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Old rockers

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Keywords: Aggregation of preferences, majority rule, Borda count, Condorcet criterion.

JEL Classification: D60, D70.

Department of Economics

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Abstract

This paper discusses three classic social evaluation procedures in a very general scenario: the majority rule, the Borda count, and the Condorcet criterion. We do so assuming no structure on individual preferences and providing a very easy way of identifying and comparing those evaluation protocols. Our informational inputs are the individual pairwise comparisons of alternatives and the informational outputs can be social pairwise comparisons or social orderings. The majority rule is obtained as the only pairwise social evaluation that satisfies informational efficiency, anonymity, monotonicity, and symmetry (a variant of the classical result by May). Arrow's impossibility theorem appears as a corollary of this result. Borda and Condorcet evaluation functions are obtained along the same lines when we require the evaluation to be a social ordering. Both social evaluation functions can be regarded as applying the same principle over different domains (individual pairwise comparisons or social pairwise comparisons).

Keywords: Aggregation of preferences, majority rule, Borda count, Condorcet criterion, informational efficiency, separability.

JEL numbers: D60, D70

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1. Introduction

Arrow's Impossibility Theorem is one of the key results in social choice theory. It establishes that, under general conditions, there is no way of finding a social ordering over a set of alternatives from individual preferences (Arrow, 1963). Collective choice requires, therefore, to accept that the social ordering may not be defined in some problems, or to modify Arrow's informational framework, in particular, renouncing to informational efficiency, or introducing interpersonal utility comparisons (see D'Aspremont 1985, or Sen 1986 for a detailed discussion). The essence of the problem can be summarized by saying that any reasonable social evaluation should respect the preferences of the majority, but the majority rule is not transitive.

From this perspective, the evaluation protocols proposed by Borda (1784) and Condorcet (1785) are very relevant (even though motivated by different concerns). Both proposals advocate for an evaluation of social alternatives based on sensible principles, closely related to majority voting. They share the idea that the social evaluation of any pair of alternatives should consider the individual orderings over all the alternatives, which implies renouncing to informational efficiency. And they differ in the way of transforming those individual preferences into social evaluations. Borda focused on how much support each alternative receives, whereas Condorcet considered how many supporters back each alternative (e.g. Moulin, 1988, Young 1995).

In spite of its ancient origin and its frequent use, the interest for those evaluation protocols and their implications has a long standing in the literature, and it is still alive (e.g., Young 1974, Nitzan & Rubinstein 1981, Mihara 2017, Dasgupta & Maskin 2020, Herrero & Villar 2021, 2023, Barberà & Bossert 2022, 2023, Maskin 2023, Villar 2023). Old rockers die hard.

The purpose of this paper is to discuss the relationships between those social evaluation criteria in a general but friendly format. Regarding generality, individual preferences are not assumed to be transitive, complete, or even consistent (i.e., the strict preference may not be asymmetric, and indifference need not be symmetric). In such an unstructured framework, we shall take the individuals' pairwise comparisons of alternatives as our informational inputs. Two different types of social evaluations (informational outputs) are considered. On the one hand, *evaluation rules*, which transform individual pairwise comparisons into social pairwise comparisons. On the other hand, *evaluation functions*, which provide overall social evaluations for each individual alternative. This approach permits one greatly simplify the discussion and helps clarifying the differences between those evaluation protocols.

We shall see that an evaluation rule satisfies the properties of informational efficiency, anonymity, and monotonicity, if and only if the social pairwise score of alternative i vis a vis alternative j corresponds to the number of agents who prefer i to j (which implies, together with symmetry, that the social preference is defined by majority voting). In the same spirit, an evaluation function satisfies separability (the evaluation of each alternative only depends on the comparison of this alternative with each other), anonymity, monotonicity, and symmetry, if and only if it is the Borda evaluation function. Interestingly, applying the Borda evaluation function to the social preferences derived from majority voting yields the Condorcet evaluation function. We also show that both Borda and Condorcet can be regarded as utilitarian social welfare orderings defined over different domains.

