, c_j - c_i\}$, we have $\rho(c, E)(x) = 0$.

In contrast to the previous pair, the next (also dual) properties provide a sure (rather than conditional) *compensation* or *sacrifice*. More precisely, the first one provides each claimant with a lower bound on the awards that depends on the endowment units, the individual claim and the number of claimants. The second one provides each claimant with a lower bound on the losses that depends on the total loss units, the individual claim and the number of claimants. To give both axioms bite, the precise value of both bounds in the statements is set at the maximum possible level. Formally,

**Sure compensation**. For each $(c, E) \in \mathcal{D}^N$, and each $x \in X(c, E)$ such that $x_i < \min\left\{\left\lfloor \frac{E}{n} \right\rfloor, \left\lfloor \frac{c_i}{n} \right\rfloor\right\}$ for some $i \in N$, we have $\rho(c, E)(x) = 0$.

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[11] Moulin and Stong (2002) use the terms of **demand monotonicity** and **demand monotonicity***.

[12] Notice the abuse of notation. By $\rho_i(c, E) + d$ we represent the probability distribution that assigns the same probability to $i$ getting $x_i + d$ as $\rho_i(c, E)$ assigns to $i$ getting $x_i$. That is, $(\rho_i(c, E) + d)(x_i + d) = \alpha$ if and only if $\rho_i(c, E)(x_i) = \alpha$.

[13] That is, a rule satisfies one property if and only if its dual rule satisfies the other.

**Sure sacrifice**. For each $(c,E) \in \mathcal{D}^N$, and each $x \in X(c,E)$ such that $c_i - x_i < \min\left\{\left\lfloor \frac{L}{n} \right\rfloor, \left\lfloor \frac{c_i}{n} \right\rfloor\right\}$ for some $i \in N$, we have $\rho(c,E)(x) = 0$.

The previous two pairs of axioms have not been considered in this model before. They, nevertheless, naturally translate into this setting existing axioms in the benchmark model (e.g., Yeh, 2006; Moreno-Ternero and Villar, 2004).

We also consider another axiom for this setting, reflecting a basic requirement of *impartiality.* It states that, for two-agent problems, if a probabilistic rule only gives positive probability to two allocations such that the awards of a person differ by only one unit in the two allocations, then the probability placed on both allocations by the rule must in fact be equal. It therefore reflects the absence of biases in favor of one of the two agents for an extra unit, when just two allocations (differing only in the recipient of the extra unit) are available. Formally,

**Bilateral evenness**. For each problem $(c,E) \in \mathcal{D}^{\{i,j\}}$ and each pair $x, x' \in X(c,E)$ such that $x'_i = x_i + 1$ and $\rho(c,E)(y) = 0$ if and only if $y \notin \{x, x'\}$, we have $\rho(c,E)(x) = \rho(c,E)(x')$.

Finally, we consider the standard notion of *consistency* to relate the choices in problems before and after the departure of a given agent. This notion has been widely considered in axiomatic research, playing a crucial role in the benchmark model of perfectly divisible endowments (e.g., Thomson, 2019). In our probabilistic setting, we follow the formulation proposed by Moulin and Stong (2002). It says that, for each agent $i \in N$, we compute first her award $x_i$ and next divide each of the $E - x_i$ units randomly among $N \setminus \{i\}$, given the conditional claims, and assuming that random draws are stochastically independent. Formally,

**Consistency**. For each $(c,E) \in \mathcal{D}^N$, each $i \in N$, and each $x \in X(c,E)$,

$$\rho(c,E)(x) = \rho\left(c_{-i}, E - x_i\right)\left(x_{-i}\right) \cdot \rho(c,E)(X_i = x_i),$$

where $\rho(c,E)(X_i = x_i) = \rho(c,E)\left(\{x_i\} \times X\left(c_{-i}, E - x_i\right)\right)$.

## 2.2 Rules

Our axiomatic study will take us to three rules. The first rule is the *probabilistic uniform awards* rule. As the name suggests, this rule makes expected awards as equal as possible, distributing sequentially each unit of the endowment with uniform probability among all agents, dropping an agent when her claim is fully honored. There exist three equivalent ways to formally define this rule (e.g., Moulin, 2002a; Moulin and Stong, 2002). We consider here one that exploits the connection with the formal definition of *constrained* *equal awards* in the benchmark model. More precisely, for each $(c, E) \in \mathcal{D}^N$, let $\lambda > 0$ be the real number solving $\sum_{i \in N} \min\{c_i, \lambda\} = E$. Let $Q(c, E) = \{i \in N : c_i \leq \lambda\} = \{i \in N : c_i \leq \lfloor \lambda \rfloor\}$, which might be an empty set, with cardinality $q \geq 0$. The rule awards $\min\{c_i, \lfloor \lambda \rfloor\}$ units for sure to each $i \in N$. The remaining $(n - q)(\lambda - \lfloor \lambda \rfloor)$ units are randomly distributed (with equal probability) over $N \setminus Q(c, E)$, giving at most one unit per agent. Formally,

$$\rho^{UA}(c, E)(x) = \frac{1}{|\overline{X}(c, E)|} \quad \forall x \in \overline{X}(c, E).$$

where

$$\overline{X}(c, E) = \left\{ y \in X(c, E) \,\middle|\, \begin{array}{l} y_i = c_i \ \forall i \in Q(c, E) \text{ and} \\ y_i \in \left\{ \left\lfloor \dfrac{\overline{E}}{n - q} \right\rfloor, \left\lfloor \dfrac{\overline{E}}{n - q} \right\rfloor + 1 \right\} \ \forall i \notin Q(c, E) \end{array} \right\},$$

with $Q(c, E) = \{i \in N : c_i \leq \lambda\}$, $\lambda$ is such that $\sum_{i \in N} \min\{c_i, \lambda\} = E$, $q = |Q(c, E)|$ and $\overline{E} = E - \sum_{i \in Q(c,E)} c_i$.

In the two-agent case, the rule has the following straightforward expression, whose representation appears in Figure 1. For ease of exposition, we assume that $(c, E) \in \mathcal{D}^{\{1,2\}}$.

If $E \geq 2c_1$ then

$$\rho^{UA}(c, E)(x) = \begin{cases} 1 & \text{if } x = (c_1, E - c_1) \\ 0 & \text{otherwise.} \end{cases}$$

If $E < 2c_1$ then

$$\rho^{UA}(c, E)(x) = \begin{cases} 1 & \text{if } E \text{ is even and } x = \left(\frac{E}{2}, \frac{E}{2}\right) \\ \frac{1}{2} & \text{if } E \text{ is odd and } x \in \left\{ \left(\lfloor \frac{E}{2} \rfloor, \lfloor \frac{E}{2} \rfloor + 1\right), \left(\lfloor \frac{E}{2} \rfloor + 1, \lfloor \frac{E}{2} \rfloor\right) \right\} \\ 0 & \text{otherwise.} \end{cases}$$

The second rule is the dual of the previous rule. Thus, it makes expected losses as equal as possible. We also define it exploiting the connection with the formal definition of *constrained equal losses* in the benchmark model. Specifically, for each $(c, E) \in \mathcal{D}^N$, let $\mu > 0$ be the real number solving the equation $\sum_{i \in N} \max\{c_i - \mu, 0\} = E$. Let $V(c, E) = \{i \in N : c_i \leq \mu\} = \{i \in N : c_i \leq \lfloor \mu \rfloor\}$, which might be an empty set, with cardinality $v \geq 0$. For each claimant $i \in N$, the rule subtracts for sure $\min\{c_i, \lfloor \mu \rfloor\}$ units from her claim. The remaining $(n - v)(\mu - \lfloor \mu \rfloor)$ units of loss are randomly spread (with equal probability) over $N \setminus V(c, E)$, with at most one unit per agent. Formally,

$$\rho^{UL}(c, E)(x) = \frac{1}{|\overline{X}(c, E)|} \quad \forall x \in \overline{X}(c, E).$$

where

$$\overline{X}(c,E) = \left\{ y \in X(c,E) \,\middle|\, y_i = 0 \; \forall i \in V(c,E) \text{ and } c_i - y_i \in \left\{ \left\lfloor \frac{\overline{L}}{n-v} \right\rfloor, \left\lfloor \frac{\overline{L}}{n-v} \right\rfloor + 1 \right\} \; \forall i \notin V(c,E) \right\},$$

with $V(c,E) = \{i \in N : c_i \leq \mu\}$, $\mu$ is such that $\sum_{i \in N} \max\{c_i - \mu, 0\} = E$, $v = |V(c,E)|$ and $\overline{L} = L - \sum_{i \in V(c,E)} c_i$.

In the two-agent case, the rule has the following straightforward expression, represented in Figure 1. For ease of exposition, we assume that $(c,E) \in \mathcal{D}^{\{1,2\}}$.

If $E \leq c_2 - c_1$ then

$$\rho^{UL}(c,E)(x) = \begin{cases} 1 & \text{if } x = (0,E) \\ 0 & \text{otherwise.} \end{cases}$$

If $E > c_2 - c_1$ then

$$\rho^{UL}(c,E)(x) = \begin{cases} 1 & \text{if } L \text{ is even and } x = \left(c_1 - \lfloor \frac{L}{2} \rfloor, c_2 - \lfloor \frac{L}{2} \rfloor\right) \\ \frac{1}{2} & \text{if } L \text{ is odd and } x \in \left\{ \left(c_1 - \lfloor \frac{L}{2} \rfloor, c_2 - \lfloor \frac{L}{2} \rfloor + 1\right), \left(c_1 - \lfloor \frac{L}{2} \rfloor + 1, c_2 - \lfloor \frac{L}{2} \rfloor\right) \right\} \\ 0 & \text{otherwise.} \end{cases}$$

The Talmud reports the following intriguing case. Two men disagree on the ownership of a garment, worth 200, say. The first man claims half of it, 100, and the other claims it all, 200. Assuming both claims to be made in good faith, the Talmud recommends the first man should get 50 whereas the second should get 150. A plausible rationale behind this solution is that the first man *concedes* the second what he does not claim, whereas the contested part is split equally. This can be generalized to solve any two-agent problem with perfectly divisible endowments, giving rise to what Thomson (2003) coined *concede-and-divide*. This amounts to conceding each agent what the other did not claim (from the endowment) and dividing equally the residual. The following definition translates this procedure to the context of this paper.

For ease of exposition, we assume that $(c,E) \in \mathcal{D}^{\{1,2\}}$.

(i) If $E \leq c_1$, then

$$\rho^{CD}(c,E)(x) = \begin{cases} 1 & \text{if } E \text{ is even and } x = \left(\frac{E}{2}, \frac{E}{2}\right) \\ \frac{1}{2} & \text{if } E \text{ is odd and } x \in \left\{ \left(\lfloor \frac{E}{2} \rfloor, \lfloor \frac{E}{2} \rfloor + 1\right), \left(\lfloor \frac{E}{2} \rfloor + 1, \lfloor \frac{E}{2} \rfloor\right) \right\} \\ 0 & \text{otherwise.} \end{cases}$$

(ii) If $c_1 \leq E \leq c_2$, then

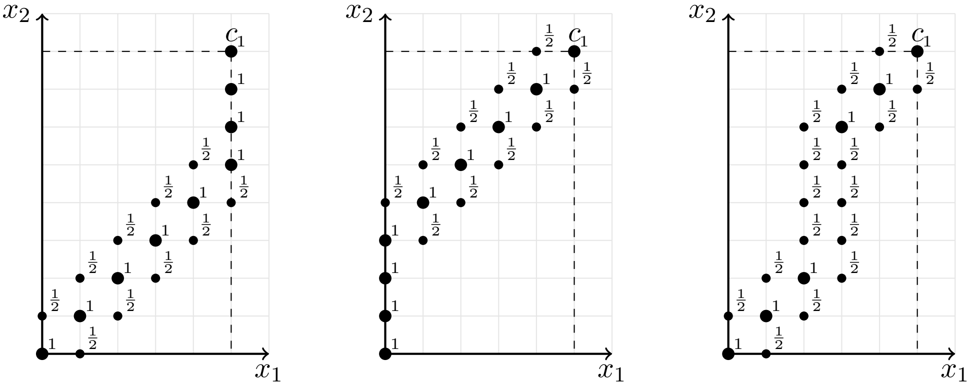
$$\rho^{CD}(c,E)(x) = \begin{cases} 1 & \text{if } c_1 \text{ is even and } x = \left(\frac{c_1}{2}, E - \frac{c_1}{2}\right) \\ \frac{1}{2} & \text{if } c_1 \text{ is odd and } x \in \left\{\left(\lfloor \frac{c_1}{2} \rfloor, E - \lfloor \frac{c_1}{2} \rfloor\right), \left(\lfloor \frac{c_1}{2} \rfloor + 1, E - \lfloor \frac{c_1}{2} \rfloor - 1\right)\right\} \\ 0 & \text{otherwise.} \end{cases}$$

(iii) If $c_2 \leq E$, then

$$\rho^{CD}(c,E)(x) = \begin{cases} 1 & \text{if } L \text{ is even and } x = \left(c_1 - \frac{L}{2}, c_2 - \frac{L}{2}\right) \\ \frac{1}{2} & \text{if } L \text{ is odd and } x \in \left\{\left(c_1 - \lfloor \frac{L}{2} \rfloor, c_2 - \lfloor \frac{L}{2} \rfloor - 1\right), \left(c_1 - \lfloor \frac{L}{2} \rfloor - 1, c_2 - \lfloor \frac{L}{2} \rfloor\right)\right\} \\ 0 & \text{otherwise.} \end{cases}$$

As we shall argue later in the text, extending this rule to the general case of an arbitrary number of agents is far from trivial.

Figure 1 represents, for the two-agent case, the three probabilistic rules introduced in this section.



(a) (b) (c)

Figure 1: **Figure 1: Rules in the two-claimant case**. This figure illustrates the "path of probabilistic allocations" of the three previous rules for $N = \{1, 2\}$ and $c \in \mathbb{R}^N_+$ with $c_1 < c_2$. Each bullet in the picture represents a deterministic allocation chosen with a positive probability, which is the number written near the corresponding bullet, for a given endowment (deterministic allocations associated to probability 1 are represented with a larger bullet). The path of probabilistic allocations for $c$ of a given rule is the collection of the deterministic allocations with non-null probabilities that the rule yields, as the endowment $E$ varies (in integer units) from 0 to the aggregate claim $c_1 + c_2$. (a) The path of awards for $c$ of the *probabilistic uniform awards* rule follows the band of width one around the $45^o$ line until the lowest claimant is fully compensated, i.e., until $E = 2c_1$, from where it is vertical until it reaches the vector of claims. For instance, if $c = (5, 8)$ and $E = 3$, it yields $(1, 2)$ and $(2, 1)$ with probability $\frac{1}{2}$ each, whereas if $E = 8$, it yields $(4, 4)$ with probability 1. (b) By duality, the path of awards for $c$ of the *probabilistic uniform losses* rule follows down from the vector of claims the band of width one around the line of slope 1 until it reaches the vertical axis, i.e., until $E = c_2 - c_1$. After that, it follows the vertical axis until the origin. For instance, if $c = (5, 8)$ and $E = 3$, it yields $(0, 3)$ with probability 1, whereas if $E = 8$, it yields $(2, 6)$ and $(3, 5)$ with probability $\frac{1}{2}$ each. (c) Finally, the path of awards of *probabilistic concede-and-divide* follows the band of width one around the $45^o$ line until $E = c_2$. Then, it follows a vertical band of width 1 until $E = c_2$, from where it follows the band of width one around the line of slope 1 until it reaches the vector of claims. For instance, if $c = (5, 8)$ and $E = 3$, it yields $(1, 2)$ and $(2, 1)$ with probability $\frac{1}{2}$ each, whereas if $E = 9$ it yields $(3, 6)$ with probability 1.

## 3 The results

The first result of our paper characterizes the *probabilistic uniform awards* rule.