2. The model

Consider a group of agents, $N = \{1,2, \dots, n\}$ and a set of alternatives $A = \{1,2, \dots, m\}$. Let $\succcurlyeq_h$ denote agent h's preference relation defined over ordered pairs $(i, j) \in A \times A, j \neq i$, with $\succ_h, \sim_h$ being the corresponding strict preference and indifference, respectively. No restriction is imposed on those preferences, so that they may be non-transitive, incomplete, or even inconsistent (i.e. strict preference need not be asymmetric nor indifference symmetric). Trivially, that also implies that we are implicitly assuming the property of universal domain, as all types of preferences are allowed.

An evaluation problem is a pair $E = (A, (\succcurlyeq_h)_{h=1}^n)$ consisting of a set of alternatives and a set of agents, represented by their preferences, which are defined over ordered pairs of those alternatives. Given a set of alternatives, A, with cardinal m, let Ω_A^n stand for the set of all evaluation problems involving n agents with preferences $(\succcurlyeq_h)_{h=1}^n$. We denote by $E_A^n = (\succcurlyeq_h)_{h=1}^n$ a generic element in that set, which corresponds to the preferences of n agents defined over A. Consequently, two problems within this framework differ in that the agents' preferences change. The question is how to obtain a social evaluation of those alternatives in A from the preferences of the agents in N.

To deal with this evaluation problem in such a general setting, our informational inputs will be the individual pairwise comparisons of alternatives. That is, for each individual $h \in N$, and each pair of ordered alternatives $(i, j) \in A \times A, i \neq j$, we consider whether this individual prefers i to j, is indifferent between i and j, or is not able to compare i and j. As for the evaluation outcomes, we consider two different procedures. The first one consists of transforming individual pairwise comparisons into social pairwise comparisons. The second one provides an overall evaluation of each individual alternative. We shall refer to the first type of procedure as evaluation *rules*, and to the second type as an evaluation *functions*.

For a given ordered pair, $(i, j) \in A \times A, j \neq i$, let p_{ij} denote the number of agents who strictly prefer alternative i to alternative j, e_{ij} the number of those who consider that

i is indifferent to j, and z_{ij} the number of those who cannot compare i and j. Now let $n_{ij} = \left(p_{ij} + \frac{e_{ij}}{2}\right)$ stand for "the number" of agents $h \in N$ such that $i \succcurlyeq_h j$.

Consider now the following social evaluation protocols:

- ***Majority rule***, $\succcurlyeq^M$. For each problem $E_A^n \in \Omega_A^n$, $i \succcurlyeq^M j \Leftrightarrow n_{ij} \geq n_{ji}$.
- ***Borda evaluation function***, $\succcurlyeq^B$. For each problem $E_A^n \in \Omega_A^n$, $i \succcurlyeq^B j \Leftrightarrow \sum_{k \neq i} n_{ik} \geq \sum_{k \neq j} n_{jk}$.
- ***Condorcet evaluation function***, $\succcurlyeq^C$. For each problem $E_A^n \in \Omega_A^n$, $i \succcurlyeq^C j \Leftrightarrow \sum_{k \neq i} q_{ik}(E_A^n) \geq \sum_{k \neq j} q_{jk}(E_A^n)$, where $q_{ik}(E_A^n) = 1$ if $n_{ik} > n_{ki}$, $q_{ik}(E_A^n) = \frac{1}{2}$ if $n_{ik} = n_{ki}$, and $q_{ik}(E_A^n) = 0$, otherwise.

According to the majority rule an alternative is better than another when there is a majority who prefer the first alternative to the second. Borda regards alternative i as socially better than alternative j when i is preferred to some other alternative more times than j in all pairwise comparisons. Finally, i precedes j in the Condorcet social ordering when the number of times that i beats another alternative by majority voting is larger than the number of times that j does (when that number is $(m-1)$ we say that this alternative is a Condorcet winner).

Evaluation rules

An ***evaluation rule*** defined on A is a mapping $F: \Omega_A^n \to \mathbb{R}^{(m-1)\times(m-1)}$ such that, for each ordered pair of alternatives (i, j), with $j \neq i$, $F_{ij}(E_A^n)$ tells us the social evaluation of alternative i with respect to alternative j. An evaluation rule is thus a procedure to aggregate individual pairwise opinions into social pairwise evaluations.