**Theorem 1.** *A probabilistic rule satisfies conditional compensation, claim monotonicity, bilateral evenness, and consistency if and only if it is the probabilistic uniform awards rule.*

By duality, we obtain the corresponding characterization of the *probabilistic uniform losses* rule.

**Corollary 1.** *A probabilistic rule satisfies conditional sacrifice, linked claim-endowment monotonicity, bilateral evenness, and consistency if and only if it is the probabilistic uniform losses rule.*

Moulin and Stong (2002) provide alternative characterizations of these two rules. More precisely, they show that the *probabilistic uniform awards* rule is characterized by the combination of *upper composition* (and axiom pertaining to the way a rule reacts with respect to tentative estimations of the endowment to be allocated), *linked claim-endowment monotonicity* and two impartiality notions: *equal treatment ex ante*, and *equal treatment ex post*. As a consequence, they also obtain that the *probabilistic uniform losses* rule is characterized *lower composition* (the dual axiom of *upper composition*), *claim monotonicity* and the same (self-dual) impartiality axioms (*equal treatment ex ante*, and *equal treatment ex post*).

The proof of Theorem 1 is provided in the appendix. Therein, it is also shown that the first three axioms in Theorem 1 characterize the two-agent version of the *probabilistic uniform awards* rule (and, therefore, the first three axioms in Corollary 1 characterize the two-agent version of the *probabilistic uniform losses* rule). Somewhat of a counterpart result for probabilistic concede-and-divide is the following. Its proof also appears in the appendix.

**Theorem 2.** *A probabilistic rule satisfies sure compensation, sure sacrifice, and bilateral evenness if and only if it is probabilistic concede-and-divide.*

Nevertheless, as the next result implies, no counterpart to Theorem 1 can be obtained when combining the axioms from Theorem 2 with *consistency*. In other words, there is no consistent extension of *probabilistic concede-and-divide*.

**Theorem 3.** *No probabilistic rule satisfies sure compensation, sure sacrifice, bilateral evenness, and consistency.*

*Proof.* Suppose, by contradiction, that there exists a probabilistic rule $\rho$ satisfying the properties named in the theorem. Consider the problem $(c, E) = ((1, 3, 5), 3) \in \mathcal{D}^{\{1,2,3\}}$. Then,

$$X(c, E) = \left\{ \begin{array}{l} x^1 = (0, 0, 3),\, x^2 = (0, 1, 2),\, x^3 = (0, 2, 1),\, x^4 = (0, 3, 0), \\ x^5 = (1, 0, 2),\, x^6 = (1, 1, 1),\, x^7 = (1, 2, 0) \end{array} \right\}$$

For the sake of simplicity, we denote $p^i = \rho(c, E)(x^i)$.

Notice that, by Theorem 2, we already know how the rule behaves for two-agent problems. We then explore the application of *consistency*, allocation by allocation.

- For $x^1$. If Agent 1 leaves, the associated reduced problem is $((3, 5), 3) \in \mathcal{D}^{\{2,3\}}$. *Consistency* implies that

  $$\rho(c, E)(x^1) = \rho\left((3, 5), 3\right)(0, 3) \cdot \rho(c, E)(X_1 = 0),$$

  Theorem 2 states that $\rho\left((3, 5), 3\right)(0, 3) = 0$. Thus,

  $$p^1 = 0 \cdot (p^1 + p^2 + p^3 + p^4).$$

  If Agent 2 leaves, the associated reduced problem is $((1, 5), 3) \in \mathcal{D}^{\{1,3\}}$. *Consistency* and Theorem 3 together imply that

  $$p^1 = \frac{1}{2} \cdot (p^1 + p^5).$$

  Thus, we obtain

  $$p^1 = 0 = p^5. \tag{1}$$

- For $x^2$. If Agent 3 leaves, the associated reduced problem is $((1, 3), 1) \in \mathcal{D}^{\{1,2\}}$. *Consistency* and Theorem 2 together imply that

  $$p^2 = \frac{1}{2} \cdot (p^2 + p^5).$$

  If, instead, Agent 2 is who leaves, we have that

  $$p^2 = \frac{1}{2} \cdot (p^2 + p^6).$$

  Thus,

  $$p^2 = 0 = p^6. \tag{2}$$

- For $x^3$. If Agent 3 leaves, the associated reduced problem is $((1,3),2) \in \mathcal{D}^{\{1,2\}}$. *Consistency* and Theorem 3 together imply that

  $$p^3 = \frac{1}{2} \cdot (p^3 + p^6).$$

  If, instead, Agent 2 leaves, we have that

  $$p^3 = \frac{1}{2} \cdot (p^3 + p^7).$$

  Thus,

  $$p^3 = 0 = p^7. \tag{3}$$

- For $x^4$. If Agent 1 leaves, the associated reduced problem is $((3,5),3) \in \mathcal{D}^{\{2,3\}}$. *Consistency* and Theorem 3 together imply that

  $$p^4 = 0 \cdot (p^1 + p^2 + p^3 + p^4).$$

  Thus,

  $$p^4 = 0. \tag{4}$$

Therefore, from $(1) - (4)$, we conclude that $\rho$ is such that

$$\rho(c,E)(x) = 0,$$

for each $x \in X(c,E)$, which is a contradiction. □

Theorem 3 can be strengthened upon considering a weakening of *bilateral evenness.* Rather than stating that, for two-agent problems, when two allocations only differ in one unit, while any other allocation has zero probability to occur, then they should be equiprobable, it is only assumed that both allocations have positive probability (but not necessarily the same).[14]

## 4 Discussion

We have explored the implications of axioms reflecting principles of *compensation* and *sacrifice* in probabilistic rationing. We have shown that, together with basic notions

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[14] The proof of Theorem 3 could easily be adapted to account for this case and, thus, the incompatibility would remain. In other words, none of the *non-degenerate* rules within the family described at Proposition 5 in the appendix admit a consistent extension. This suggests that only *degenerate* rules within that family are potential candidates to escape from the impossibility. Nevertheless, we can also show that some of them do not admit a consistent extension. Thus, we can only state that whether the trio made of *sure compensation*, *sure sacrifice* and *consistency* is incompatible remains an open question.

of *monotonicity* and *consistency*, those axioms characterize the focal (and polar) *probabilistic uniform rules. Compensation* leads to the *probabilistic uniform awards*, whereas *sacrifice* leads to the *probabilistic uniform losses.* A basic impartiality requirement singles out the *equiprobable* version of those two rules. Otherwise, weighted versions of the rules (in which deterministic allocations in the support can be assigned unequal probabilities) can also be obtained. The results contrast with their existing counterparts in the benchmark model of rationing perfectly divisible endowments, where the so-called *constrained equal awards* and *constrained equal losses* rules are characterized without adding *consistency* and *impartiality.*

In the two-agent case, a compromise between both principles of *compensation* and *sacrifice* characterizes *probabilistic concede-and-divide.* Surprisingly, the result cannot be extended to an arbitrary number of agents. As a matter of fact, we show that the principles are incompatible with *consistency.* This is in stark contrast with the benchmark model of rationing perfectly divisible endowments, where the so-called *Talmud* rule is characterized by the combination of *consistency* and those two principles compromising between *compensation* and *sacrifice.*

One might be tempted to argue that our central axioms are highly specific/descriptive of the rules they characterize. We can, nevertheless, provide many examples of rules satisfying each of these axioms individually. In Appendix 5.4, we also show our characterizations are tight. In that sense, our results are of a similar nature to those in the benchmark model.

Our paper modifies the benchmark model in two directions: to discrete endowments and to probabilistic rules. Probabilistic rules could also be defined for continuous claims problems and deterministic rules could also be defined for discrete claims problems. To the best of our knowledge, the former case (namely, probabilistic rules for continuous claims problems) has not been explored in the literature. As for the latter case (namely, deterministic rules could also be defined for discrete claims problems), we have already mentioned throughout the paper the counterparts of the results we obtain here.