We now present some familiar properties that an evaluation rule may satisfy. The first one, informational efficiency (originally "independence of irrelevant alternatives"), says that the social evaluation of a pair of alternatives only depends on the agents' evaluations of those two alternatives. That is,

Informational efficiency: Let $E_A^n, (E_A^n)' \in \Omega_A^n$ two problems such that, for all $h \in N$, $i \succ'_h j$ if and only if $i \succ_h j$, and $i \sim'_h j$ if and only if , $i \sim_h j$. Then, $F_{ij}\left((E_A^n)'\right) = F_{ij}(E_A^n)$.

The second property, anonymity, establishes that only the agents' preferences matter for the social evaluation. Formally,

Anonymity: Let $E_A^n, (E_A^n)' \in \Omega_A^n$ two problems such that $(E_A^n)'$ obtains from E_A^n by permuting the agents' preferences. Then, $F_{ij}(E_A^n) = F_{ij}\left((E_A^n)'\right)$.

The third property, monotonicity, says that more favourable opinions regarding an alternative increase its social worth. Namely,

Monotonicity: Let $E_A^n, (E_A^n)' \in \Omega_A^n$ two problems such that: (i) $i \succ'_h j$ whenever $i \succ_h j$, for all h; and (ii) $i \succcurlyeq'_h j$ whenever $i \sim_h j$, for all h, with $i \succ'_{h^*} j$, for some h^*. Then, $F_{ij}\left((E_A^n)'\right) > F_{ij}(E_A^n)$.

The last property, symmetry, establishes that if two alternatives cannot be distinguished in the agents' preferences, they cannot be distinguished by the evaluation rule.

Symmetry: Let $E_A^n \in \Omega_A^n$ be a problem in which, for all $h \in N$, either $i \sim_h j$ and $j \sim_h i$, or both (i, j) and (j, i) are incomparable. Then, $F_{ij}(E_A^n) = F_{ji}(E_A^n)$.

The following result is obtained:

Proposition 1: (i) *An evaluation rule F satisfies informational efficiency, anonymity, and monotonicity, if and only if $F_{ij}(E_A^n) = g_{ij}(n_{ij})$, for some increasing function g_{ij}.* (ii) *If symmetry also holds, then F is the majority rule. Moreover, all those properties are independent.*

Proof

The direct implications of (i) and (ii) are immediate. To see the reciprocal, note that informational efficiency and anonymity imply that the social evaluation of the pair (i, j) only depends on the number n_{ij}. That is, $F_{ij}(E_A^n) = g_{ij}(n_{ij})$. It follows from monotonicity that function g_{ij} must be increasing.

Now consider the family of problems $E_A^n \in \Omega_A^n$, in which, for all $h \in N$, either $i \sim_h j$ and $j \sim_h i$, or both (i, j) and (j, i) are incomparable. Let x denote half the number of the agents who are indifferent between alternatives i and j. By symmetry it must be the case that $g_{ij}(x) = g_{ji}(x)$. But this holds for all possible values of x. As g_{ij} is an increasing function, $g_{ij} = g_{ji} = g_{\{i,j\}}$. Hence, for any problem $E_A^n \in \Omega_A^n$, by monotonicity, $n_{ij} > n_{ji} \Leftrightarrow g_{\{i,j\}}(n_{ij}) > g_{\{i,j\}}(n_{ji})$, so that F is the majority rule.

To see that those properties are independent, consider the following evaluation function:

(a) $F_{ij}(E_A^n) = \sum_{j \neq i} n_{ij}$ satisfies all properties but informational efficiency.

(b) $F_{ij}(E_A^n) = n_{ij} + \delta_{h^*}$, where $\delta_{h^*} = 1$ when individual $h^* \in N$ prefers i to j, satisfies all properties but anonymity.

(c) $F_{ij}(E_A^n) = -n_{ij}$ satisfies all properties but monotonicity.