As mentioned above, we have obtained here counterpart results to those obtained by Moreno-Ternero and Villar (2004) and Yeh (2006). Now, we acknowledge that there are many other rules for the benchmark model, and even many other axiomatic results for the *constrained equal awards* rule, the *constrained equal losses* rule, and the *Talmud* rule. Instances appear in Dagan (1996), Herrero and Villar (2001), Yeh (2004), Moreno-Ternero and Villar (2006), or Yeh (2008), among others. It is left for further research to explore counterpart results to those. This would shed light on a general reformulation of rules and properties in the new model.

We framed our paper into the context of probabilistic rationing. As mentioned by Moulin and Stong (2002), an equivalent interpretation of the model is that of *scheduling* a list of homogeneous indivisible tasks, where agents have deterministic and fixed demands (claims) of possibly various sizes. With that interpretation, the *probabilistic uniform rules* are referred as *fair queuing methods.*[15]

To conclude, we mention that the principles of *compensation* and *sacrifice* we consider in this paper are all violated by the *proportional* rule, which plays a central role in this model (and related ones) and it is overwhelmingly used in real life. One might interpret this as a negative feature of these principles. We believe otherwise. In our view, the principles have normative appeal and connect to standard notions in distributive justice (such as lower bounds conditions). Therefore, they might constitute one of the very few aspects over which one might challenge the ubiquitous use of the *proportional* rule in resource allocation. In a sense, we hope our work conveys a message, especially to practitioners and outsiders alike to the field of normative economics (where this point has been made repeatedly for decades in the huge literature on claims problems): when it comes to resource allocation and distributive justice, there is life beyond proportionality!

# 5 Appendix

## 5.1 Proof of Theorem 1

In order to provide the proof of Theorem 1, we start uncovering the implications of *conditional compensation* and *claim monotonicity*, when applied together in the two-agent case.

**Proposition 1.** *If a probabilistic rule $\rho$ satisfies conditional compensation and claim monotonicity then*

*(i) If $E \leq 2c_1$, then*

$$\rho(c,E)(x) = 0 \quad \forall x \in X(c,E) \text{ such that } |x_1 - x_2| > 1.$$

*(ii) If $E \geq 2c_1$, then*

$$\rho(c,E)(x) = \begin{cases} 1 & \text{if } x = (c_1, E - c_1) \\ 0 & \text{otherwise.} \end{cases}$$

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[15] See also Moulin (2007, 2008) for a variant of the problem without "processor sharing".

*Proof.* Let $(c, E) \in \mathcal{D}^{\{1,2\}}$. If $E \geq 2c_1$, *conditional compensation* implies the result. If $E < 2c_1$, we distinguish two cases: **Case 1**. $E$ is even.

In this case, $\left(\frac{E}{2}, \frac{E}{2}\right) \in X(c, E)$. We define two new claims vectors: $c' = \left(\frac{E}{2}, c_2\right)$ and $c'' = \left(c_1, \frac{E}{2}\right)$. By *conditional compensation*,

$$\rho(c', E)(x) = \rho(c'', E)(x) = \begin{cases} 1 & \text{if } x = \left(\frac{E}{2}, \frac{E}{2}\right) \\ 0 & \text{otherwise.} \end{cases}$$

Now, by *claim monotonicity*,

$$\rho_1(c, E) \succeq_{\mathrm{SD}} \rho_1(c', E) \quad \text{and} \quad \rho_2(c, E) \succeq_{\mathrm{SD}} \rho_2(c'', E).$$

As $\rho_1(c, E) \succeq_{\mathrm{SD}} \rho_1(c', E)$, it follows that $\mathrm{Prob}\left(\rho_1(c, E) \geq \frac{E}{2}\right) \geq \mathrm{Prob}\left(\rho_1(c', E) \geq \frac{E}{2}\right) = 1$. This implies that $\rho_1(c, E)(a) = 0$ for each $a \in \mathbb{N}$ such that $a < \frac{E}{2}$. And, then, $\rho_2(c, E)(a) = 0$ for each $a \in \mathbb{N}$ such that $a > \frac{E}{2}$. Similarly, as $\rho_2(c, E) \succeq_{\mathrm{SD}} \rho_2(c'', E)$, it follows that $\mathrm{Prob}\left(\rho_2(c, E) \geq \frac{E}{2}\right) \geq \mathrm{Prob}\left(\rho_2(c', E) \geq \frac{E}{2}\right) = 1$. This implies that $\rho_2(c, E)(a) = 0$ for each $a \in \mathbb{N}$ such that $a < \frac{E}{2}$. Thus,

$$\rho(c, E)(x) = \begin{cases} 1 & \text{if } x = \left(\frac{E}{2}, \frac{E}{2}\right) \\ 0 & \text{otherwise} \end{cases},$$

which implies that $\rho(c, E)(x) = 0$ for each $x \in X(c, E)$ such that $|x_1 - x_2| > 1$.

**Case 2**. $E$ is odd.

Let $y = \left(\lfloor \frac{E}{2} \rfloor, \lfloor \frac{E}{2} \rfloor + 1\right)$ and $z = \left(\lfloor \frac{E}{2} \rfloor + 1, \lfloor \frac{E}{2} \rfloor\right)$. Let $c' = (y_1, c_2) = (\lfloor \frac{E}{2} \rfloor, c_2)$ and $c'' = (c_1, z_2) = (c_1, \lfloor \frac{E}{2} \rfloor)$. By *conditional compensation*,

$$\rho(c', E)(x) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \rho(c'', E)(x) = \begin{cases} 1 & \text{if } x = z \\ 0 & \text{otherwise.} \end{cases}$$

Now, by *claim monotonicity*,

$$\rho_1(c, E) \succeq_{\mathrm{SD}} \rho_1(c', E) \quad \text{and} \quad \rho_2(c, E) \succeq_{\mathrm{SD}} \rho_2(c'', E).$$

As $\rho_1(c, E) \succeq_{\mathrm{SD}} \rho_1(c', E)$, it follows that $\mathrm{Prob}\left(\rho_1(c, E) \geq y_1\right) \geq \mathrm{Prob}\left(\rho_1(c', E) \geq y_1\right) = 1$. Then, $\rho_1(c, E)(a) = 0$ for each $a \in \mathbb{N}$ such that $a < y_1$, and $\sum_{a=y_1}^{E} \rho_1(c, E)(a) = 1$. Similarly, as $\rho_2(c, E) \succeq_{\mathrm{SD}} \rho_2(c'', E)$, it follows that $\mathrm{Prob}\left(\rho_2(c, E) \geq z_2\right) \geq \mathrm{Prob}\left(\rho_2(c', E) \geq z_2\right) = 1$, and then $\rho_2(c, E)(a) = 0$ for each $a \in \mathbb{N}$ such that $a < z_2$, and $\sum_{a=z_2}^{E} \rho_2(c, E)(a) = 1$.

Notice that, by definition, $\rho_1(c, E)(x_1) = \rho_2(c, E)(E - x_1)$. Because $\rho_1(c, E)(a) = 0$ for each $a < y_1$, we also have that $\rho_2(c, E)(E - a) = 0$ for each $a < y_1$ and, equivalently, that

$\rho_2(c,E)(a) = 0$ for each $a > E - y_1 = y_2$. Dually, we can obtain that $\rho_1(c,E)(a) = 0$ for each $a > E - z_2 = z_1$. Now,

$$\begin{aligned} 1 &= \textstyle\sum_{a=y_1}^{E} \rho_1(c,E)(a) \\ &= \rho_1(c,E)(y_1) + \rho_1(c,E)(z_1) + \textstyle\sum_{a=z_1+1}^{E} \rho_1(c,E)(a) \\ &= \rho_1(c,E)(y_1) + \rho_1(c,E)(z_1). \end{aligned}$$

In other words, $\rho(c,E)(y_1,y_2) + \rho(c,E)(z_1,z_2) = 1$, which implies that $\rho(c,E)(x) = 0$ for each $x \in X(c,E)$ such that $|x_1 - x_2| > 1$. □

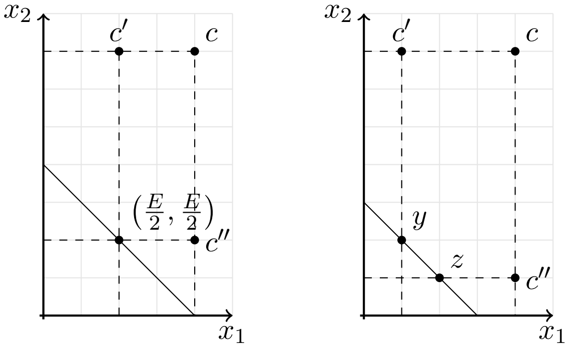

Figure 2: Illustration of the proof of both cases for Proposition 1.

As the next result states, *bilateral evenness* closes the gap from Proposition 1 to characterize the *probabilistic uniform awards* rule in the two-agent case.

**Proposition 2.** *If $|N| = 2$, a probabilistic rule satisfies conditional compensation, claim monotonicity, and bilateral evenness if and only if it is the probabilistic uniform awards rule.*

*Proof.* It is straightforward to show that the *probabilistic uniform awards* rule satisfies the three axioms of the theorem. Conversely, let $\rho$ be a probabilistic rule that satisfies the three axioms of the theorem. Let $(c,E) \in \mathcal{D}^{\{1,2\}}$. By Proposition 1, if $E \geq 2c_1$,

$$\rho(c,E)(x) = \begin{cases} 1 & \text{if } x = (c_1, E - c_1) \\ 0 & \text{otherwise.} \end{cases}$$

Assume now that $E < 2c_1$. By Proposition 1, if $|x_1 - x_2| > 1$ then $\rho(c,E)(x) = 0$. If $E$ is even, the previous condition implies that $\rho(c,E)(x) = 1$ if and only if $x = \left(\frac{E}{2}, \frac{E}{2}\right)$, as desired. If $E$ is odd, the previous condition implies that $\rho(c,E)(y) + \rho(c,E)(z) = 1$, where $y = \left(\lfloor\frac{E}{2}\rfloor, \lfloor\frac{E}{2}\rfloor + 1\right)$ and $z = \left(\lfloor\frac{E}{2}\rfloor + 1, \lfloor\frac{E}{2}\rfloor\right)$. By *bilateral evenness*, $\rho(c,E)(y) = \rho(c,E)(z) = \frac{1}{2}$. □

We then extend Proposition 1, exploring the implications of *conditional compensation* and *claim monotonicity*, when imposed together for probabilistic rules in the general case of an arbitrary number of agents. Proposition 3 states that, under those circumstances, some deterministic allocations are excluded (have zero probability to occur). They are those in which the awards to two non-fully compensated claimants differ by no more than a certain threshold. More precisely, agents with "small" claims are fully compensated, whereas the rest receive, *with positive probability*, at least an equal share of the remaining units.

Before introducing Proposition 3, we provide the following lemma, which provides an alternative description of the set $Q(c,E)=\{i\in N : c_i\leq\lambda\}$ (comprising agents that are fully compensated) in the definition of the probabilistic uniform awards rule.

**Lemma 1.** *For each* $(c,E)\in\mathcal{D}^N$,

$$Q(c,E)=\left\{i\in N:\sum_{j=1}^{n}\min\{c_i,c_j\}\leq E\right\}.$$

*Proof.* Let $(c,E)\in\mathcal{D}^N$ be given. And let $\lambda\in\mathbb{R}_+$ be such that $\sum_{i=1}^{n}\min\{c_i,\lambda\}=E$. As $c_1\leq c_2\leq\ldots\leq c_n$, there exists $k\in N$ such that

$$\min\{c_i,\lambda\}=\begin{cases}c_i & \text{if } i\in N_k=\{1,\ldots,k-1\}\\ \lambda=\frac{E-\sum_{j=1}^{k-1}c_j}{n-k+1} & \text{otherwise.}\end{cases}$$

Note that if $k=1$ above, then $N_1=\emptyset$ and $\lambda=\frac{E}{n}$. Note also that $Q(c,E)=N_k$.

Let $i\in Q(c,E)$. Then,

$$\sum_{j=1}^{n}\min\{c_i,c_j\}=\sum_{j=1}^{i}c_j+\sum_{j=i+1}^{k-1}c_i+\sum_{j=k}^{n}c_i\leq\sum_{j=1}^{i}c_j+\sum_{j=i+1}^{k-1}c_j+(n-k+1)\lambda=E.$$

Thus, $i$ is such that $\sum_{j=1}^{n}\min\{c_i,c_j\}\leq E$, as desired.

Conversely, let $i$ be such that $\sum_{j=1}^{n}\min\{c_i,c_j\}\leq E$. As $\sum_{j=1}^{n}\min\{\lambda,c_j\}=E$, we conclude that $c_i\leq\lambda$, and thus, $i\in Q(c,E)$. $\square$

**Proposition 3.** *For each* $(c,E)\in\mathcal{D}^N$, *let* $Q(c,E)=\left\{i\in N:\sum_{j=1}^{n}\min\{c_i,c_j\}\leq E\right\}$, $q=|Q(c,E)|$, $\overline{E}=E-\sum_{i\in Q(c,E)}c_i$, *and* $T=\overline{E}-\left\lfloor\frac{\overline{E}}{n-q}\right\rfloor(n-q)$. *Let*

$$\overline{X}(c,E)=\Bigg\{y\in X(c,E)\,\Bigg|\,y_i=c_i\ \forall i\in Q(c,E)\ \textit{and}$$
$$y_i\in\left\{\left\lfloor\frac{\overline{E}}{n-q}\right\rfloor,\left\lfloor\frac{\overline{E}}{n-q}\right\rfloor+1,\ldots,\left\lfloor\frac{\overline{E}}{n-q}\right\rfloor+T\right\}\ \forall i\notin Q(c,E)\Bigg\}.$$

*If a probabilistic rule* $\rho$ *satisfies conditional compensation and claim monotonicity, then* $\rho(c,E)(x) = 0$ *for each* $x \notin \overline{X}(c,E)$.

*Proof.* Let $\rho$ be a probabilistic rule that satisfies *conditional compensation* and *claim monotonicity*. Let $(c,E) \in \mathcal{D}^N$, $Q(c,E) = \left\{ i \in N : \sum_{j=1}^{n} \min\{c_i, c_j\} \leq E \right\}$, $q = |Q(c,E)|$, $\overline{E} = E - \sum_{i \in Q(c,E)} c_i$, and $T = \overline{E} - \left\lfloor \frac{\overline{E}}{n-q} \right\rfloor (n-q)$. Also, let

$$\widehat{X}(c,E) = \left\{ x \in X(c,E) \middle| \begin{array}{l} x_i = c_i \ \forall i \in Q(c,E) \text{ and} \\ x_i \in \left\{ \left\lfloor \dfrac{\overline{E}}{n-q} \right\rfloor, \left\lfloor \dfrac{\overline{E}}{n-q} \right\rfloor + 1 \right\} i \notin Q(c,E) \end{array} \right\}.$$

Let $i \in N \setminus Q(c,E)$. There exists $x^i \in \widehat{X}(c,E)$ such that $x^i_i = \left\lfloor \frac{\overline{E}}{n-q} \right\rfloor$. Let us define the claims vector $c^i$ as follows

$$c^i_j = \begin{cases} x^i_i & \text{if } j = i \\ c_i & \text{otherwise.} \end{cases}$$

For this claims vector, agent $i$ is considered “small”, in the sense that replacing all higher claims by $i$’s claim, the endowment would be sufficient to cover them all:

$$\begin{aligned} \sum_{j \in N} \min\left\{c^i_i, c^i_j\right\} &= \sum_{j \in Q(c,E)} \min\left\{c^i_i, c^i_j\right\} + \sum_{j \notin Q(c,E)} \min\left\{c^i_i, c^i_j\right\} \\ &= \sum_{j \in Q(c,E)} c^i_j + \sum_{j \notin Q(c,E)} \min\left\{ \left\lfloor \frac{\overline{E}}{n-q} \right\rfloor, c^i_j \right\} \\ &\leq \sum_{j \in Q(c,E)} c_j + \left\lfloor \frac{\overline{E}}{n-q} \right\rfloor (n-q) \\ &= E - \left( \overline{E} - \left\lfloor \frac{\overline{E}}{n-q} \right\rfloor (n-q) \right) \\ &\leq E. \end{aligned}$$