(d) $F_{ij}(E_A^n) = n_{ij} + \gamma_{k^*}$, where $\gamma_{k^*} = 1$ when alternative $k^* \in A$ is preferred by some agent to alternative j, satisfies all properties but symmetry.

Q.e.d.

Let us briefly comment on this result, which can be regarded as a minor variant of May's characterization of the majority rule (May, 1952). The first part of Proposition 1 says that assuming informational efficiency, anonymity, and monotonicity, is equivalent to apply an evaluation rule $F_{ij}(E_A^n) = g_{ij}(n_{ij})$, where g_{ij} is an increasing function. Even though this is an immediate outcome, note that, for each alternative $i \in A$ the evaluation rule F provides, an $(m-1)$ vector of pairwise comparisons, $[F_{ij}(E_A^n)]_{j \neq i}$. This means that, in general, we cannot compare globally two alternatives in A, which is a different way of stating that we cannot rank socially alternatives from individual preferences with this criterion. Formally,

Corollary 1: *There is no evaluation rule F, that satisfies informational efficiency, anonymity, and monotonicity, that provides a social ordering of the alternatives in A.*

This Corollary simply says that part (i) of Proposition 1 can also be regarded as an expression of Arrow's impossibility theorem: there is no social welfare ordering that satisfies informational efficiency, anonymity, and monotonicity (universal domain being implicit). Note that this is still true when individual preferences are regular, i.e., transitive and complete.

As the g_{ij} functions in part (i) of Proposition 1 are specific to each ordered pair (i, j), we cannot make any comparison between different pairs, not even between (i, j) and (j, i). Assuming symmetry, though, permits one comparing complementary pairs, (i, j) and (j, i), as $g_{ij} = g_{ji} = g_{\{i,j\}}$ compatible with informational efficiency. This, in turn, yields the majority rule as the only social evaluation that satisfies those four criteria.

Evaluation functions

An ***evaluation function*** defined on A is a mapping $\phi: \Omega_A^n : \to \mathbb{R}^m$ such that $\phi_i(E_A^n) > \phi_j(E_A^n)$ means that alternative i is regarded as socially better than alternative j. This is clearly a more demanding evaluation protocol and implies a social ordering of the alternatives. The following properties that an evaluation function may satisfy, can be regarded as translations to this context of those presented above.

The first one, separability, establishes that the social evaluation of an alternative only depends on how the agents value this alternative vis a vis the others. The remaining properties are the same as above, applied to the case of evaluation functions.

Separability: An evaluation function $\phi: \Omega_A^n \to \mathbb{R}^m$ is separable if, for all $i \in A$, $\phi_i(E_A^n)$ only depends on the individual comparisons of the pairs $(i,1),(i,2),\dots,\big(i,(i-1)\big),\big(i,(i+1)\big),\dots,(i,m)$.

Anonymity: Let $E_A^n,(E_A^n)' \in \Omega_A^n$ two problems such that $(E_A^n)'$ obtains from E_A^n by permuting the agents' preferences. Then, $\phi_i(E_A^n) = \phi_i\big((E_A^n)'\big)$.

Monotonicity: Let $E_A^n,(E_A^n)' \in \Omega_A^n$ two problems such that: (i) $i \succ_h' j$ whenever $i \succ_h j$, for all h; and (ii) $i \succcurlyeq_h' j$ whenever $i \sim_h j$, for all h, with $i \succ_{h^*}' j$, for some h^*. Then, $\phi_i\big((E_A^n)'\big) > \phi_j(E_A^n)$.

Symmetry: Let $E_A^n \in \Omega_A^n$ be a problem in which, for all $h \in N$, all $j \neq i$, either $i \sim_h j$ and $j \sim_h i$, or both (i,j) and (j,i) are incomparable. Then, $\phi_i(E_A^n) = \phi_j(E_A^n)$.

The following result is obtained:

Proposition 2: *An evaluation function $\phi: \Omega_A^n \to \mathbb{R}^m$ satisfies separability, anonymity, monotonicity, and symmetry, if and only if it is the Borda social evaluation function* (*i.e.,* $\phi_i(E_A^n) > \phi_j(E_A^n) \Leftrightarrow \sum_{k\neq i} n_{ik} > \sum_{k\neq j} n_{jk}$). Besides, those properties are independent.