By *conditional compensation*, $\rho_i(c^i,E)(x^i_i) = \rho_i(c^i,E)(c^i_i) = 1$. Now, as $c_i \geq x^i_i = c^i_i$, by *claim monotonicity*, $\rho_i(c,E) \succeq_{\mathrm{SD}} \rho_i(c^i,E)$, which means that

$$\operatorname{Prob}\left(\rho_i(c,E) \geq x^i_i\right) \geq \operatorname{Prob}\left(\rho_i(c^i,E) \geq x^i_i\right).$$

As $\rho_i(c^i,E)(x^i_i) = 1$, we conclude that $\operatorname{Prob}\left(\rho_i(c,E) \geq x^i_i\right) = 1$, and, thus, $\operatorname{Prob}\left(\rho_i(c,E) \geq \left\lfloor \frac{\overline{E}}{n-q} \right\rfloor\right) = 1$. Therefore, $\rho_i(c,E)(a) = 0$ for each $a < \left\lfloor \frac{\overline{E}}{n-q} \right\rfloor$. On the other hand, as $\sum_{i \in N \setminus Q(c,E)} y_i = \overline{E}$ for each $y \in X(c,E)$, it follows that $\rho_i(c,E)(a) = 0$ for each $a > \left\lfloor \frac{\overline{E}}{n-q} \right\rfloor + T$. □

The next proposition says that, if in addition to *conditional compensation* and *claim monotonicity*, we require *consistency*, then the awards to the non-fully compensated agents differ by, at most, one unit. In other words, the units from the endowment that are not allocated to "small" agents are distributed as equally as possible.

**Proposition 4.** *Let* $(c,E) \in \mathcal{D}^N$, $Q(c,E) = \left\{ i \in N : \sum_{j=1}^{n} \min\{c_i, c_j\} \leq E \right\}$, $q = |Q(c,E)|$ *and* $\overline{E} = E - \sum_{i \in Q(c,E)} c_i$. *Also, let*

$$\overline{X}(c,E) = \left\{ y \in X(c,E) \middle| \, y_i = c_i \forall i \in Q(c,E) \text{ and } y_i \in \left\{ \left\lfloor \frac{\overline{E}}{n-q} \right\rfloor, \left\lfloor \frac{\overline{E}}{n-q} \right\rfloor + 1 \right\} \forall i \notin Q(c,E) \right\}.$$

*If a probabilistic rule* $\rho$ *satisfies conditional compensation, claim monotonicity, and consistency, then* $\rho(c,E)(x) = 0$ *for each* $x \notin \overline{X}(c,E)$.

*Proof.* Let $(c,E) \in \mathcal{D}^{\{i,j\}}$. We know that $\rho(c,E)(x) = 0$ when $x \notin \overline{X}(c,E)$, where

$$\overline{X}(c,E) = \left\{ y \in X(c,E) \middle| \, y_k = c_k \forall k \in Q(c,E) \text{ and } y_k \in \left\{ \left\lfloor \frac{\overline{E}}{2-q} \right\rfloor, \left\lfloor \frac{\overline{E}}{2-q} \right\rfloor + 1 \right\} \forall k \notin Q(c,E) \right\},$$

with $Q(c,E) = \left\{ k \in \{i,j\} : \sum_{h=1}^{2} \min\{c_k, c_h\} \leq E \right\}$, $q = |Q(c,E)|$, $\overline{E} = E - \sum_{i \in Q(c,E)} c_i$.

Let $(c,E) \in \mathcal{D}^N$. Applying *consistency* iteratively, we obtain that, for each pair $\{i,j\} \subseteq N$ and each $x \in X(c,E)$,

$$\begin{aligned} \rho(c,E)(x) = & \rho\left(c_{\{i,j\}}, E - \sum_{t \in N \setminus \{i,j\}} x_t\right)\left(x_{\{i,j\}}\right) \\ & \cdot \rho\left(c_{\{i,j,k\}}, E - \sum_{t \in N \setminus \{i,j,k\}} x_t\right)(X_k = x_k) \\ & \cdot \ldots \\ & \cdot \rho\left(c,E\right)\left(X_{i_n} = x_{i_n}\right). \end{aligned}$$

As $\rho\left(c_{\{i,j\}}, E - \sum_{t \in N \setminus \{i,j\}} x_t\right)\left(x_{\{i,j\}}\right) = 0$, when $x \notin \overline{X}(c,E)$, the desired conclusion follows. □

If, in addition to the previous three axioms, we require *bilateral evenness*, then we obtain that the probability distribution on the set $\overline{X}(c,E)$ is uniform; namely, we characterize the *probabilistic uniform awards rule*, as stated in Theorem 1, whose proof we conclude now.

*Proof.* It is well known that the probabilistic uniform awards rule satisfies *claim monotonicity* and *consistency* (e.g., Moulin and Stong, 2002). It is straightforward to show that it also satisfies *conditional compensation* and *bilateral evenness.* We then focus on the converse implication. Let $\rho$ be a probabilistic rule that satisfies *conditional compensation*, *claim monotonicity*, *bilateral evenness*, and *consistency.* By Proposition 4, we already know that $\rho(c,E)(x) = 0$ when $x \notin \overline{X}(c,E)$. We then concentrate on $\overline{X}(c,E)$ and proceed by induction on the cardinality of $N$. If $|N| = 2$, Proposition 1 yields the result. Let us suppose now that the result holds for each $N'$ with $|N'| = n-1$ and let us show that it is also true for $N$ when $|N| = n$. Let $x \in \overline{X}(c,E)$. Let $N^+(x) = \left\{ i \in N : x_i = \left\lfloor \frac{\overline{E}}{n-q} \right\rfloor + 1 \right\}$ and $n^+(x) = |N^+(x)|$.

**Case 1** $n^+(x) = 0$. Then $|\overline{X}(c,E)| = 1$. Proposition 4 says that the probability of any other allocation different from $x$ is zero, and therefore $\rho(c,E)(x) = 1$.

**Case 2** $n^+(x) = 1$. Then $|\overline{X}(c,E)| = n-q$. For each $i \in N \setminus Q(c,E)$, let $x^i \in \overline{X}(c,E)$ such that $x^i_i = \left\lfloor \frac{\overline{E}}{n-q} \right\rfloor + 1$ and $p^i = \rho(c,E)(x^i)$. Notice that, by construction, $x$ is one of those $x^i$. By *consistency*, for each $j \in N \setminus (Q(c,E) \cup \{i\})$, we have that

$$p^i = \rho(c,E)(x^i) = (1-p^j)\rho\left(c_{N\setminus\{j\}}, E - x^i_j\right)\left(x^i_{N\setminus\{j\}}\right).$$

By induction, as $|N \setminus \{j\}| < n$, we already know that

$$\rho\left(c_{N\setminus\{j\}}, E - x^i_j\right)\left(x^i_{N\setminus\{j\}}\right) = \frac{1}{\left|\overline{X}\left(c_{N\setminus\{j\}}, E - x^i_j\right)\right|}.$$

Then,

$$p^i = (1-p^j)\frac{1}{\left|\overline{X}\left(c_{N\setminus\{j\}}, E - x^i_j\right)\right|},$$

for each $j \in N \setminus (Q(c,E) \cup \{i\})$, and each $i \in N \setminus Q(c,E)$. Therefore,

$$p^i = p^{i'},$$

for each pair $i, i' \in N \setminus Q(c,E)$, and, thus,

$$p^i = \frac{1}{n-q} = \frac{1}{|\overline{X}(c,E)|},$$

for each $i \in N \setminus Q(c,E)$. As $x = x^i$ for some $i \in N \setminus Q(c,E)$, we conclude that

$$\rho(c,E)(x) = \frac{1}{|\overline{X}(c,E)|}.$$

**Case 3** $n^+(x) = k$ (with $1 < k < n-q-1$). Then $|\overline{X}(c,E)| = \binom{n-q}{k}$.

Given $S \subset N \setminus Q(c,E)$, we define $x^S \in \overline{X}(c,E)$ such that $x_i^S = \left\lfloor \frac{\overline{E}}{n-q} \right\rfloor + 1$ for each $i \in S$, and let $p^S = \rho(c,E)(x^S)$. Let $i \in S$. By *consistency*,

$$
\begin{aligned}
p^S &= \rho(c,E)(x^S) \\
&= \left( \sum_{\substack{M \subset N \setminus Q(c,E) \\ j \in M}} p^M \right) \cdot \rho\left(c_{N \setminus \{j\}}, E - x_j^S\right)\left(x^S_{N \setminus \{j\}}\right),
\end{aligned}
$$

for each $j \in S \setminus \{i\}$, and

$$
\begin{aligned}
p^S &= \rho(c,E)(x^S) \\
&= \left( 1 - \sum_{\substack{M \subset N \setminus Q(c,E) \\ j \in M}} p^M \right) \cdot \rho\left(c_{N \setminus \{j\}}, E - x_j^S\right)\left(x^S_{N \setminus \{j\}}\right),
\end{aligned}
$$

for each $j \notin S \cup Q(c,E)$.

By induction, as $|N \setminus \{j\}| < n$, we already know that

$$
\rho\left(c_{N \setminus \{j\}}, E - x_j^S\right)\left(x^S_{N \setminus \{j\}}\right) = \frac{1}{\left|\overline{X}\left(c_{N \setminus \{j\}}, E - x_j^S\right)\right|}.
$$

Then

$$
\sum_{\substack{M \subset N \setminus Q(c,E) \\ j \in M}} p^M = \sum_{\substack{M \subset N \setminus Q(c,E) \\ j' \in M}} p^M,
$$

for each pair $j, j' \in S \setminus \{i\}$, and

$$
\sum_{\substack{M \subset N \setminus Q(c,E) \\ j \in M}} p^M = \sum_{\substack{M \subset N \setminus Q(c,E) \\ j' \in M}} p^M,
$$

for each pair $j, j' \notin S \cup Q(c,E)$. Therefore,

$$
p^S = p^{S'},
$$

for each pair $S, S' \subset N \setminus Q(c,E)$.

**Case 4** $n^+(x) = n - q - 1$. Then $|\overline{X}(c,E)| = n - q$. For each $i \in N \setminus Q(c,E)$ let $x^i \in \overline{X}(c,E)$ be such that $x_i^i = \left\lfloor \frac{\overline{E}}{n-q} \right\rfloor$ and $p^i = \rho(c,E)(x^i)$. By construction, $x$ is one of those $x^i$. By *consistency*, for each $j \in N \setminus (Q(c,E) \cup \{i\})$, we have that

$$
p^i = \rho(c,E)(x^i) = (1 - p^j)\rho\left(c_{N \setminus \{j\}}, E - x_j^i\right)\left(x^i_{N \setminus \{j\}}\right).
$$

By induction, as $|N \setminus \{j\}| < 1$, we already know that

$$\rho\left(c_{N\setminus\{j\}}, E - x_j^i\right)\left(x^i_{N\setminus\{j\}}\right) = \frac{1}{\left|\overline{X}\left(c_{N\setminus\{j\}}, E - x_j^i\right)\right|}.$$

Using an argument analogous to the previous case, we obtain that

$$\rho(c,E)(x) = \frac{1}{|\overline{X}(c,E)|}.$$

□

## 5.2 Proof of Theorem 2

In order to provide the proof of Theorem 2, we start uncovering the implications of *sure compensation* and *sure sacrifice*, when applied together in the two-agent case.

**Proposition 5.** *If $|N| = 2$, a probabilistic rule $\rho$ satisfies sure compensation and sure sacrifice then $\rho$ is such that*

*(i) If $E \leq c_1$, then*

$$\rho(c,E)(x) = 0 \quad \forall x \in X(c,E) \text{ such that } |x_1 - x_2| > 1.$$

*(ii) If $c_1 \leq E \leq c_2$ and $c_1$ is even, then*

$$\rho(c,E)\left(\frac{c_1}{2}, E - \frac{c_1}{2}\right) = 1.$$

*(iii) If $c_1 \leq E \leq c_2$ and $c_1$ is odd, then*

$$\rho(c,E)(x) = 0 \quad \forall x \in X(c,E) \text{ such that } x_1 \notin \left\{\left\lfloor\frac{c_1}{2}\right\rfloor, \left\lfloor\frac{c_1}{2}\right\rfloor + 1\right\}.$$

*(iv) If $E \geq c_2$, then*

$$\rho(c,E)(x) = 0 \quad \forall x \in X(c,E) \text{ such that } |(c_1 - x_1) - (c_2 - x_2)| > 1.$$

*Proof.* Let $(c,E) \in \mathcal{D}^{\{1,2\}}$. We distinguish several cases:

**Case 1**. $E \leq c_1$.

In this case, $SC(c,E) = \left\{x \in X(c,E) : x_1 \geq \left\lfloor\frac{E}{2}\right\rfloor \text{ and } x_2 \geq \left\lfloor\frac{E}{2}\right\rfloor\right\}$. By *sure compensation* we know that $\sum_{x\in SC(c,E)} \rho(c,E)(x) = 1$. As $x_1 + x_2 = E$, $\rho(c,E)(x) = 0$ for each $x \in X(c,E)$ such that $|x_1 - x_2| > 1$.

**Case 2**. $c_1 \leq E \leq c_2$ and $c_1$ is even.

In this case, $SC(c,E) = \left\{x \in X(c,E) : x_1 \geq \frac{c_1}{2} \text{ and } x_2 \geq \left\lfloor \frac{E}{2} \right\rfloor\right\}$ and $SS(c,E) = \left\{x \in X(c,E) : x_1 \leq \frac{c_1}{2} \text{ and } x_2 \leq c_2 - \left\lfloor \frac{L}{2} \right\rfloor\right\}$. By *sure compensation* and *sure sacrifice*,

$$\sum_{x \in SC(c,E)} \rho(c,E)(x) = \sum_{x \in SS(c,E)} \rho(c,E)(x) = 1.$$

As $SC(c,E) \cap SS(c,E) = \left(\frac{c_1}{2}, E - \frac{c_1}{2}\right)$, then

$$\rho(c,E)\left(\frac{c_1}{2}, E - \frac{c_1}{2}\right) = 1.$$

**Case 3**. $c_1 \leq E \leq c_2$ and $c_1$ is odd.

In this case, $SC(c,E) = \left\{x \in X(c,E) : x_1 \geq \left\lfloor \frac{c_1}{2} \right\rfloor \text{ and } x_2 \geq \left\lfloor \frac{E}{2} \right\rfloor\right\}$ and $SS(c,E) = \left\{x \in X(c,E) : x_1 \leq c_1 - \left\lfloor \frac{c_1}{2} \right\rfloor \text{ and } x_2 \leq c_2 - \left\lfloor \frac{L}{2} \right\rfloor\right\}$. By *sure compensation* and *sure sacrifice*,

$$\sum_{x \in SC(c,E)} \rho(c,E)(x) = \sum_{x \in SS(c,E)} \rho(c,E)(x) = 1.$$

As $SC(c,E) \cap SS(c,E) = \{y,z\}$ where

$$y = \left(\left\lfloor \frac{c_1}{2} \right\rfloor, E - \left\lfloor \frac{c_1}{2} \right\rfloor\right) \text{ and } z = \left(\left\lfloor \frac{c_1}{2} \right\rfloor + 1, E - \left\lfloor \frac{c_1}{2} \right\rfloor - 1\right),$$

then

$$\rho(c,E)(x) = 0 \quad \forall x \in X(c,E) \text{ such that } x_1 \notin \left\{\left\lfloor \frac{c_1}{2} \right\rfloor, \left\lfloor \frac{c_1}{2} \right\rfloor + 1\right\}.$$

**Case 4**. $E > c_2$.

Notice that $SS(c,E) = \left\{x \in X(c,E) : x_1 \leq c_1 - \left\lfloor \frac{L}{2} \right\rfloor \text{ and } x_2 \leq c_2 - \left\lfloor \frac{L}{2} \right\rfloor\right\}$. By *sure sacrifice* we know that $\sum_{x \in SS(c,E)} \rho(c,E)(x) = 1$. As $(c_1 - x_1) + (c_2 - x_2) = L$, $\rho(c,E)(x) = 0$ for each $x \in X(c,E)$ such that $|(c_1 - x_1) - (c_2 - x_2)| > 1$. □

As stated in Theorem 2, whose proof we complete now, *bilateral evenness* closes the gap from Proposition 5 to characterize *probabilistic concede-and-divide*.