Proof

By anonymity and separability, $\phi_i(E_A^n) = g_i\big(n_{i1},\dots,n_{i(i-1)},n_{i(i+1)},\dots,n_{im}\big)$. Monotonicity implies that g_i is increasing and symmetry that $\phi_i(E_A^n)$ only depends on $\sum_{k\neq i} n_{ik}$, with $\phi_i(E_A^n) = \phi_j(E_A^n)$ whenever $\sum_{k\neq i} n_{ik} = \sum_{k\neq i} n_{jk}$. As this happens for all possible values of $\sum_{k\neq i} n_{ik}$ and g_i is increasing for all i, it must be the case that $g_i = g_j = g$, for all i,j, with g increasing. Hence, $\phi_i(E_A^n) > \phi_j(E_A^n) \Leftrightarrow g(\sum_{k\neq i} n_{ik}) > g\big(\sum_{k\neq j} n_{jk}\big) \Leftrightarrow \sum_{k\neq i} n_{ik} > \sum_{k\neq j} n_{jk}$.

To see that those properties are independent, we replicate the argument in Proposition 1. That is,

(e) $\phi_i(E_A^n) = \sum_{i=1}^m \sum_{j\neq i} n_{ij}$ satisfies all properties but separability.

(f) $\phi_i(E_A^n) = \sum_{j\neq i} n_{ij} + \delta_{h^*}$, where δ_{h^*} is the number of times that individual $h^* \in N$ prefers i to another alternative, satisfies all properties but anonymity.

(g) $\phi_i(E_A^n) = -\sum_{j\neq i} n_{ij}$ satisfies all properties but monotonicity.

(h) $\phi_i(E_A^n) = \sum_{j\neq i} n_{ij} + \gamma_{k^*}$, where γ_{k^*} is the number of times that alternative $k^* \in A$ is preferred by some agent to another alternative, satisfies all properties but symmetry.

Q.e.d.

Anonymity and separability make the evaluation of each alternative a function of the number of agents who prefer that alternative to any other. And symmetry, as it happened in Proposition 1, introduces a way of comparing alternatives. Indeed, the Borda evaluation function can be regarded as a double transformation of the agents' preferences. First, we compare socially pairs of alternatives by associating to each ordered pair the corresponding number of favourable votes. Then we add up those numbers (by losing informational efficiency while so doing). From this perspective we can also think of the Borda evaluation function as associating to each alternative a majority criterion defined over the number of times that an alternative is preferred to some other.[1] To see this, let $\#_i(h)$ denote the number of times that alternative i is preferred to some other alternative, by agent h, plus half the number of times that this agent is indifferent between i and j, in all pairwise comparisons. Then, $\sum_{k\neq i} n_{ik} = \sum_{h=1}^{n} \#_i(h)$, so that we can write: $i \succcurlyeq^B j \Leftrightarrow \sum_{h=1}^{n} \#_i(h) \geq \sum_{h=1}^{n} \#_j(h)$. That is, alternative i is socially better than or equal to alternative j if the number of times that i is better than or equal to some other alternative, in the agents' pairwise comparisons, is larger than or equal to that of j.

The case of a single-agent society

Let us consider the case in which the set N is a singleton, that is, the family of problems Ω_A^1. Even though it might seem bizarre discussing the social evaluation problem when there is a single agent, there are some elements of that particular setting that help the comparison between those evaluation protocols. Note that we can interpret this case as that of a society made with n identical agents, under anonymity.

It is immediate to observe that a social evaluation rule defined over Ω_A^1 that satisfies informational efficiency and monotonicity corresponds to the identity mapping over the agent's pairwise comparisons. This might be regarded as implementing the majority rule, as the social evaluation of the alternatives corresponds to the opinions of the agent. The impossibility result in this context appears when the agent's preferences are not regular

[1] This also tells us that this evaluation function satisfies the principle of neutrality, which is the equivalent to anonymity with respect to the alternatives (this is a different way of stating $g_i = g_j = g$ in the proof o Proposition 2).