*Proof.* It is straightforward to show that *probabilistic concede-and-divide* satisfies the three axioms of the theorem. Conversely, let $\rho$ be a rule satisfying the three axioms of the theorem. Let $(c,E) \in \mathcal{D}^{\{1,2\}}$. By Proposition 5, Cases (ii) and (iii) are straightforward. We consider the other two cases. If $E \leq c_1$, by Proposition 5, if $\rho(c,E)(x) > 0$, then $x = \left(\frac{E}{2}, \frac{E}{2}\right)$ when $E$ is even, or $x \in \left\{\left(\lfloor \frac{E}{2} \rfloor, \lfloor \frac{E}{2} \rfloor + 1\right), \left(\lfloor \frac{E}{2} \rfloor + 1, \lfloor \frac{E}{2} \rfloor\right)\right\}$ when $E$ is odd. *Bilateral evenness* then concludes. If $c_2 \leq E$, by Proposition 5, if $\rho(c,E)(x) > 0$, then $x = \left(c_1 - \frac{L}{2}, c_2 - \frac{L}{2}\right)$ when $L$ is even, or $x \in \left\{\left(c_1 - \lfloor \frac{L}{2} \rfloor, c_2 - \lfloor \frac{L}{2} \rfloor - 1\right), \left(c_1 - \lfloor \frac{L}{2} \rfloor - 1, c_2 - \lfloor \frac{L}{2} \rfloor\right)\right\}$ when $L$ is odd. *Bilateral evenness* then concludes. □

### 5.3 A table

In Table 1 we provide the performance of the rules considered in the paper with respect to the mentioned axioms. $Y^*$ and $Y^{MS}$ mean that the axiom is used for the characterization of the corresponding rule in this paper and in Moulin and Stong (2002), respectively.

| | $\rho^{UA}$ | $\rho^{UL}$ | $\rho^{CD}$ |
|---|---|---|---|
| Claim monotonicity | $Y^*$ | $Y^{MS}$ | Y |
| Linked claim-endowment monotonicity | $Y^{MS}$ | $Y^*$ | Y |
| Conditional compensation | $Y^*$ | N | N |
| Conditional sacrifice | N | $Y^*$ | N |
| Sure compensation | Y | N | $Y^*$ |
| Sure sacrifice | N | Y | $Y^*$ |
| Bilateral Evenness | $Y^*$ | $Y^*$ | $Y^*$ |
| Consistency | $Y^*$ | $Y^*$ | - |
| Equal treatment ex-ante | $Y^{MS}$ | $Y^{MS}$ | Y |
| Equal treatment ex-post | $Y^{MS}$ | $Y^{MS}$ | Y |
| Lower composition | N | $Y^{MS}$ | N |
| Upper composition | $Y^{MS}$ | N | N |

Table 1: Axioms satisfied by the rules analyzed in this paper.

### 5.4 On the tightness of the characterization results

**Example 1.** *Let* $\sigma$ *be a linear ordering of the set of all pairs* $(i,a) \in \mathbb{N} \setminus \{0\} \times \mathbb{N}$ *such that (i) for each* $i \in \mathbb{N} \setminus \{0\}$*, and each* $a \in \mathbb{N} \setminus \{0\}$*,* $(i, a+1)\sigma(i,a)$*, and (ii)* $a > b$*, then* $(i,a)\sigma(j,b)$ *for any* $\{i,j\} \subset \mathbb{N}$*. For each problem* $(N,c,E) \in \mathcal{D}$ *proceed as follows. Start by fully compensating all of the agents. Identify each agent with her claim. Subtract one unit from the agent corresponding to the pair with the highest priority, according to* $\sigma$ *among all those involved in the problem. Remove the pair. Identify the agent corresponding to the pair with the next highest priority, according to* $\sigma$*, and proceed in the same way. Repeat this process until the endowment has been reached. This process identifies a unique deterministic allocation* $x^\sigma \in X(N,c,E)$*.*[16] *We define the probabilistic*

---

[16] We adapt the *monotonic standards of comparison* used in Herrero and Martínez (2008) to our probabilistic setting.

*rule* $\rho^\sigma$ *as*

$$\rho^\sigma(N,c,E)(x) = \begin{cases} 1 & \text{if } x = x^\sigma \\ 0 & \text{otherwise.} \end{cases}$$

*The rule* $\rho^\sigma$ *satisfies claim monotonicity, conditional compensation, and consistency, but violates bilateral evenness.*

**Example 2.** *Let* $\overline{\sigma}$ *be a linear ordering of the potential set of agents* $\mathbb{N}$. *For each problem* $(N,c,E) \in \mathcal{D}$ *identify the deterministic allocation* $x^{\overline{\sigma}} \in X(N,c,E)$ *such that*

$$[i \,\overline{\sigma}\, j \text{ and } x_j > 0] \;\Rightarrow\; x_i = c_i.$$

*We define the probabilistic rule* $\rho^\sigma$ *as*

$$\rho^{\overline{\sigma}}(N,c,E)(x) = \begin{cases} 1 & \text{if } x = x^{\overline{\sigma}} \\ 0 & \text{otherwise.} \end{cases}$$

*The rule* $\rho^{\overline{\sigma}}$ *satisfies claim monotonicity, bilateral evenness, and consistency, but violates conditional compensation.*

**Example 3.** *Let* $M^k(c) = \{j \in N \mid c_j = k.th \min_{i \in N} c_i\}$ *and* $m^k(c) = |M^k(c)|$.[17] *We define* $\rho'$ *as follows,*

$$\rho'(c,E)(x) = \frac{1}{|\overline{X}(c,E)|} \quad \forall x \in \overline{X}(c,E).$$

*where* $\overline{X}(c,E) \subset X(c,E)$ *is such that* $y \in \overline{X}(c,E)$ *if and only if, for each* $i \in M^k(c)$

$$y_i \in \begin{cases} \{0\} & \text{if } E \leq \sum_{\substack{j \in M^s(c) \\ s<k}} c_j \\ \left\{ \left\lfloor \frac{E - \sum_{\substack{j \in M^s(c) \\ s<k}} c_j}{m^k(c)} \right\rfloor, \left\lfloor \frac{E - \sum_{\substack{j \in M^s(c) \\ s<k}} c_j}{m^k(c)} \right\rfloor + 1 \right\} & \text{if } \sum_{\substack{j \in M^s(c) \\ s<k}} c_j \leq E \leq \sum_{\substack{j \in M^s(c) \\ s\leq k}} c_j \\ \{c_i\} & \text{otherwise} \end{cases}$$

*The rule* $\rho'$ *satisfies conditional compensation, bilateral evenness, and consistency, but violates claim monotonicity.*

**Example 4.** *Let* $\rho$ *be the probabilistic rule that coincides with* $\rho^{UG}$ *when* $n = 2$, *and with* $\rho^\sigma$ *(from Example 1) otherwise. This rule satisfies claim monotonicity, conditional compensation, and bilateral evenness, but violates consistency.*

---

[17] We adapt the example used in Herrero and Villar (2002) to our probabilistic setting.

The previous four examples show that Theorem 1 is tight as none of the axioms used therein is redundant. As for Corollary 1, it suffices to consider the dual rules of the ones presented in the examples. Finally, for Theorem 2, the *probabilistic uniform gains* is an example of a rule satisfying *sure compensation* and *bilateral evenness*, but violating *sure sacrifice*. Likewise, the *probabilistic uniform losses* is an example of a rule satisfying *sure sacrifice* and *bilateral evenness*, but violating *sure compensation*. Finally, the rules from Proposition 5 different from *probabilistic concede-and-divide* are examples of rules satisfying *sure compensation* and *sure sacrifice* but violating *bilateral evenness*.

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