(e.g., because the agent might be applying several criteria to the evaluation of the alternatives).[2]

A social evaluation function defined over Ω_A^1 induces a complete ordering and can thus be interpreted as the agent's utility function. In particular,

Corollary 2: *An evaluation function* $u: \Omega_A^1 \to \mathbb{R}^m$ *satisfies separability, monotonicity, and symmetry, if and only if* $u(i) = \#_i, \forall i \in A$*.*

Proof

Anonymity holds trivially on Ω_A^1 so that, by Proposition 2, $u(i)$ must be the Borda evaluation function. That is, $u(i) > u(j) \Leftrightarrow \sum_{k \neq i} n_{ik} > \sum_{k \neq j} n_{jk}$. But in this case, there is a single agent so that $\sum_{k \neq i} n_{ik} = \#_i$.

Q.e.d.

The Borda evaluation function in the case of a single agent society corresponds to a utility function in which the evaluation of each alternative is given by the number of alternatives that are below in the preference ranking. That is, those preferences can be described by the following pairwise comparison: $s_{ij} = 1$ if $i \succ j$, $s_{ij} = \frac{1}{2}$, if $i \sim j$, and $s_{ij} = 0$, otherwise. Then, $\#_i = \sum_{j \neq i} s_{ij}$.

Also observe that this utility function also corresponds to the Condorcet evaluation function, so that in this particular case both evaluation functions coincide.

The Condorcet evaluation function

The single-agent society is also useful to relate the three evaluation protocols from a different viewpoint. Now consider an evaluation function defined over the social preferences derived from the majority rule, $\succcurlyeq^M$. That is, our informational inputs are now the following: $i \succcurlyeq^M j \Leftrightarrow n_{ij} \geq n_{ji}$. This can be described, without loss of generality, as follows: $q_{ij}(E_A^n) = 1$ if $if\ n_{ij} > n_{ji}$, $q_{ij}(E_A^n) = \frac{1}{2}$, if $n_{ij} = n_{ji}$, and $q_{ij}(E_A^n) = 0$, otherwise.

Let $\hat{E}_A^n$ denote the derived evaluation problem in in which we compare alternatives pairwise in terms of the majority rule. Then:

Corollary 3: *An evaluation function* ϕ *defined on the set of derived problems* $\hat{E}_A^n$ *satisfies separability, monotonicity, and symmetry, if an only if it is the Condorcet evaluation function.*

Proof

[2] "Each one of us is a confederacy of souls" wrote Antonio Tabucchi in *Sostiene Pereira*.

First note that anonymity is trivially satisfied as this derived problem is equivalent to a single-agent social evaluation problem. Then, it suffices to note that the Borda evaluation function is equivalent to those properties (Proposition 2) and that it corresponds to the Condorcet evaluation function when applied to those social preferences.

Q.e.d.

The Condorcet evaluation function associates to each alternative the number of alternatives that it dominates, according to the majority rule, whereas the Borda evaluation function associates to each alternative the number of times that this alternative dominates some other in the agents' preference relations. An alternative i is Condorcet maximal if $\phi_i(\hat{E}_A^n) \geq \phi_j(\hat{E}_A^n), \forall j \neq i$. When $\phi_i(\hat{E}_A^n) = m - 1$, we say that alternative i is a Condorcet winner. If $m - 1 = \phi_i(\hat{E}_A^n) > \phi_j(\hat{E}_A^n), \forall j \neq i$, then alternative i is a strong Condorcet winner. Clearly, a strong Condorcet winner may well not exist, even if the Condorcet evaluation function is well defined.

Corollary 3 can be rephrased as saying that the Borda evaluation function applied to the social preference derived from the majority rule yields the Condorcet evaluation function.

Utilitarianism

Let us now assume that the properties of separability, monotonicity, and symmetry hold for the individual evaluation function of each $h \in N$, considered as a single-agent society. Then, according to Corollary 2, each agent is endowed with a utility function $u_h(i) = \#_i(h)$. That is, the utility of alternative i for agent h is given by the number of alternatives that this agent finds worse than i, plus half of those who finds indifferent. In general, any monotonous transformation of this utility function is a valid representation of the same preferences. Consider the informational framework of *cardinal unit comparability* (D'Aspremont 1985), that is, only transformations of the form $v_h = au_h + b_h, a > 0$, are admissible representations of agent h's preferences, with the parameter a common to all agents. Then, a social welfare function that only depends on those utilities and satisfies anonymity must be utilitarian (D'Aspremont & Gevers, 1977), that is, $W(i) = \sum_{h=1}^{n} u_h(i)$. But this is precisely the Borda evaluation function, as $\sum_{j \neq i} n_{ij} = \sum_{h=1}^{n} \#_i(h)$.

The Borda evaluation function thus corresponds to the utilitarian social welfare function applied to the individual utilities under cardinal unit comparability. And the Condorcet evaluation function corresponds to the utilitarian social welfare function applied to the social pairwise comparisons derived from the majority rule. Which means that Borda and Condorcet apply the same social evaluation criterion over different domains. From this perspective, in the Borda evaluation function individual utilities aggregate pairwise comparisons in terms of the number of alternatives dominated, whereas the social welfare ordering adds up those utilities under the assumption of cardinal unit comparability. The Condorcet criterion aggregates agents' pairwise comparisons into pairwise comparisons of a collective agent, and then the social welfare ordering corresponds to the utility of this collective agent. We can say informally that Borda plus Borda yields Borda, whereas majority voting plus Borda yields Condorcet.

3. Final comments

The aggregation of individual preferences requires comparing alternatives and comparing individuals from a social viewpoint. Anonymity is the property that introduces the comparability across individuals, by assuming that they are equal in all respects other than their preferences. Hence, all individuals enter the social evaluation process on an equal foot. The property of informational efficiency, as defined in Section 2, is incompatible with going beyond the comparison of pairs of alternatives, so that, when combined with anonymity, no social ordering is to be expected. Yet the property of symmetry enables the social comparison of complementary pairs, (i, j) and (j, i), a feature that derives from the impossibility of socially differentiating those alternatives that agents cannot distinguish. The combination of informational efficiency with symmetry (together with anonymity and monotonicity) yields the majority rule as the social evaluation criterion that permits comparing socially pairs of alternatives.

An evaluation function is a much more demanding protocol, as it implies a proper social ordering (i.e., a complete and transitive criterion, with asymmetric strict preference and symmetric indifference). Substituting informational efficiency by separability and keeping the other properties we obtain the Borda evaluation function. Here again symmetry is the property that permits comparing alternatives from a social perspective.[3]

[3] Note, though, that looking at the Borda evaluation function as the sum of agents' utilities in Corollary 2, an informational efficiency property over utilities also holds. That is, the evaluation of two alternatives only depends on the agents' utilities regarding those two alternatives. Indeed, assuming that the social welfare ordering only depends on the agents' utilities amounts to assuming the property of "neutrality", which is equivalent to universal domain, the Pareto principle, and informational efficiency (over utilities), as shown in D'Aspremont & Gevers (1977). Yet informational efficiency defined over the utility functions is a very different property of that defined over the alternatives in Section 2. This is so because the existence of a utility function already implies an aggregation through the alternatives at individual level, as it requires comparing all the alternatives in one way or another.

The relationships between the majority rule and the Borda and the Condorcet evaluation functions, that derive from the characterizations presented above, can be summarized as follows:

- The Borda evaluation function can be regarded as applying a majority criterion over the number of times that an alternative is preferred to some other.
- The Condorcet evaluation function corresponds to the Borda evaluation function applied to the social preference derived from the majority rule.
- The Condorcet evaluation function, consequently, can be regarded as a majority criterion defined on the number of times that each alternative has a pairwise majority.
- The Borda evaluation function applied to a single-agent society corresponds to the agent's utility function.
- In a single-agent society the three evaluation protocols coincide.
- Borda and Condorcet evaluation functions can be interpreted as the utilitarian social welfare function applied to different domains (individual utilities under cardinality and comparability and the social pairwise comparisons derived from the majority rule, respectively).

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